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A CHEVALLEY TYPE RESTRICTION THEOREM FOR A PROPER COMPLEX EQUIFOCAL SUBMANIFOLD

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Abstract

In this paper, we prove a Chevalley type restriction theorem for a proper complex equifocal submanifold. The proof is performed by showing the same type restriction theorem for an infinite dimensional proper anti-Kaehlerian isoparametric submanifold and using it.

1. Introduction

In 1985, C. L. Terng ([T1]) proved that the ring of all polynomials over a Euclidean space which are constant along parallel submanifolds of a given isoparametric submanifold in the space is isomorphic to that of all polynomials over a section of the submanifold which are invariant with respect to the associated Coxeter group. In fact, the restriction map to the section gives an isomorphism between these rings. This fact is similar to the so-called Chevalley restriction theorem for semi-simple Lie groups (see [W] for example). Also, in 1989, C. L. Terng ([T2]) proved that the ring of all C^∞-functions over a Hilbert space which are constant along parallel submanifolds of a given isoparametric submanifold in the space is isomorphic to that of all C^∞-functions over a section of the submanifold which are invariant with respect to the associated affine Coxeter group. From this result, she showed a similar restriction theorem for equifocal submanifolds in a symmetric space of compact type through a Riemannian submersion of a Hilbert space onto the symmetric space. For non-compact submanifolds in a symmetric space of non-compact type, the equifocality is a rather weak condition. So, we [K1] introduced the stricter condition of the complex equifocality. Furthermore, we [K2] introduced the notion of an infinite dimensional anti-Kaehlerian isoparametric submanifold and showed that the investigation of a complete, real analytic and complex equifocal submanifold is replaced by that of an infinite dimensional anti-Kaehlerian isoparametric submanifold. In the sequel, we assume that all complex equifocal submanifolds are complete and real anlytic. On the other hand, Heintze-Liu-Olmos [HLO] has recently introduced the notion of an isoparametric submanifold with flat section

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in a general Riemannian manifold. Isoparametric submanifolds with flat section in a symmetric space of non-compact type are complex equifocal and conversely, all complex equifocal submanifolds satisfying certain condition are isoparametric one with flat section (see Theorem 15 of [K2]). Also, we ([K2], [K3]) introduced the notions of an infinite dimensional proper anti-Kaehlerian isoparametric submanifold and a proper complex equifocal submanifold, where we note that principal orbits of Hermann type actions (i.e., the actions of (not necessarily compact) symmetric subgroups of G) on a symmetric space G/K of non-compact type are proper complex equifocal (see [K3]). We [K4] defined the notion of the complex Coxeter groups associated with these submanifolds. In the sequel, we assume that all infinite dimensional proper anti-Kaehlerian isoparametric submanifolds are complete. In this paper, we first prove the slice theorem for an infinite dimensional proper anti-Kaehlerian isoparametric submanifold, which states that the intersections of the submanifold with suitable finite dimensional anti-Kaehlerian affine subspaces are finite dimensional anti-Kaehlerian isoparametric ones. Also, we prove the Chevalley type restriction theorem for a finite dimensional proper anti-Kaehlerian isoparametric submanifold, which states that the ring of complex polynomials over the ambient space which are constant along parallel submanifolds of a given finite dimensional proper anti-Kaehlerian isoparametric submanifold is isomorphic to that of all complex poylnomials over a section of the submanifold which is invariant with respect to the associated complex Coxeter group. In fact, the restriction map to the section gives an isomorphism between these rings. By using these theorems, we prove the following Chevalley type restriction theorem for an infinite dimensional proper anti-Kaehlerian isoparametric submanifold.

THEOREM A. *Let M be a infinite dimensional proper anti-Kaehlerian isoparametric submanifold in an anti-Kaehlerian space V, Σ be a section of M and W be the complex Coxeter group* (*which acts on* Σ) *associated with M. Assume that a foliation $\mathfrak{F}$* (*which may have singular leaves*) *consisting of parallel submanifolds of M is defined on the whole of V. Then the restriction map $r : C^\infty(V, \mathbf{C})^{\mathfrak{F}} \to C^\infty(\Sigma, \mathbf{C})^W$ is an isomorphism, where $C^\infty(V, \mathbf{C})^{\mathfrak{F}}$ is the ring of all complex-valued C^∞-functions on V which are constant along leaves of $\mathfrak{F}$ and $C^\infty(\Sigma, \mathbf{C})^W$ is that of all W-invariant complex-valued C^∞-functions on Σ.*

The main theorem of this paper is the following Chevalley type restriction theorem for a proper complex equifocal submanifold.

THEOREM B. *Let M be a proper complex equifocal submanifold in a symmetric space G/K of non-compact type, Σ be a section of M and W be the complex Coxeter group* (*which acts on the extrinsic complexification $\Sigma^{\mathbf{c}}$ of* Σ) *associated with M. Let $\mathfrak{F}$* (*resp.* $\mathfrak{F}^{\mathbf{c}}$) *be a foliation* (*which may have singular leaves*) *consisting of parallel submanifolds of M* (*resp. the extrinsic complexification $M^{\mathbf{c}}$ of M*). *Assume that $\mathfrak{F}^{\mathbf{c}}$ is defined on the whole of the anti-Kaehlerian sym-*

metric space $G^{\mathbf{c}}/K^{\mathbf{c}}$ *associated with* G/K (*hence* $\mathfrak{F}$ *is defined on the whole of* G/K) *and* $G^{\mathbf{c}}$ *is simply connected. Then the restriction map* $r : C^{\infty}(G/K)^{\mathfrak{F}} \to C^{\infty}(\Sigma)^{W_{\Sigma}}$ *is an isomorphism, where* $C^{\infty}(G/K)^{\mathfrak{F}}$ *is the ring of all* (*real-valued*) C^{∞}-*functions on* G/K *which are constant along leaves of* $\mathfrak{F}$ *and* $C^{\infty}(\Sigma)^{W_{\Sigma}}$ *is that of all* (*real-valued*) C^{∞}-*functions on* Σ *which are invariant with respect to* $W_{\Sigma} := \langle \{R|_{\Sigma} \mid R \in W \text{ s.t. } R(\Sigma) = \Sigma\} \rangle$ ($\langle * \rangle$: *the group generated by the set* $*$).

Remark 1.1. The principal orbits of the Hermann type action (in the sense of [K3]) on a symmetric space of non-compact type are proper complex equifocal submanifolds satisfying the conditions of Theorem B.

2. Basic notions and facts

In this section, we first recall the notion of a proper complex equifocal submanifold. Let M be an immersed submanifold with abelian normal bundle in a symmetric space $N = G/K$ of non-compact type. Denote by A the shape tensor of M. Let $v \in T_x^{\perp}M$ and $X \in T_xM$ ($x = gK$). Denote by γ_v the geodesic in N with $\dot{\gamma}_v(0) = v$. The Jacobi field Y along γ_v with $Y(0) = X$ and $Y'(0) = -A_vX$ is given by

$$Y(s) = (P_{\gamma_v|_{[0,s]}} \circ (D^{co}_{sv} - sD^{si}_{sv} \circ A_v))(X),$$

where $Y'(0) = \tilde{\nabla}_v Y$, $P_{\gamma_v|_{[0,s]}}$ is the parallel translation along $\gamma_v|_{[0,s]}$,

$$D^{co}_{sv} = g_* \circ \cos(\sqrt{-1}\ \mathrm{ad}(sg_*^{-1}v)) \circ g_*^{-1}$$

and

$$D^{si}_{sv} = g_* \circ \frac{\sin(\sqrt{-1}\ \mathrm{ad}(sg_*^{-1}v))}{\sqrt{-1}\ \mathrm{ad}(sg_*^{-1}v)} \circ g_*^{-1}.$$

