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ON UNIQUENESS OF MEROMORPHIC FUNCTIONS IN AN ANGULAR DOMAIN

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Abstract

In this article, we investigate the uniqueness of meromorphic functions dealing with two shared values and a shared set in an angular domain. Results are obtained extending some results given by W. C. Lin and S. Mori.

1. Introduction and statement of results

In this paper, we assume that the reader is familiar with the standard notations of the Nevanlinna's value distribution theory (see e.g. [6], [11]), such as $T(r,f)$, $\sigma(f)$, the characteristic function and the order of a meromorphic function f respectively. Recall the hyper order of f is defined by

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

We denote $M(\sigma_2)$ by the set of transcendental meromorphic functions of finite hyper order.

For the sake of convenience, we use the following notations (see e.g. [9]). Let S be a nonempty subset of $\mathbf{C}_\infty := \mathbf{C} \cup \{\infty\}$, we put $E(S,f) = \bigcup_{a\in S}\{z \in \mathbf{C} \mid f(z) = a\}$, where all the roots of $f(z) = a$ in $E(S,f)$ are counted according to its multiplicities (CM).

Given a domain $X \subset \mathbf{C}$, we denote $E_X(S,f) = \bigcup_{a\in S}\{z \in \overline{X} \mid f(z) = a, CM\}$, where $\overline{X}$ is the closure of X in $\mathbf{C}$. When $X = \mathbf{C}$, $E_C(S,f) = E(S,f)$. Let f and g be two nonconstant meromorphic functions defined in $\mathbf{C}$. If $E_X(S,f) = E_X(S,g)$, we say f and g share the set S CM (counting multiplicities) in X. When $S = a$, we also say f and g share a CM. Throughout this paper, we set S_j $(j = 1,2,3)$ as $S_1 = \{0\}$; $S_2 = \{\infty\}$; $S_3 = \{w \mid w^n(w+a) - b = 0\}$, where $n \in \mathbf{N}$, and the algebraic equation $w^n(w+a) - b = 0$ has no multiple roots.

Since R. Nevanlinna proved his 'four-CM' and 'five-IM' theorems, there have been many results on the uniqueness of meromorphic functions in the complex

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plane (see e.g. [11]). In [14], J. H. Zheng firstly took into account the uniqueness dealing with five shared values in some angular domains of **C**. After that, J. H. Zheng [13] investigated the uniqueness of transcendental meromorphic functions dealing with shared values in an angular domain instead of the whole complex plane and prove the following.

Theorem A. *Let $f(z)$ and $g(z)$ be both transcendental meromorphic functions. Given an angular domain $X = \{z : \alpha < \arg z < \beta\}$ with $0 \le \alpha < \beta \le 2\pi$ and for some positive number ε and for some $a \in \mathbf{C}$*

$$\limsup_{r \to +\infty} \frac{\log n(r, \theta, \varepsilon, a)}{\log r} > \omega,$$

where $n(r, \theta, \varepsilon, a)$ is the number of zeros of $f(z) - a$ in $X(r) = \{|z| < r\} \cap X$ and $\omega = \dfrac{\pi}{\beta - \alpha}$. We assume that $f(z)$ and $g(z)$ share five distinct values a_j, $j = 1, 2, \ldots, 5$ IM in X, then $f \equiv g$.

Zheng [15] indicated that the proof of Theorem A used $R_{\alpha,\beta}(r, g) = O(\log r S_{\alpha,\beta}(r, g))$ but it is not clear that the equality would always hold. Hence, he add the following condition $\lim_{r \notin E \to +\infty} \dfrac{S_{\alpha,\beta}(r, g)}{\log rT(r, g)} = \infty$ to theorem A in [15]. For the uniqueness of meromorphic functions in the whole complex plane, H. X. Yi [12] established the following theorem for answering a question posed by Gross [5].

Theorem B. *Let $n \in \mathbf{N} - \{1\}$. If f and g are two entire functions satisfying, $E_C(S_j, f) = E_C(S_j, g)$, $j = 1, 3$, then $f \equiv g$.*

W. C. Lin and S. Mori [9] deal with Theorem B under certain value/set-sharing condition in a sector instead of the plane **C** and prove the following theorem.