Here ad is the adjoint representation of the Lie algebra $\mathfrak{g}$ of G. All focal radii of M along γ_v are obtained as real numbers s_0 with $\mathrm{Ker}(D^{co}_{s_0v} - s_0D^{si}_{s_0v} \circ A_v) \neq \{0\}$. So, we call a complex number z_0 with $\mathrm{Ker}(D^{co}_{z_0v} - z_0D^{si}_{z_0v} \circ A^{\mathbf{c}}_v) \neq \{0\}$ a *complex focal radius of* M *along* γ_v and call $\dim \mathrm{Ker}(D^{co}_{z_0v} - z_0D^{si}_{z_0v} \circ A^{\mathbf{c}}_v)$ the *multiplicity* of the complex focal radius z_0, where $D^{co}_{z_0v}$ (resp. $D^{si}_{z_0v}$) implies the complexification of a map $(g_* \circ \cos(\sqrt{-1}z_0\ \mathrm{ad}(g_*^{-1}v)) \circ g_*^{-1})|_{T_xM}$ (resp. $\left.\left(g_* \circ \frac{\sin(\sqrt{-1}z_0\ \mathrm{ad}(g_*^{-1}v))}{\sqrt{-1}z_0\ \mathrm{ad}(g_*^{-1}v)} \circ g_*^{-1}\right)\right|_{T_xM}$) from T_xM to $T_xN^{\mathbf{c}}$. Also, for a complex focal radius z_0 of M along γ_v, we call z_0v ($\in T_x^{\perp}M^{\mathbf{c}}$) a *complex focal normal vector of* M *at* x. Furthermore, assume that M has globally flat normal bundle. Let $\tilde{v}$ be a parallel unit normal vector field of M. Assume that the number (which may be 0 or ∞) of distinct complex focal radii along $\gamma_{\tilde{v}_x}$ is independent of the choice of $x \in M$. Furthermore assume that this number is not equal to 0. Let $\{r_{i,x} \mid i = 1, 2, \ldots\}$ be the set of all complex focal radii along $\gamma_{\tilde{v}_x}$, where $|r_{i,x}| < |r_{i+1,x}|$ or "$|r_{i,x}| = |r_{i+1,x}|$ & $\mathrm{Re}\, r_{i,x} > \mathrm{Re}\, r_{i+1,x}$" or "$|r_{i,x}| = |r_{i+1,x}|$ & $\mathrm{Re}\, r_{i,x} = \mathrm{Re}\, r_{i+1,x}$ & $\mathrm{Im}\, r_{i,x} - \mathrm{Im}\, r_{i+1,x} > 0$". Let r_i $(i = 1, 2, \ldots)$ be complex valued functions on M defined by assigning $r_{i,x}$ to each $x \in M$. We call these functions r_i $(i = 1, 2, \ldots)$ *complex focal radius functions for* $\tilde{v}$. We call $r_i\tilde{v}$ a *complex focal normal vector field for* $\tilde{v}$. If, for each parallel unit normal vector field $\tilde{v}$ of M, the number of distinct complex focal radii along $\gamma_{\tilde{v}_x}$ is independent of the choice of $x \in M$, each complex focal radius function for $\tilde{v}$ is constant on M and it has constant multiplicity, then we call M a *complex equifocal submanifold*. Let $\phi : H^0([0,1], \mathfrak{g}) \to G$ be the parallel transport map for G (see [K1] about this definition) and $\pi : G \to G/K$ be the natural projection. It is shown that M is complex equifocal if and only if each component of $(\pi \circ \phi)^{-1}(M)$ is complex isoparametric (see [K1] about this definition). In particular, if each component of $(\pi \circ \phi)^{-1}(M)$ is proper complex isoparametric (see [K1] about this definition), then we call M a *proper complex equifocal submanifold*.

Next we recall the notion of an infinite dimensional anti-Kaehlerian isoparametric submanifold. Let M be an anti-Kaehlerian Fredholm submanifold in an infinite dimensional anti-Kaehlerian space V and A be the shape tensor of M. See [K2] about the definitions of an infinite dimensional anti-Kaehlerian space and anti-Kaehlerian Fredholm submanifold. Denote by the same symbol J the complex structures of M and V. Fix a unit normal vector v of M. If there exists $X(\neq 0) \in TM$ with $A_vX = aX + bJX$, then we call the complex number $a + b\sqrt{-1}$ a *J-eigenvalue of A_v* (or a *complex principal curvature of direction v*) and call X a *J-eigenvector for $a + b\sqrt{-1}$*. Also, we call the space of all J-eigenvectors for $a + b\sqrt{-1}$ a *J-eigenspace for $a + b\sqrt{-1}$*. The J-eigenspaces are orthogonal to one another and each J-eigenspace is J-invariant. We call the set of all J-eigenvalues of A_v the *J-spectrum of A_v* and denote it by $\mathrm{Spec}_J\, A_v$. The set $\mathrm{Spec}_J\, A_v \backslash \{0\}$ is described as follows:

$$\mathrm{Spec}_J\, A_v \backslash \{0\} = \{\lambda_i \mid i = 1, 2, \ldots\}$$
$$\left(\begin{array}{c} |\lambda_i| > |\lambda_{i+1}| \text{ or } \text{“}|\lambda_i| = |\lambda_{i+1}| \ \& \ \mathrm{Re}\, \lambda_i > \mathrm{Re}\, \lambda_{i+1}\text{”} \\ \text{or “}|\lambda_i| = |\lambda_{i+1}| \ \& \ \mathrm{Re}\, \lambda_i = \mathrm{Re}\, \lambda_{i+1} \ \& \ \mathrm{Im}\, \lambda_i - \mathrm{Im}\, \lambda_{i+1} > 0\text{”} \end{array}\right).$$

Also, the J-eigenspace for each J-eigenvalue of A_v other than 0 is of finite dimension. We call the J-eigenvalue λ_i the *i-th complex principal curvature of direction v*. Assume that M has globally flat normal bundle. Fix a parallel normal vector field $\tilde{v}$ of M. Assume that the number (which may be ∞) of distinct complex principal curvatures of direction $\tilde{v}_x$ is independent of the choice of $x \in M$. Then we can define functions $\tilde{\lambda}_i$ $(i = 1, 2, \ldots)$ on M by assigning the i-th complex principal curvature of direction $\tilde{v}_x$ to each $x \in M$. We call this function $\tilde{\lambda}_i$ the *i-th complex principal curvature function of direction $\tilde{v}$*. We consider the following condition:

(AKI) For each parallel normal vector field $\tilde{v}$, the number of distinct complex principal curvatures of direction $\tilde{v}_x$ is independent of the choice of $x \in M$, each complex principal curvature function of direction $\tilde{v}$ is constant on M and it has constant multiplicity.

If M satisfies this condition (AKI), then we call M an *anti-Kaehlerian isoparametric submanifold.* Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal system of T_xM. If $\{e_i\}_{i=1}^{\infty} \cup \{Je_i\}_{i=1}^{\infty}$ is an orthonormal base of T_xM, then we call $\{e_i\}_{i=1}^{\infty}$ a *J-orthonormal base*. If there exists a J-orthonormal base consisting of J-eigenvectors of A_v, then A_v is said to *be diagonalized with respect to the J-orthonormal base*. If M is anti-Kaehlerian isoparametric and, for each $v \in T^{\perp}M$, the shape operator A_v is diagonalized with respect to a J-orthonormal base, then we call M a *proper anti-Kaehlerian isoparametric submanifold.* For arbitrary two unit normal vector v_1 and v_2 of a proper anti-Kaehlerian isoparametric submanifold, the shape operators A_{v_1} and A_{v_2} are simultaneously diagonalized with respect to a J-orthonormal base. Note that the notion of a proper anti-Kaehlerian isoparametric submanifold in a finite dimensional anti-Kaehlerian space is defined similarly. Let M be a proper anti-Kaehlerian isoparametric submanifold in an infinite dimensional anti-Kaehlerian space V. Let $\{E_i \mid i \in I\}$ be the family of distributions on M such that, for each $x \in M$, $\{E_i(x) \mid i \in I\}$ is the set of all common J-eigenspaces of A_v's $(v \in T_x^{\perp}M)$. The relation $T_xM = \overline{\bigoplus_{i \in I} E_i}$ holds. Let λ_i $(i \in I)$ be the section of $(T^{\perp}M)^* \otimes \mathbf{C}$ such that $A_v = \mathrm{Re}\,\lambda_i(v)\,\mathrm{id} + \mathrm{Im}\,\lambda_i(v)J$ on $E_i(\pi(v))$ for each $v \in T^{\perp}M$, where π is the bundle projection of $T^{\perp}M$. We call λ_i $(i \in I)$ *complex principal curvatures* of M and call distributions E_i $(i \in I)$ *complex curvature distributions* of M. It is shown that there uniquely exists a normal vector field v_i of M with $\lambda_i(\cdot) = \langle v_i, \cdot \rangle - \sqrt{-1}\langle Jv_i, \cdot \rangle$ (see Lemma 5 of [K2]). We call v_i $(i \in I)$ the *complex curvature normals* of M. Note that v_i is parallel with respect to the normal connection $\nabla^{\perp}$. The focal set of (M, x) $(x \in M)$ coincides with the sum $\bigcup_{i \in I} (\lambda_i)_x^{-1}(1)$ of the complex hyperplanes $(\lambda_i)_x^{-1}(1)$ $(i \in I)$. The Coxeter group generated by te complex reflections of order two with respect to $(\lambda_i)_x^{-1}(1)$ $(\in I)$ is discrete (see Proposition 3.7 of [K4]). We call this group the *complex Coxeter group associated with M at x* and denote it by W_x. This group is independent of the choice of $x \in M$ up to isomorphism. Hence we simply denote it by W.

Let M be a submanifold with globally flat and abelian normal bundle in a symmetric space $N = G/K$ of non-compact type. We assume that G admits the complexification $G^{\mathbf{c}}$. In [K2], we defined the extrinsic complexification $M^{\mathbf{c}}$ of M as an anti-Kaehlerian submanifold in the anti-Kaehlerian symmetric space $G^{\mathbf{c}}/K^{\mathbf{c}}$ associated with G/K. Let $\phi^{\mathbf{c}} : H^0([0,1], \mathfrak{g}^{\mathbf{c}}) \to G^{\mathbf{c}}$ be the parallel transport map for $G^{\mathbf{c}}$ (see [K2] about this definition) and $\pi^{\mathbf{c}} : G^{\mathbf{c}} \to G^{\mathbf{c}}/K^{\mathbf{c}}$ be the natural projection. Let $\tilde{M}^{\mathbf{c}}$ be the complete extension of $(\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})^{-1}(M^{\mathbf{c}})$. It is shown that M is a proper complex equifocal one if and only if each component of $\tilde{M}^{\mathbf{c}}$ is a proper anti-Kaehlerian isoparametric one. Let W be the complex Coxeter group associated with $\tilde{M}^{\mathbf{c}}$. We call *W the complex Coxeter group associated with M*. Here we note that W is determined by only complex focal datas of M without the use of $M^{\mathbf{c}}$.

3. The associated complex Coxeter groups

Let M be a proper anti-Kaehlerian isoparametric submanifold in a finite dimensional anti-Kaehlerian space V. Denote by the same symbol J the complex structures of M and V. Let $\{\lambda_i \mid i \in I\}$ (resp. $\{v_i \mid i \in I\}$) be the set of all complex principal curvatures (resp. the set of all complex curvature normals) of M and E_i $(i \in I)$ be the complex curvature distribution for λ_i. Let T_i^x be the complex reflection of order two with respect to the complex hyperplane $l_i^x := (\lambda_i)_x^{-1}(1)$ of $T_x^\perp M$, which is an affine transformation of $T_x^\perp M$. When T_i^x is regarded as a linear transformation of $T_x^\perp M$, we denote it by R_i^x. Also, when l_i^x is rgarded as a linear subspace of $T_x^\perp M$, we denote it by the same symbol l_i^x. The complex Coxeter group W associated with M is generated by T_i^x's $(i \in I)$. Let W^L be the group generated by R_i^x's $(i \in I)$. According to Proposition 3.7 of [K4], it is shown that W is discrete and hence W^L is finite. Also, we can show the following fact.

Proposition 3.1. *The group W is isomorphic to W^L.*

Proof. Let $L_x^{E_i}$ be the leaf of E_i through $x \in M$, which is a complex sphere in V (see [K2]). Let o_x be the center of the complex sphere L_x^E. Define a diffeomorphism ϕ_i of M by assigning the antipodal point of x in the complex sphere $L_x^{E_i}$ to each $x \in M$. Then we have $\{E_j(x) \mid j \in I\} = \{E_j(\phi_i(x)) \mid j \in I\}$ $(i \in I)$ as a family of linear subspaces of V (see Section 3 of [K4]). Let $(E_j)_{\phi_i(x)} = (E_{\sigma_i(j)})_x$. According to Lemma 3.2 in [K4], we have $(\lambda_j)_{\phi_i(x)} = \dfrac{(\lambda_{\sigma_i(j)})_x}{1 - (\lambda_{\sigma_i(j)})_x(\phi_i(x) - x)}$, that is, $(v_j)_{\phi_i(x)} = \dfrac{(v_{\sigma_i(j)})_x}{1 - (\lambda_{\sigma_i(j)})_x(\phi_i(x) - x)}$. For simplicity, we denote T_i^x (resp. R_i^x) by T_i (resp. R_i). We have $\phi_i(x) = T_i(x)$. Hence we have $(v_j)_{T_i(x)} = \dfrac{(v_{\sigma_i(j)})_x}{1 - (\lambda_{\sigma_i(j)})_x(T_i(x) - x)}$. In more general, we can show

$$(v_j)_{(T_{i_1} \circ \cdots \circ T_{i_r})(x)} = \frac{(v_{(\sigma_{i_1} \circ \cdots \circ \sigma_{i_r})(j)})_x}{1 - (\lambda_{(\sigma_{i_1} \circ \cdots \circ \sigma_{i_r})(j)})_x((T_{i_1} \circ \cdots \circ T_{i_r})(x) - x)} \tag{3.1}$$

$((i_1, \ldots, i_r) \in I^r)$, where r is an arbitrary positive integer. On the other hand, it is clear that

$$(v_j)_{(T_{i_1} \circ \cdots \circ T_{i_r})(x)} = (R_{i_1} \circ \cdots \circ R_{i_r})((v_j)_x). \tag{3.2}$$

Assume that $R_{i_1} \circ \cdots \circ R_{i_r} = \mathrm{id}$. Then, by using (3.2), we can show $(E_j)_{(\phi_{i_1} \circ \cdots \circ \phi_{i_r})(x)} = (E_j)_x$. On the other hand, we have $(E_j)_{(\phi_{i_1} \circ \cdots \circ \phi_{i_r})(x)} = (E_{(\sigma_{i_r} \circ \cdots \circ \sigma_{i_1})(j)})_x$. Hence we have $\sigma_{i_r} \circ \cdots \circ \sigma_{i_1} = \mathrm{id}$, that is, $\sigma_{i_1} \circ \cdots \circ \sigma_{i_r} = \mathrm{id}$. Furthermore, it follows from (3.1) and (3.2) that $\langle (v_j)_x, (T_{i_1} \circ \cdots \circ T_{i_r})(x) - x \rangle = 0$. Since M is contained in the affine subspace $x + T_xM \oplus \mathrm{Span}\{(v_i)_x \mid i \in I\}$ and $(T_{i_1} \circ \cdots \circ T_{i_r})(x) - x \in T_x^\perp M$, it follows from the arbitrariness of j that $(T_{i_1} \circ \cdots \circ T_{i_r})(x) = x$, which implies $T_{i_1} \circ \cdots \circ T_{i_r} = \mathrm{id}$. Conversely, it is clear that $T_{i_1} \circ \cdots \circ T_{i_r} = \mathrm{id}$ implies $R_{i_1} \circ \cdots \circ R_{i_r} = \mathrm{id}$. Since M is of finite dimension, I is finite. Hence W (resp.

W^L) is finitely generated by T_i's (resp. R_i's). These facts imply that W is isomorphic to W^L. q.e.d.

From this proposition, we can show the following fact.

PROPOSITION 3.2. *The set $\bigcap_{i\in I} l_i^x$ is not empty.*

Proof. Since W^L is finite, so is also W by Proposition 3.1. Let $a := \dfrac{1}{|W|}\sum_{g\in W} g(x)$, where $|W|$ is the order of W. Clearly we have $g(a) = a$ for each $g \in W$. Hence we have $a \in \bigcap_{i\in I} l_i^x$. This completes the proof. q.e.d.

4. Basic results

In this section, we prepare basic results to prove Theorem A. We first prepare the following slice theorem for a proper anti-Kaehlerian isoparametric submanifold.