Theorem C. *Let $f(z) \in M(\sigma_2)$, $\rho(f) = \infty$, and $\delta(\infty, f) > 0$. Then there exists a direction $\arg z = \theta$ $(0 \le \theta < 2\pi)$ such that for any ε $\left(0 < \varepsilon < \dfrac{\pi}{2}\right)$, if a meromorphic function $g(z) \in M(\sigma_2)$ satisfies the condition $E_C(S_1, f) = E_C(S_1, g)$ and $E_X(S_j, f) = E_X(S_j, g)$ for $j = 2, 3$, where $n \ge 3$ and $X = \{z : |\arg z - \theta| < \varepsilon\}$, then $f \equiv g$.*

Theorem C only discussed the transcendental meromorphic functions of finite hyper order. In this paper, we shall prove that Theorem C is valid for any transcendental meromorphic functions of infinite order. In order to establish our main results, we recall the following definitions and Lemma 1.

Lemma 1. *Let $B(r)$ be a positive and continuous function in $[0, +\infty)$ which satisfies* $\limsup_{r\to\infty} \dfrac{\log B(r)}{\log r} = \infty$, *then there exists continuously differentiable functions $\rho(r)$, which satisfies the following condition.*

(i) *$\rho(r)$ is continuous and nondecreasing for $r \geq r_0$ $(r_0 > 0)$ and tends to $+\infty$ as $r \to +\infty$.*

(ii) *The function $U(r) = r^{\rho(r)}$ $(r \geq r_0)$ satisfies the condition*

$$\lim_{r\to+\infty} \frac{\log U(R)}{\log U(r)} = 1, \quad R = r + \frac{r}{\log U(r)}.$$

(iii) $\limsup_{r\to+\infty} \dfrac{\log B(r)}{\log U(r)} = 1.$

Lemma 1 is due to K. L. Hiong [7]. A simple proof of the existence of $\rho(r)$ was given by Chuang [3].

Definition 1. We define $\rho(r)$ and $U(r)$ in Lemma 1 by the order and type function of $B(r)$, respectively. For a transcendental meromorphic function $f(z)$ of infinite order, we define its order and type function as the order and type function of $T(r, f)$. We denote $M(\rho(r))$ by the set of all meromorphic functions $f(z)$ in $\mathbf{C}$ such that $\limsup_{r\to+\infty} \dfrac{\log T(r, f)}{\log U(r)} = 1.$

Definition 2 (see e.g. [2]). Let $H(r)$ be a positive and continuous function in $[0, +\infty)$. Let $\rho(r)$ and $U(r)$ be a pair of real functions satisfying Lemma 1. We say that $H(r)$ is of order less than $\rho(r)$ if $\limsup_{r\to\infty} \dfrac{\log H(r)}{\log U(r)} < 1$. In order that $H(r)$ is of order less than $\rho(r)$, it is necessary and sufficient that we can fined a number μ $(0 < \mu < 1)$ such that $H(r) < U^{\mu}(r)$, when r is sufficiently large.

The main purpose of this paper is to prove the following theorems.

Theorem 1. *Let $f(z), g(z) \in M(\rho(r))$, and $\delta(\infty, f) > 0$. For given small ε $(0 < \varepsilon < \pi)$, let $X = \{z : |\arg z - \theta| < \varepsilon\}$, where $0 \leq \theta < 2\pi$. Suppose that for some $a \in \mathbf{C}$,*

$$(*) \qquad \limsup_{r\to+\infty} \frac{\log n\left(r, \theta, \frac{\varepsilon}{3}, a\right)}{\log U(r)} = 1,$$

where $n\left(r, \theta, \frac{\varepsilon}{3}, a\right)$ denotes the number of zeros of $f(z) - a$ in $X_{\varepsilon/3}(r) = \{|z| < r\} \cap \left\{z : |\arg z - \theta| < \frac{\varepsilon}{3}\right\}$. Assume that $f(z)$ and $g(z)$ satisfy the conditions $E_C(S_1, f) = E_C(S_1, g)$ and $E_X(S_j, f) = E_X(S_j, g)$ for $j = 2, 3$, where $n \geq 3$. Then $f \equiv g$.