THEOREM 4.1. *Let M be a proper anti-Kaehlerian isoparametric submanifold in an infinite dimensional anti-Kaehlerian space V, λ_i $(i \in I)$ be its complex principal curvatures, v_i $(i \in I)$ be its complex curvature normals and E_i be its complex curvature distributions. Let I' be a subset of I with $\bigcap_{i\in I'} l_i^{x_0} \setminus \bigcup_{i\in I\setminus I'} l_i^{x_0} \neq \emptyset$, where $l_i^{x_0} := (\lambda_i)_{x_0}^{-1}(1)$. Then the following statements* (i)$\sim$(iii) *hold*:

(i) *$\bigoplus_{i\in I'} E_i$ is integrable,*

(ii) *The leaf $L_{x_0}^{I'}$ of $\bigoplus_{i\in I'} E_i$ through x_0 is contained in the complex affine subspace $V_{I'} := x_0 + (\bigoplus_{i\in I'}(E_i)_{x_0}) \oplus \mathrm{Span}_J\{(v_i)_{x_0} \mid i \in I'\}$, where $\mathrm{Span}_J(*)$ implies the complex subspace of V spanned by $(*)$,*

(iii) *$L_{x_0}^{I'}$ is a proper anti-Kaehlerian isoparametric submanifold in $V_{I'}$ and its complex Coxeter group $W_{I'}$ is generated by the complex reflections of order two with respect to the complex hyperplanes $l_j^{x_0} \cap \mathrm{Span}_J\{(v_i)_{x_0} \mid i \in I'\}$ $(j \in I')$ in $\mathrm{Span}_J\{(v_i)_{x_0} \mid i \in I'\}$.*

Proof. Let $N_{I'} := \mathrm{Span}_J\{v_i \mid i \in I'\}$. Take $v_0 \in \bigcap_{i\in I'} l_i^{x_0} \setminus \bigcup_{i\in I\setminus I'} l_i^{x_0}$ and let v be the parallel normal vector field of M with $v_{x_0} = v_0$. Define a submersion $\pi_v : M \to V$ by $\pi_v(x) := x + v_x$ $(x \in M)$. Set $M_v := \pi_v(M)$ and $F_v^{x_0} := \pi_v^{-1}(\pi_v(x_0))$. Easily we can show $\pi_{v*}|_{E_i} = (1 - \lambda_i(v))\,\mathrm{id}_{E_i}$ $(i \in I)$, where id_{E_i} is the identity transformation of E_i. Hence we have $\mathrm{Ker}\,\pi_{v*} = \bigoplus_{i\in I'} E_i$, which implies that $\bigoplus_{i\in I'} E_i$ is integrable and that $L_{x_0}^{I'} = F_v^{x_0}$. Thus the statement (i) is shown. Denote by A (resp. $\nabla^\perp$) the shape tensor (resp. the normal connection) of M and denote by A' (resp. $\nabla^{\perp'}$) the shape tensor (resp. the normal connection) of $L_{x_0}^{I'}$ (in V). For $w \in T^\perp M \ominus N_{I'}$, we have $A_w|_{E_j} = 0$ $(j \in I')$ and hence $A_w|_{TL_{x_0}^{I'}} = 0$. That is, we have $A'_w = 0$. Hence the first normal space of $L_{x_0}^{I'}$ is contained in $N_{I'}$. For any $X \in TL_{x_0}^{I'}$, we have $\tilde\nabla_X v_i \in TL_{x_0}^{I'}$ $(i \in I')$, that is, $\nabla_X^{\perp'} v_i = 0$. This implies that $N_{I'}|_{L_{x_0}^{I'}}$ is parallel with respect to $\nabla^{\perp'}$. Therefore, it follows from the reduction theorem that $L^{I'}_{x_0}$ is contained in the complex affine subspace $x_0 + T_{x_0}L^{I'}_{x_0} \oplus (N_{I'})_{x_0} (= V_{I'})$. Thus the statement (ii) is shown. Since $A_{v_i}(TL^{I'}_{x_0}) = \bigoplus_{j \in I'} A_{v_i}(E_j) \subset TL^{I'}_{x_0}$ $(i \in I')$, we have $A_{v_i}|_{TL^{I'}_{x_0}} = A'_{v_i}$ $(i \in I')$. Hence $A'_{v_i}|_{E_j} = \lambda_j(v_i)\,\mathrm{id}_{E_j}$ $(i, j \in I')$, where we note that A'_{v_i} is regarded as the shape operator of $L^{I'}_{x_0}$ in $V_{I'}$. Thus we see that $L^{I'}_{x_0}$ is a proper anti-Kaehlerian isoparametric submanifold in $V_{I'}$ and that $\{\lambda_i|_{N_{I'}} \mid i \in I'\}$ is the set of all the complex principal curvatures of $L^{I'}_{x_0}$. Since $(\lambda_i|_{N_{I'}})^{-1}(1) = l_i^{x_0} \cap (N_{I'})_{x_0}$ $(i \in I')$, the complex Coxeter group $W_{I'}$ associated with $L^{I'}_{x_0}$ is given as in the statement (iii). q.e.d.

Thus an infinite dimensional proper anti-Kaehlerian isoparametric submanifold is multi-foliated by finite dimensional proper anti-Kaehlerian isoparametric ones. Hence the study of the finite dimensional proper anti-Kaehlerian isoparametric submanifold leads to that of the infinite dimensional proper anti-Kaehlerian isoparametric one. Next, by imitating the proof (see Section 3 of [T1]) of Theorem C of [T1], we shall prove the following Chevalley type restriction theorem for a finite dimensional proper anti-Kaehlerian isoparametric submanifold.

THEOREM 4.2. *Let M be a finite dimensional proper anti-Kaehlerian isoparametric submanifold in a finite dimensional anti-Kaehlerian space V, Σ be a section of M and W be the complex Coxeter group (which acts on Σ) associated with M. Assume that a foliation $\mathfrak{F}$ (which may have singular leaves) consisting of parallel submanifolds of M is defined on the whole of V. Then the restriction map $r : \mathrm{Pol}_{\mathbf{c}}(V)^{\mathfrak{F}} \to \mathrm{Pol}_{\mathbf{c}}(\Sigma)^W$ is an isomorphism, where $\mathrm{Pol}_{\mathbf{c}}(V)^{\mathfrak{F}}$ is the ring of all complex polynomials on V which are constant along leaves of $\mathfrak{F}$ and $\mathrm{Pol}_{\mathbf{c}}(\Sigma)^W$ is that of all W-invariant complex polynomials on Σ.*

Remark 4.1. Let G/K be a symmetric space of non-compact type and H be a symmetric subgroup of G, where G can be assumed to be a connected semisimple Lie group and have its complexification, and K can be assumed to be a maximal compact subgroup of G. Then principal orbits of the $P(G^{\mathbf{c}}, H^{\mathbf{c}} \times K^{\mathbf{c}})$-action on $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ are infinite dimensional proper anti-Kaehlerian isoparametric submanifolds (see Theorems 1, 3 of [K2] and Theorem B of [K3]) and those slices as in Theorem 4.1 are finite dimensional proper anti-Kaehlerian isoparametric ones satisfying the assumptions of Theorem 4.2, where $G^{\mathbf{c}}$, $K^{\mathbf{c}}$ and $H^{\mathbf{c}}$ are the complexifications of G, K and H, respectively, $P(G^{\mathbf{c}}, H^{\mathbf{c}} \times K^{\mathbf{c}}) := \{g \in H^1([0,1], G^{\mathbf{c}}) \mid (g(0), g(1)) \in H^{\mathbf{c}} \times K^{\mathbf{c}}\}$ and $\mathfrak{g}^{\mathbf{c}}$ is the Lie algebra of $G^{\mathbf{c}}$.

The proof of Theorem C of [T1] is written too smartly. Since we need to prove Theorem 4.2 carefully, we shall give the proof in detail comparatively by dividing into some lemmas. Let M, V, W, $\mathfrak{F}$ and Σ be as in the statement of Theorem 4.2. Let $n := \dim_{\mathbf{c}} M$ and $m := \dim_{\mathbf{c}} V$. Fix $x_0 \in M \cap \Sigma$. We identify Σ with $T^{\perp}_{x_0}M$ through $\exp^{\perp}$. Let D be a contractible open neighborhood of x_0

in $T^{\perp}_{x_0}M$ contained in a fundamental domain of the complex Coxeter group W containing x_0 and B be a tubular neighborhood of the 0-section of the normal bundle $T^{\perp}M$ obtained by parallel translating D with respect to the normal connection. First we prepare the following lemma.

LEMMA 4.3. *Let u be a W-invariant holomorphic function over Σ. Then there uniquely exists a holomorphic function f over* $\exp^{\perp}(B)$ *such that $f = u$ on* $\exp^{\perp}(B) \cap \Sigma$ *and that f is constant along each leaf of $\mathfrak{F}|_B$.*

Proof. Define a function f over $\exp^{\perp}(B)$ by $f(\exp^{\perp}(v)) := u(\tilde{v}_{x_0})$ $(v \in B)$, where $\tilde{v}$ is the parallel normal vector field of M with $\tilde{v}_x = v$ (x: the base point of v). It is clear that f satisfies two conditions in the statement. Since $\exp^{\perp}|_B$ is an embedding, $\mathfrak{F}|_{\exp^{\perp}(B)}$ posseses no singular leaf. From this fact, the holomorphicity of f follows. The uniqueness is trivial. q.e.d.

Take a J-orthonormal base $\xi_1, J\xi_1, \ldots, \xi_{m-n}, J\xi_{m-n}$ of $T^{\perp}_{x_0}M$ with $\langle \xi_\alpha, \xi_\alpha \rangle = 1$ $(\alpha = 1, \ldots, m-n)$ and $a \in \bigcap_{i\in I} l_i$, where $\{l_i\}_{i\in I}$ is the family of all focal complex hyperplanes of (M, x_0). Let $a = \sum_{\alpha=1}^{m-n}(a_\alpha\xi_\alpha + b_\alpha J\xi_\alpha)$. Identify Σ $(= T^{\perp}_{x_0}M)$ with $\mathbf{C}^{m-n}$ under the correspondence $\sum_{\alpha=1}^{m-n}(x_\alpha\xi_\alpha + y_\alpha J\xi_\alpha) \leftrightarrow ((x_1 - a_1) + \sqrt{-1}(y_1 - b_1), \ldots, (x_{m-n} - a_{m-n}) + \sqrt{-1}(y_{m-n} - b_{m-n}))$. In similar to the statement (ii) of Lemma 3.3 of [T1], we have the following fact.

LEMMA 4.4. *Let $u : \Sigma \to \mathbf{C}$ be a W-invariant homogeneous* (*complex*) *polynomial of degree k and $f : \exp^{\perp}(B) \to \mathbf{C}$ be the extension of $u|_{\exp^{\perp}(B)\cap\Sigma}$ as in Lemma* 4.3. *Then $\langle \mathrm{grad}\, f, \mathrm{grad}\, f \rangle^{\mathbf{c}}$ is an extension* (*as in Lemma* 4.3) *of the restriction of a W-invariant homogeneous polyomial of degree* $2(k-1)$ *over Σ to* $\exp^{\perp}(B) \cap \Sigma$.

Proof. Let $\bar{\xi}_\alpha$ $(\alpha = 1, \ldots, m-n)$ be a parallel unit normal vector field on M with $(\bar{\xi}_\alpha)_{x_0} = \xi_\alpha$. Define a unit vector field $\tilde{\bar{\xi}}_\alpha$ $(\alpha = 1, \ldots, m-n)$ on $\exp^{\perp}(B)$ by $(\tilde{\bar{\xi}}_\alpha)_{\exp^{\perp}(v)} = (\bar{\xi}_\alpha)_{\pi(v)}$ $(v \in B)$, where π is the bundle projection of $T^{\perp}M$. Easily we can show $df_{\exp^{\perp}(v)}((\tilde{\bar{\xi}}_\alpha)_{\exp^{\perp}(v)}) = du_{\tilde{v}_{x_0}}((\tilde{\xi}_\alpha)_{\tilde{v}_{x_0}})$ $(\alpha = 1, \ldots, m-n)$, where $v \in B$ and $\tilde{v}$ is as in the proof of Lemma 4.3. From this relation and the fact that f is constant along each leaf of $\mathfrak{F}|_{\exp^{\perp}(B)}$, we have $\langle (\mathrm{grad}\, f)_{\exp^{\perp}(v)}, (\mathrm{grad}\, f)_{\exp^{\perp}(v)} \rangle^{\mathbf{c}} = \langle (\mathrm{grad}\, u)_{\tilde{v}_{x_0}}, (\mathrm{grad}\, u)_{\tilde{v}_{x_0}} \rangle^{\mathbf{c}}$. Thus $\langle \mathrm{grad}\, f, \mathrm{grad}\, f \rangle^{\mathbf{c}}$ is an extension (as in Lemma 4.3) of the restriction of a W-invariant homogeneous polynomial $\langle \mathrm{grad}\, u, \mathrm{grad}\, u \rangle^{\mathbf{c}}$ of degree $2(k-1)$ over Σ to $\exp^{\perp}(B) \cap \Sigma$. q.e.d.

Also, we prepare the following lemma.

LEMMA 4.5. *Let u and f be as in Lemma* 4.3. *Then we have*

$$(\triangle f)(\exp^{\perp}(v)) = (\triangle u)(\tilde{v}_{x_0}) + \sum_{i=1}^{g} \frac{m_i \langle (\mathrm{grad}\, u)_{\tilde{v}_{x_0}}, (v_i)_{x_0} \rangle^{\mathbf{c}}}{1 - \lambda_i(\tilde{v}_{x_0})}$$

($v \in B$), *where* $\triangle f$ *is defined by* $\triangle f := -\sum_{B=1}^{2m} \varepsilon_B(\tilde{\nabla}_{e_B}(\tilde{\nabla}_{e_B} f) - \tilde{\nabla}_{\tilde{\nabla}_{e_B} e_B} f)$ ($\tilde{\nabla}$: *the connection of* V, $\{e_1, \dots, e_{2m}\}$: *an orthonormal base of* V, $\varepsilon_B := \langle e_B, e_B \rangle$), $\triangle u$ *is also defined similarly*, $\tilde{v}$ *is as in the proof of Lemma* 4.3, $\{\lambda_i \mid i = 1, \dots, g\}$ *is the set of all complex principal curvatures of* M, $m_i := \frac{1}{2} \dim E_i$ (E_i: *the complex curvature distribution for* λ_i) *and* v_i *is the complex curvature normal for* λ_i.

Proof. Take an arbitrary $x_1 \in M$. Let U be a contractible open set of M containing x_0 and x_1, $\{e_1^i, Je_1^i, \dots, e_{m_i}^i, Je_{m_i}^i\}$ be a J-orthonormal frame field of $E_i|_U$ ($i = 1, \dots, g$). Take $v \in B_x (:= B \cap T_x^{\perp} M)$ ($x \in U$). Let $c : (-\varepsilon, \varepsilon) \to U$ be a curve with $\dot{c}(0) = (e_j^i)_x$ and $\tilde{v}$ be a parallel normal vector field along c with $\tilde{v}(0) = v$. Define a vector field $\tilde{e}_j^i$ on $\exp^{\perp}(B|_U)$ by $(\tilde{e}_j^i)_{\exp^{\perp}(v)} := \exp_*^{\perp}(\dot{\tilde{v}}(0))$ ($v \in B|_U$). It is clear that $\bigcup_{i=1}^{g} \{\tilde{e}_1^i, J\tilde{e}_1^i, \dots, \tilde{e}_{m_i}^i, J\tilde{e}_{m_i}^i\}$ is a tangent frame field of leaves of $\mathfrak{F}|_{\exp^{\perp}(B|_U)}$. Also, let $\tilde{\xi}_\alpha$ ($\alpha = n+1, \dots, m$) be unit vector fields on $\exp^{\perp}(B)$ as in the proof of Lemma 4.4, which are normal to leaves of $\mathfrak{F}|_{\exp^{\perp}(B)}$. Let $v \in B_x$ ($x \in U$). Let γ_v be the normal geodesic of M with $\dot{\gamma}_v(0) = v$. Since $\tilde{e}_j^i|_{\gamma_v}$ is the Jacobi field along γ_v satisfying $\tilde{\nabla}_v \tilde{e}_j^i = -A_v(e_j^i)_x = -\lambda_i(v)(e_j^i)_x$, we have $(\tilde{e}_j^i)_{\gamma_v(s)} = (1 - s\lambda_i(v))(e_j^i)_x$ and hence

$$(4.1) \qquad (\triangle f)(\exp^{\perp}(v)) = (\triangle u)(\tilde{v}_{x_0}) + \sum_{i=1}^{g} \sum_{j=1}^{m_i} \frac{2}{(1 - \lambda_i(v))^2} (\tilde{\nabla}_{(\tilde{\nabla}_{\tilde{e}_j^i} \tilde{e}_j^i)_{\perp}} f)_{\exp^{\perp}(v)},$$

where $(\cdot)_{\perp}$ is the normal component (to leaves of $\mathfrak{F}|_{\exp^{\perp}(B|_U)}$) of $(\cdot)$. On the other hand, we have $((\tilde{\nabla}_{\tilde{e}_j^i} \tilde{e}_j^i)_{\perp})_{\exp^{\perp}(v)} = \sum_{\alpha=1}^{m-n} \lambda_i((\bar{\xi}_\alpha)_x)(1 - \lambda_i(\tilde{v}_{x_0}))(\tilde{\xi}_\alpha)_{\exp^{\perp}(v)}$, where we use $(\tilde{\nabla}_{\tilde{e}_j^i} \tilde{\xi}_\alpha)_{\exp^{\perp}(v)} = (\tilde{\nabla}_{e_j^i} \bar{\xi}_\alpha)_x$, $\lambda_i(v) = \lambda_i(\tilde{v}_{x_0})$ and $\tilde{\nabla} J = 0$. Substituting this relation into (4.1), we have $(\triangle f)(\exp^{\perp}(v)) = (\triangle u)(\tilde{v}_{x_0}) + \sum_{i=1}^{g} \frac{m_i}{1 - \lambda_i(\tilde{v}_{x_0})} \langle (\operatorname{grad} u)_{\tilde{v}_{x_0}}, (v_i)_{x_0} \rangle^{\mathbf{c}}$, where $\tilde{v}_i$ is the vector field on $\exp^{\perp}(B|_U)$ arising from $v_i|_U$. q.e.d.