It is well known that a meromorphic function $f(z) \in M(\rho(r))$ has at least one direction $\arg z = \theta$, $0 \le \theta < 2\pi$ from the origin such that for arbitrary small $\varepsilon > 0$, we have

$$\limsup_{r \to +\infty} \frac{\log n(r, \theta, \varepsilon, a)}{\log U(r)} = 1,$$

for all but at most two $a \in \mathbf{C}_\infty$ (see e.g. [3], [10]). From the Theorem 1, any meromorphic function $g(z) \in M(\rho(r))$ has at least one direction $\arg z = \theta$, $0 \le \theta < 2\pi$ under the value/set-sharing condition in Theorem C coincides with $f(z)$. Hence Theorem 1 extend the result give by [9].

Furthermore, we shall prove that Theorem 1 is valid for some transcendental meromorphic functions of finite order and prove the following theorem.

THEOREM 2. *Let $f(z)$, $g(z)$ be meromorphic functions of finite order growth. Suppose that $\delta(\infty, f) > 0$. Given one angular domain $X = \{z : \alpha < \arg z < \beta\}$, where $0 \le \alpha < \beta \le 2\pi$ and for some positive number ε and for some $a \in \mathbf{C}$*

$$\limsup_{r \to +\infty} \frac{\log n(r, X_\varepsilon, a)}{\log r} > \omega, \tag{1}$$

where $n(r, \theta, \varepsilon, a)$ is the number of zeros of $f(z) - a$ in $X_\varepsilon(\alpha, \beta)(r) = \{|z| < r\} \cap \{z : \alpha + \varepsilon < \arg z < \beta - \varepsilon\}$ and $\omega = \dfrac{\pi}{\beta - \alpha}$. We assume that $f(z)$ and $g(z)$ satisfy the condition $E_C(S_1, f) = E_C(S_1, g)$ and $E_X(S_j, f) = E_X(S_j, g)$ for $j = 2, 3$, where $n \ge 3$, then f and g satisfy one of the following two relations: (i) $f \equiv g$; (ii) $f^n(f + a)g^n(g + a) \equiv b^2$.

2. Some lemmas

Our proof requires the Nevanlinna theory in an angular domain. For the sake of convenience, we recall some notations and definitions. Let $f(z)$ be a meromorphic function. Consider an angular domain $\Omega(\alpha, \beta) = \{z \mid \alpha \le \arg z \le \beta\}$, where $0 < \beta - \alpha < 2\pi$. Nevanlinna defined the following notations (see e.g. [1], [8]).

$$A_{\alpha\beta}(r, f) = \frac{k}{\pi} \int_1^r \left(\frac{1}{t^k} - \frac{t^k}{r^{2k}} \right) \{\log^+ |f(te^{i\alpha})| + \log^+ |f(te^{i\beta})|\} \frac{dt}{t};$$

$$B_{\alpha\beta}(r, f) = \frac{2k}{\pi r^k} \int_\alpha^\beta \log^+ |f(te^{i\alpha})| \sin k(\theta - \alpha)\, d\theta;$$

$$C_{\alpha\beta}(r, f) = 2 \sum_{b \in \triangle} \left(\frac{1}{|b_v|^k} - \frac{|b_v|^k}{r^{2k}} \right) \sin k(\beta_v - \alpha),$$

where $k = \dfrac{\pi}{\beta - \alpha}$, $1 \le r < \infty$ and the summation $\sum_{b \in \triangle}$ is taken over all poles $b = |b|e^{i\theta}$ of the function $f(z)$ in the sector $\triangle : 1 < |z| < r$, $\alpha < \arg z < \beta$, each pole b occurs in the sum $\sum_{b\in\triangle}$ as many times as it's multiplicity, when pole b occurs only once in the sum $\sum_{b\in\triangle}$, we denote it $\overline{C}(r,f)$. Furthermore, for $r>1$, we define

$$D_{\alpha\beta}(r,f)=A_{\alpha\beta}(r,f)+B_{\alpha\beta}(r,f),\quad S_{\alpha\beta}(r,f)=C_{\alpha\beta}(r,f)+D_{\alpha\beta}(r,f).$$