By identifying $T_{x_0}^{\perp} M$ with Σ through $\exp^{\perp}$, we regard $\lambda_i|_{T_{x_0}^{\perp} M}$ as a $\mathbf{C}$-linear function over Σ.

LEMMA 4.6. *The function* $\dfrac{1}{1 - \lambda_i|_{T_{x_0}^{\perp} M}} \langle \operatorname{grad} u, (v_i)_{x_0} \rangle^{\mathbf{c}}$ *over* Σ *is a homogeneous polynomial of degree* $(k-2)$ *as the function over* $\mathbf{C}^{m-n}$ *identified with* $\Sigma = T_{x_0}^{\perp} M$ *as stated in this section.*

Proof. Assume that $v \in (\lambda_i|_{T_{x_0}^{\perp} M})^{-1}(1)$. Since the complex reflection of order two with respect to $(\lambda_i|_{T_{x_0}^{\perp} M})^{-1}(1)$ is an element of the complex Coxeter group W of M and u is W-invariant, we have $\langle (\operatorname{grad} u)_v, (v_i)_{x_0} \rangle^{\mathbf{c}} = 0$. This implies that $\langle \operatorname{grad} u, (v_i)_{x_0} \rangle^{\mathbf{c}}$ is divisible by $1 - \lambda_i|_{T_{x_0}^{\perp} M}$. Hence the statement follows. q.e.d.

Furthermore, we can show the following fact.

Lemma 4.7. *The function* $\sum_{i=1}^{g} \dfrac{m_i}{1-\lambda_i|_{T^{\perp}_{x_0}M}} \langle \operatorname{grad} u, (v_i)_{x_0} \rangle^{\mathbf{c}}$ *over* Σ *is* W-*invariant*.

Proof. For simplicity, let $\hat{u} := \sum_{i=1}^{g} \dfrac{m_i \langle \operatorname{grad} u, (v_i)_{x_0} \rangle^{\mathbf{c}}}{1-\lambda_i|_{T^{\perp}_{x_0}M}}$ and denote $T_j^{x_0}$ (resp. $R_j^{x_0}$) by T_j (resp. R_j), where $T_j^{x_0}$ and $R_j^{x_0}$ are as in Section 3. Take $v \in \Sigma (= T^{\perp}_{x_0}M)$. Since u is W-invariant, we have

$$\hat{u}(T_j(v)) = \sum_{i=1}^{g} \frac{m_i \langle (\operatorname{grad} u)_v, R_j((v_i)_{x_0}) \rangle^{\mathbf{c}}}{1-\lambda_i(T_j(v))}. \tag{4.2}$$

Take $a \in \bigcap_{k=1}^{g} l_k^{x_0}$, where we note that $\bigcap_{k=1}^{g} l_k^{x_0} \neq \emptyset$ by Proposition 3.2. The complex reflection T_j is expressed as $T_j = P_a \circ R_j \circ P_{-a}$, where $P_{\pm a}$ is defined by $P_{\pm a}(v) = v \pm a$ $(v \in \Sigma = T^{\perp}_{x_0}M)$. Let ϕ_j and σ_j $(j = 1, \dots, g)$ be as in Section 3. Since $R_j((v_i)_{x_0}) = (v_i)_{\phi_j(x_0)}$ and $(\lambda_i)_{\phi_j(x_0)} = \dfrac{(\lambda_{\sigma_j(i)})_{x_0}}{1-(\lambda_{\sigma_j(i)})_{x_0}(\phi_j(x_0)-x_0)}$, we have

$$\lambda_i(T_j(v)) = \frac{(\lambda_{\sigma_j(i)})_{x_0}(v) - 1}{1-(\lambda_{\sigma_j(i)})_{x_0}(\phi_j(x_0)-x_0)} + 1. \tag{4.3}$$

On the other hand, we have $\langle (\operatorname{grad} u)_v, R_j((v_i)_{x_0}) \rangle^{\mathbf{c}} = \dfrac{1}{2} \dfrac{(\lambda_{\sigma_j(i)})_{x_0}((\operatorname{grad} u)_v)}{1-(\lambda_{\sigma_j(i)})_{x_0}(\phi_j(x_0)-x_0)}$. Substituting this relation and (4.3) into (4.2), we have $\hat{u}(T_j(v)) = \sum_{i=1}^{g} \cdot \dfrac{m_i (\lambda_i)_{x_0}((\operatorname{grad} u)_v)}{2(1-(\lambda_i)_{x_0}(v))} = \hat{u}(v)$, where we note $m_{\sigma_j(i)} = m_i$. Thus $\hat{u}$ is W-invariant.

q.e.d.

From Lemmas 4.5, 4.6 and 4.7, we have the following fact.

Lemma 4.8. *Let* u *and* f *be as in Lemma* 4.3. *Then* $\triangle f$ *is an extension* (*as in Lemma* 4.3) *of the restriction of a* W-*invariant homogeneous polynomial of degree* $(k-2)$ *over* Σ *to* $\exp^{\perp}(B) \cap \Sigma$.

Proof. According to Lemma 4.5, $\triangle f$ is an extension of $(\triangle u + \hat{u})|_{\exp^{\perp}(B)\cap\Sigma}$, where $\hat{u}$ is as in the proof of Lemma 4.7. According to Lemmas 4.6 and 4.7, $\hat{u}$ is a W-invariant homogeneous polynomial of degree $(k-2)$. Also, it is clear that so is also $\triangle u$. Hence the statement follows. q.e.d.

By using Lemmas 4.4 and 4.8, we shall prove Theorem 4.2.

Proof of Theorem 4.2. It is clear that the restriction map $r : \mathrm{Pol}_{\mathbf{c}}(V)^{\mathfrak{F}} \to \mathrm{Pol}_{\mathbf{c}}(\Sigma)^{W}$ is an injective homomorphism. By the induction for the degree, we shall show that r is surjective. It is clear that all elements of degree zero of $\mathrm{Pol}_{\mathbf{c}}(\Sigma)^{W}$ belong to the image of r. Assume that all elements of degree at most

$(k-1)$ of $\mathrm{Pol}_{\mathbf{c}}(\Sigma)^W$ belong to the image of r. Take an element u of degree k of $\mathrm{Pol}_{\mathbf{c}}(\Sigma)^W$. From Lemmas 4.4 and 4.8, we can show that u belongs to the image of r by imitating the proof of Theorem C of [T1]. Therefore, by the induction, we see that r is surjective. q.e.d.

Let V be a finite dimensional anti-Kaehlerian space and $f = (f_1, \dots, f_k) : V \to \mathbf{C}^k$ be a polynomial map. In this paper, we call f a *proper anti-Kaehlerian isoparametric polynomial map* if it satisfies the following conditions:

(i) f has a regular point,

(ii) $\langle \operatorname{grad} f_i, \operatorname{grad} f_j \rangle^{\mathbf{c}}$ $(1 \le i \le k,\ 1 \le j \le k)$ and $\triangle f_i$ $(1 \le i \le k)$ are functions of $f_1, \dots, f_k$,

(iii) $[\operatorname{grad} f_i, \operatorname{grad} f_j]^{\mathbf{c}}$ $(1 \le i \le k,\ 1 \le j \le k)$ are linear combinations of $\operatorname{grad} f_1, \dots, \operatorname{grad} f_k$ having functions of $f_1, \dots, f_k$ as the coefficients,

where $\operatorname{grad} f_i$ is the complex vector field defined by $\operatorname{grad} f_i := \operatorname{grad}(f_i)_{\mathbf{R}} + \sqrt{-1} \cdot \operatorname{grad}(f_i)_I$ ($(f_i)_{\mathbf{R}}$ (resp. $(f_i)_I$): the real (resp. imaginary) part of f_i), $\langle\ ,\ \rangle^{\mathbf{c}}$ is the complexification of the non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$ of V and $[\ ,\ ]^{\mathbf{c}}$ is that of the bracket product $[\ ,\ ]$. By using Theorem 4.2, we can prove the following fact for existence of a proper anti-Kaehlerian isoparametric polynomial map having a given proper anti-Kaehlerian isoparametric submanifold as a regular level set.