For sake of simplicity, we omit the subscript in all notations and use $A(r,f)$, $B(r,f)$, $C(r,f)$, $D(r,f)$ and $S(r,f)$ instead of $A_{\alpha\beta}(r,f)$, $B_{\alpha\beta}(r,f)$, $C_{\alpha\beta}(r,f)$, $D_{\alpha\beta}(r,f)$ and $S_{\alpha\beta}(r,f)$.

Lemma 2 (see e.g. [13]). *Let $f(z)$ be a nonconstant meromorphic function in the plane and $\Omega(\alpha,\beta)$ be an angular domain, where $0<\beta-\alpha\le 2\pi$. Then,*

(i) *For any value $a\in\mathbf{C}$, we have*

$$S\left(r,\frac{1}{f-a}\right)=S(r,f)+O(1),$$

holds for any $r>1$.

(ii) *If $f(z)$ is of finite order, then $Q(r,f)=A\left(r,\frac{f'}{f}\right)+B\left(r,\frac{f'}{f}\right)=O(1)$.*

If $f(z)\in M(\rho(r))$, then (see e.g. [8], [10]) $Q(r,f)=A\left(r,\frac{f'}{f}\right)+B\left(r,\frac{f'}{f}\right)=O(\log U(r))$.

Lemma 3 (see e.g. [4], [9]). *Let $P(z)$ be a polynomial of degree $d>0$, and $f(z)$ be a nonconstant meromorphic function on $\overline{X}=\overline{\Omega}(\alpha,\beta)$. Then, $S(r,P(f))=dS(r,f)+O(1)$.*

For the end of this section, we recall the following notations (see e.g. [9]). Let $f(z)$ be a meromorphic funtion in an angular domain $\Omega(\alpha,\beta)$, we denote by $C_2(r,f)$ the counting function of poles of f in $\{z\in\Omega(\alpha,\beta):|z|<r\}$, where a simple pole is counted once and a multiple pole is counted twice. In the same way, we can define $C_2\left(r,\frac{1}{f}\right)$.

Lemma 4 (see e.g. [9]). *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions such that $f(z)$ and $g(z)$ share 1, ∞ CM in $X=\Omega(\alpha,\beta)$. Then, one of the following three cases holds:*

(i) $S(r)=C_2\left(r,\frac{1}{f}\right)+C_2\left(r,\frac{1}{g}\right)+2\overline{C}(r,f)+Q(r,f)+Q(r,g)$;

(ii) $f\equiv g$;

(iii) $fg\equiv 1$, *where $S(r)=\max\{S(r,f),S(r,g)\}$, $Q(r,f)$ and $Q(r,g)$ as defined in Lemma 2.*

3. Proof of theorems

Under the conditions of Theorem 1 and Theorem 2, suppose that $f\not\equiv g$. Let

$$F = \frac{f^n(f+a)}{b}, \quad G = \frac{g^n(g+a)}{b}.$$

Then F and G share 1 and ∞ CM in X. Some process of the proof in Lin and Mori [9] also valid for our theorems, so we recall their proof in step 1 as the following.