THEOREM 4.9. *Let M, V, Σ and W be as in Theorem* 4.2. *Then there exists a proper anti-Kaehlerian isoparametric polynomial map $f = (f_1, \dots, f_k) : V \to \mathbf{C}^k$ satisfying the following conditions*:

(i) *f has M as a regular level set*,

(ii) *the focal set of M coincides with the set of all critical points of f*,

(iii) $f(V) = f(\Sigma)$,

(iv) *for $v \in \Sigma$, v is a regular point of f if and only if v is a W-regular point.*

Proof. Let W be the complex Coxeter group associated with M. According to the theorem of Chevalley [Ch] for the ring of all polynomials on $\mathbf{K}^m$ ($\mathbf{K}$: a field of characteristic zero) which are invariant with respect to a finite reflection group, we can show that the ring of all W-invariant (complex) polynomials on $\Sigma(=\mathbf{C}^{m-n})$ is generated by $(m-n)$ pieces of generators $u_1, \dots, u_{m-n}$. Let $f_1, \dots, f_{m-n}$ be their extended polynomials on V, whose existences are assured by Theorem 4.2. Let $f := (f_1, \dots, f_{m-n})(: V \to \mathbf{C}^{m-n})$. According to Lemma 4.4, the polynomial $\langle \operatorname{grad} f_i, \operatorname{grad} f_j \rangle^{\mathbf{c}}$ is the extension of a W-invariant polynomial u on Σ. Let $u = \sum_{k_1, \dots, k_{m-n}} c_{k_1 \cdots k_{m-n}} u_1^{k_1} \cdots u_{m-n}^{k_{m-n}}$. Then it is clear that $\langle \operatorname{grad} f_i, \operatorname{grad} f_j \rangle^{\mathbf{c}} = \sum_{k_1, \dots, k_{m-n}} c_{k_1 \cdots k_{m-n}} f_1^{k_1} \cdots f_{m-n}^{k_{m-n}}$. Thus $\langle \operatorname{grad} f_i, \operatorname{grad} f_j \rangle^{\mathbf{c}}$ is a function of $f_1, \dots, f_{m-n}$. Similarly, by using Lemma 4.8, we can show that $\triangle f_i$ is a function of $f_1, \dots, f_{m-n}$. Also, since f_i's are polynomials, it is shown that $[\operatorname{grad} f_i, \operatorname{grad} f_j]^{\mathbf{c}}$'s are linear combinations of $\operatorname{grad} f_1, \dots, \operatorname{grad} f_{m-n}$ having functions of $f_1, \dots, f_{m-n}$ as the coefficients. Thus f is a proper anti-Kaehlerian isoparametric polyomial map. Furthermore, it is easy to show that f satisfies the conditions (i)$\sim$(v). This completes the proof. q.e.d.

5. Proofs of Theorems A and B

In this section, we prove Theorems A and B. First we prove Theorem A. For its purpose, we prepare the following lemma.

LEMMA 5.1. *Let G be a compact subgroup of the affine transformation group $(O(2n)\cap GL(n,\mathbf{C}))\ltimes\mathbf{C}^n$ of $\mathbf{C}^n(=\mathbf{R}^{2n})$.*

(i) *The ring $\mathrm{Pol}_{\mathbf{c}}(\mathbf{C}^n)^G$ of all G-invariant complex polynomials over $\mathbf{C}^n$ is finitely generated.*

(ii) *Any G-invariant complex-valued C^∞-function f over $\mathbf{C}^n$ is expressed as $f=\phi\circ(u_1,\dots,u_k)$ in terms of some complex-valued C^∞-function ϕ on $\mathbf{C}^k$, where $u_1,\dots,u_k$ are generators of $\mathrm{Pol}_{\mathbf{c}}(\mathbf{C}^n)^G$.*

Proof. The statement (i) is trivial. Let u_i^R (resp. u_i^I) be the real (resp. imaginary) part of u_i $(i=1,\dots,k)$. It is clear that $u_1^R,u_1^I,\dots,u_k^R,u_k^I$ are generators of the ring $\mathrm{Pol}_{\mathbf{R}}(\mathbf{R}^{2n})^G$ of G-invariant real-valued polynomials over $\mathbf{R}^{2n}(=\mathbf{C}^n)$. Let f^R (resp. f^I) be the real (resp. imaginary) part of f. Since f^R and f^I are G-invariant, it follows from the Schwarz's theorem [Sc] that they are expressed as $f^R=\phi_1\circ(u_1^R,u_1^I,\dots,u_k^R,u_k^I)$ and $f^I=\phi_2\circ(u_1^R,u_1^I,\dots,u_k^R,u_k^I)$ in terms of real-valued C^∞-functions ϕ_1 and ϕ_2 over $\mathbf{R}^{2k}$, respectively. Under the identification of $\mathbf{R}^{2k}$ and $\mathbf{C}^k$, we regard $\phi_1+\sqrt{-1}\phi_2$ as the complex-valued C^∞-function over $\mathbf{C}^k$. Then we have $f=(\phi_1+\sqrt{-1}\phi_2)\circ(u_1,\dots,u_k)$. q.e.d.

By imitating the proof of Theorem 10.1 of [T2], we prove Theorem A, where we use Theorems 4.1, 4.2 and the above lemma.

Proof of Theorem A. Let $x_0\in M\cap\Sigma$. Then we have $\Sigma=\exp^\perp(T^\perp_{x_0}M)$, where $\exp^\perp$ is the normal exponential map of M and $T^\perp_{x_0}M$ is the normal space of M at x_0. We identify Σ with $T^\perp_{x_0}M$. It is clear that the restriction map r in the statement is injective. We shall show that r is surjective. Take $f\in C^\infty(\Sigma,\mathbf{C})^W$. Since f is W-invariant, it is uniquely extended to the function on V which is constant along each leaf of $\mathfrak{F}$. Denote by $\tilde f$ its extension. We have only to show that $\tilde f$ is of class C^∞. First we shall show that $\tilde f$ is of class C^∞ at a non-focal point p_1 of M. Let $\triangle$ be a fundamental domain of W containing x_0 and $\triangle^0$ be its interior, where we note that the choice of $\triangle$ is not unique. Define a map Φ of $M\times\triangle^0$ into V by $\Phi(x,v):=x+\tilde v_x$ $((x,v)\in M\times\triangle^0)$, where $\tilde v$ is the global parallel normal vector field of M with $\tilde v_{x_0}=v$. It is easy to show that $\Phi(M\times\triangle^0)$ is open and dense in V and that Φ is a C^∞-diffeomorphism of $M\times\triangle^0$ onto $\Phi(M\times\triangle^0)$. We may assume that $p_0\in\Phi(M\times\triangle^0)$ by retaking $\triangle$ if necessary. Since $(\tilde f\circ\Phi)(x,v)=f(v)$ $((x,v)\in M\times\triangle^0)$, $\tilde f\circ\Phi$ is a C^∞-function over $M\times\triangle^0$. Hence $\tilde f$ is of class C^∞ over $\Phi(M\times\triangle^0)$. In particular, $\tilde f$ is of class C^∞ at p_0. Next we shall show that $\tilde f$ is of class C^∞ at a focal point p_1 of M. Let $p_1=x_1+v$ $(x_1\in M, v\in T^\perp_{x_1}M)$ and