By Lemma 6 in [9], we deduce $F \not\equiv G$. Thus Lemma 7 in [9] implies that

$$\overline{C}\left(r, \frac{1}{f}\right) = \overline{C}\left(r, \frac{1}{g}\right) = Q(r,f) + Q(r,g). \tag{3}$$

Therefore, by the expression of F and G and (3) we have

$$\begin{aligned} &C_2\left(r, \frac{1}{F}\right) + C_2\left(r, \frac{1}{G}\right) + 2\overline{C}(r,F) \\ &\quad \le C\left(r, \frac{1}{f+a}\right) + C\left(r, \frac{1}{g+a}\right) + 2\overline{C}(r,f) + Q(r,f) + Q(r,g). \end{aligned} \tag{4}$$

Set $S_1(r) := \max\{S(r,f), S(r,g)\}$. Then, from the expression of F and G and Lemma 3, we have

$$S(r) = (n+1)S_1(r) + O(1), \tag{5}$$

where $S(r) := \max\{S(r,F), S(r,G)\}$. By (4) and Lemma 8 in [9] we deduce that

$$C_2\left(r, \frac{1}{F}\right) + C_2\left(r, \frac{1}{G}\right) + 2\overline{C}(r,F) \le \left(2 + \frac{4}{n}\right)S_1(r) + Q(r,f) + Q(r,g). \tag{6}$$

Proof of Theorem 1. Suppose that $FG \equiv 1$. Then

$$F = f^n(f+a)g^n(g+a) \equiv b^2,$$

which implies that 0, $-a$ and ∞ are all Picard exceptional values of f in X. This contradicts with $(*)$.

In fact, we deduce from $(*)$ that there exists a direction $L : \arg z = \theta_0$ in X, such that for any $\eta > 0$, $\{z : |\arg z - \theta_0| < \eta\} \subset X$, we have

$$\limsup_{r \to +\infty} \frac{\log n(r, \theta_0, \eta, a)}{\log U(r)} = 1. \tag{7}$$

In 1938, Valiron prove that (7) imply that $L : \arg z = \theta_0$ is a Borel direction of $f(z)$ (see [8, P132]). Hence there at most exists two Picard exceptional values of f in X.

Therefore, $FG \not\equiv 1$, and hence, by Lemma 4 and noting that $n \ge 3$, we have from (5) and (6), $S_1(r) \le Q(r,f) + Q(r,g)$. By Lemma 2 (ii), we have

$$S(r,f) = O(\log U(r)). \tag{8}$$

We deduce from (8) that the order of $S(r,f)$ is less than that of $\rho(r)$. Thus Definition 2 implies that we can fined a number μ $(0 < \mu < 1)$ such that

$$S(r,f) < (U(r))^{\mu}, \tag{9}$$

when r is sufficiently large.

For any $a \in \mathbf{C}$, let $b_v = |b_v|e^{i\beta_v}$ $(v = 1, 2, \ldots)$ be the roots of $f = a$ in the angular domain $\Omega(\theta - \varepsilon, \theta + \varepsilon)$, counting multiplicities. We set $n(r) = n\left(r, \theta, \frac{\varepsilon}{3}, f = a\right)$. In the angular domain $\Omega\left(\theta - \frac{\varepsilon}{3}, \theta + \frac{\varepsilon}{3}\right)$, we have $\theta - \frac{\varepsilon}{3} < \beta_v < \theta + \frac{\varepsilon}{3}$, $v = 1, 2, \ldots$. Hence, we deduce that $\frac{\varepsilon}{6} < \beta_v - \theta + \frac{\varepsilon}{2} < \frac{5\varepsilon}{6}$. From the Lemma 2 (i), it follows that

$$\begin{aligned}
S_{\theta-\varepsilon,\theta+\varepsilon}(R,f) &\geq C_{\theta-\varepsilon,\theta+\varepsilon}(R,a) + O(1) \geq C_{\theta-\varepsilon/2,\theta+\varepsilon/2}(R,a) + O(1) \\
&\geq 2 \sum_{1<|b_v|<r,\,\theta-\varepsilon/2<\beta_v<\theta+\varepsilon/2} \left(\frac{1}{|b_v|^k} - \frac{|b_v|^k}{R^{2k}}\right) \sin\frac{\pi}{\varepsilon}\left(\beta_v - \theta + \frac{\varepsilon}{2}\right) + O(1) \\
&\geq 2 \sum_{1<|b_v|<r,\,\theta-\varepsilon/3<\beta_v<\theta+\varepsilon/3} \left(\frac{1}{|b_v|^k} - \frac{|b_v|^k}{R^{2k}}\right) \sin\frac{\pi}{\varepsilon}\left(\beta_v - \theta + \frac{\varepsilon}{2}\right) + O(1) \\
&\geq \sum_{1<|b_v|<r,\,\theta-\varepsilon/3<\beta_v<\theta+\varepsilon/3} \left(\frac{1}{|b_v|^k} - \frac{|b_v|^k}{R^{2k}}\right) + O(1),
\end{aligned}$$

where $k = \frac{\pi}{\varepsilon}$ and R as defined in Lemma 1. We write above sum as a Stieltjes-integral and applying the integration by parts of the Stieltjes-integral

$$\begin{aligned}
S_{\theta-\varepsilon,\theta+\varepsilon}(R,f) &\geq \int_1^r \frac{1}{t^k}\, dn(t) - \frac{1}{R^{2k}} \int_1^r t^k\, dn(t) + O(1) \\
&\geq k \int_1^r \frac{1}{t^{k+1}} n(t)\, dt + \frac{n(r)}{r^k} - \frac{r^k n(r)}{R^{2k}} + \frac{k}{R^{2k}} \int_1^r t^{k-1} n(t)\, dt + O(1) \\
&\geq \frac{n(r)}{r^k} - \frac{r^k n(r)}{R^{2k}} + O(1) \\
&\geq \frac{n(r)}{r^k} - \frac{R^k n(r)}{R^{2k}} + O(1) \\
&\geq \left(\frac{1}{r^k} - \frac{1}{R^k}\right) n(r) + O(1).
\end{aligned} \tag{10}$$

For any $\alpha > 0$, we have

$$\limsup_{r\to\infty} \frac{\dfrac{1}{\frac{1}{r^k} - \frac{1}{R^k}}}{U^{\alpha}(r)} = 0. \tag{11}$$

From (9)–(11), we deduce that there exists a number μ' $(0 < \mu' < 1)$ such that for any $a \in \mathbf{C}$,

$$n\left(r, \theta, \frac{\varepsilon}{3}, f = a\right) < U^{\mu'}(r), \tag{12}$$

if r is sufficiently large. This contradicts with hypothesis (1) and Theorem 1 follows.

Proof of Theorem 2. Suppose that (ii) does not hold, then $FG \not\equiv 1$, and hence, by Lemma 4 and noting that $n \geq 3$, we have from (5) and (6), $S_1(r) \leq Q(r, f) + Q(r, g)$. By Lemma 2 (ii), we have

$$S(r, f) = O(1). \tag{13}$$

For any $a \in \mathbf{C}$, let $b_v = |b_v| e^{i\beta_v}$ $(v = 1, 2, \ldots)$ be the root of $f = a$ in the angular domain X_ε, counting multiplicities. We set $n(r) = n(r, X_\varepsilon, f = a)$. From the Lemma 2 (i) and using the same argument of [13], it follows that

$$\begin{aligned} S(2r, f) &\geq C(2r, a) + O(1) \\ &= 2 \sum_{1 < |b_v| < 2r,\, \alpha < \beta_v < \beta} \left(\frac{1}{|b_v|^k} - \frac{|b_v|^k}{(2r)^{2k}} \right) \sin k(\beta_v - \alpha) + O(1) \\ &\geq 2 \sin(k\varepsilon) \sum_{1 < |b_v| < 2r,\, \alpha + \varepsilon < \beta_v < \beta - \varepsilon} \left(\frac{1}{|b_v|^k} - \frac{|b_v|^k}{(2r)^{2k}} \right) + O(1) \\ &\geq 2(1 - 4^{-k}) \sin(k\varepsilon) \frac{n(r)}{r^k} + O(1), \end{aligned}$$

where $k = \dfrac{\pi}{\beta - \alpha} = \omega$. Then on combining (13), we have for any $a \in \mathbf{C}$,

$$n(r, X_\varepsilon, f = a) = O(r^k) = O(r^\omega), \tag{14}$$

if r is sufficiently large. This contradicts with hypothesis and Theorem 2 follows.

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