$\tilde{v}$ be the global parallel normal vector field of M with $\tilde{v}_{x_1} = v$. The focal map $\pi_{\tilde{v}} : M \to V$ is defined by $\pi_{\tilde{v}}(x) := x + \tilde{v}_x$ $(x \in M)$ and the focal submanifold $M_{\tilde{v}}$ is defined by $M_{\tilde{v}} := \pi_{\tilde{v}}(M)$. Let σ be a section of $\pi_{\tilde{v}}$ through x_1 over a neighborhood U of p_1 in $M_{\tilde{v}}$. Let $\mathfrak{O}$ be the complex Banach space of all compact operators of V which are commutative with the complex structure J of V and $O_J(V)$ be the Banach group of all linear isometries of V which are commutative with J. The action of $O_J(V)$ on $\mathfrak{O}$ is defined by $B \cdot C := B \circ C \circ B^{-1}$ $(B \in O_J(V), C \in \mathfrak{O})$. For each $x \in M$, we define $\tilde{A}_{\tilde{v}_x} \in \mathfrak{O}$ by $\tilde{A}_{\tilde{v}_x}|_{T_x M} = A_{\tilde{v}_x}$ and $\tilde{A}_{\tilde{v}_x}|_{T^\perp_x M} = 0$, where A is the shape tensor of M. Since M is proper anti-Kaehlerian isoparametric, for any $x \in M$, $\tilde{A}_{\tilde{v}_x}$ belongs to the $O_J(V)$-orbit through $\tilde{A}_v$. The orbit map $\psi : O_J(V) \to O_J(V) \cdot \tilde{A}_v$ is defined by $\psi(B) = B \circ \tilde{A}_v \circ B^{-1}$ $(B \in O_J(V))$. Take a C^∞-section γ of ψ through the identity transformation id_V of V over a neighborhood of $\tilde{A}_v$ in $O_J(V) \cdot \tilde{A}_v$ containing $\{\tilde{A}_{\tilde{v}_{\sigma(x)}} \,|\, x \in U\}$, where we shrink U if necessary. Let $\{\lambda_i \,|\, i \in I\}$ be the set of all complex principal curvatures of M and E_i $(i \in I)$ be the complex curvature distribution for λ_i. Set $I_v := \{i \in I \,|\, v \in (\lambda_i)_{x_1}^{-1}(1)\}$, which is finite. Define a map $\Psi : U \times T^\perp_{x_1} M \times (\bigoplus_{i \in I_v} (E_i)_{x_1}) \to T^\perp M_{\tilde{v}}|_U$ by $\Psi(p, u, w) := \tilde{u}_{\sigma(p)} + \gamma(\tilde{A}_{\tilde{v}_{\sigma(p)}})(w)$. The map Ψ is a bundle isomorphism. Take a tubular neighborhood T of the 0-section of $T^\perp M_{\tilde{v}}|_U$ such that the restriction $\exp^\perp|_T$ of the normal exponential map of $M_{\tilde{v}}$ to T is a diffeomorphism. Since $(\tilde{f} \circ \exp^\perp \circ \Psi)(p, u, w) = (\tilde{f} \circ \exp^\perp \circ \Psi)(p_1, u, w)$ $((p, u, w) \in \Psi^{-1}(T))$, we have only to show that $\tilde{f}$ is of class C^∞ over $\exp^\perp(T \cap T^\perp_{p_1} M_{\tilde{v}})$ in order to show that it is class C^∞ over $\exp^\perp(T)$ (hence at p_1). We shall show that $\tilde{f}$ is of class C^∞ over $V' := \exp^\perp(T^\perp_{p_1} M_{\tilde{v}})$. Since I_v is finite, we have $\dim V' < \infty$. According to Theorem 4.1, $\pi_{\tilde{v}}^{-1}(p_1)$ is a proper anti-Kaehlerian isoparametric submanifold in V'. Let $\mathfrak{F}'$ be a foliation (which have singular leaves) consisting of parallel submanifolds and focal submanifolds of $\pi_{\tilde{v}}^{-1}(p_1)$. Since the foliation $\mathfrak{F}$ in the statement is defined on the whole of V by the assumption and leaves of $\mathfrak{F}'$ are the intersections of those of $\mathfrak{F}$ with V', $\mathfrak{F}'$ is defined on the whole of V'. Let $\Sigma' := \exp^\perp(T^\perp_{x_1} M)$, which is a section of $\pi_{\tilde{v}}^{-1}(p_1)$. Let $k := \operatorname{codim} M$, that is, $\dim \Sigma' = k$. Let W' be the (finite) complex Coxeter group associated with the proper anti-Kaehlerian isoparametric submanifold $\pi_{\tilde{v}}^{-1}(p_1)$. The ring $\mathrm{Pol}_{\mathbf{c}}(\Sigma')^{W'}$ of all W'-invariant complex polynomials over Σ' have k pieces of generators $u_1, \ldots, u_k$. Since $\tilde{f}$ is constant along leaves of $\mathfrak{F}'$, $\tilde{f}|_{\Sigma'}$ is invarinat with respect to W'. According to Lemma 5.1, $\tilde{f}|_{\Sigma'}$ is expressed as $\tilde{f}|_{\Sigma'} = \phi \circ (u_1, \ldots, u_k)$ in terms of some complex-valued C^∞-function ϕ over $\mathbf{C}^k$. According to Theorem 4.2, each u_i $(i = 1, \ldots, k)$ is extended to the complex valued polynomial over V' which is constant along leaves of $\mathfrak{F}'$. Denote by $\tilde{u}_i$ this extension. Since both $\tilde{f}|_{V'}$ and $\phi \circ (\tilde{u}_1, \ldots, \tilde{u}_k)$ are constant along leaves of $\mathfrak{F}'$ and those restrictions to Σ' coincide with each other, we have $\tilde{f}|_{V'} = \phi \circ (\tilde{u}_1, \ldots, \tilde{u}_k)$. Thus $\tilde{f}|_{V'}$ is of class C^∞. Hence so is $\tilde{f}|_{\exp^\perp(T)}$. In particular, $\tilde{f}$ is of class C^∞ at p_1. Therefore, $\tilde{f}$ is of class C^∞ over V. This completes the proof. q.e.d.

Now we prepare the following lemma to prove Theorem B.

LEMMA 5.2. *Let M, W and W_Σ be as in the statement of Theorem* B. *Regard $\Sigma^{\mathbf{c}}$ as the section of $\tilde{M}^{\mathbf{c}}$ through a fixed point $u_0 \in \tilde{M}^{\mathbf{c}}$. Let $\{l_i \mid i \in I\}$ be the set of all focal complex hyperplanes (in $\Sigma^{\mathbf{c}} = T^\perp_{u_0}\tilde{M}^{\mathbf{c}}$) of $(\tilde{M}^{\mathbf{c}}, u_0)$ and R_i $(i \in I)$ be the complex reflection of order two with respect to l_i, where we note that $W = \langle\{R_i \mid i \in I\}\rangle$. Then the following statements* (i), (ii) *and* (iii) *hold*:

(i) $W_\Sigma = \langle\{R_i|_\Sigma \mid i \in I \text{ s.t. } \operatorname{codim}(l_i \cap \Sigma) = 1\}\rangle$ (*this group is a Coxeter group*), *where* $\operatorname{codim}(l_i \cap \Sigma)$ *is the codimension of* $l_i \cap \Sigma$ *in* Σ,

(ii) *each W_Σ-orbit is contained in the intersection of one leaf of $\mathfrak{F}$ with Σ, where $\Sigma(= \exp^\perp(T^\perp_{x_0}M))$ $(x_0 \in M \cap \Sigma)$ is identified with $T^\perp_{x_0}M$*,

(iii) *the image of the restriction map $r_\Sigma : C^\infty(\Sigma^{\mathbf{c}})^W \to C^\infty(\Sigma)$ to Σ coincides with $C^\infty(\Sigma)^{W_\Sigma}$*.

Proof. Easily we have $l_i \cap \Sigma = \emptyset$ or $\operatorname{codim}(l_i \cap \Sigma) \in \{1, 2\}$. By the elementary geometric methods, we can show

$$(5.1)\qquad \begin{cases} R_i(\Sigma) \cap \Sigma = \emptyset & (\text{when } l_i \cap \Sigma = \emptyset) \\ R_i(\Sigma) = \Sigma & (\text{when } \operatorname{codim}(l_i \cap \Sigma) = 1) \\ R_i(\Sigma) \cap \Sigma = l_i \cap \Sigma & (\text{when } \operatorname{codim}(l_i \cap \Sigma) = 2). \end{cases}$$

Hence the statement (i) is shown. In case of $\operatorname{codim}(l_i \cap \Sigma) = 1$, it follows from $R_i(\Sigma) = \Sigma$ that $l_i \cap \Sigma$ is contained in the focal set of (M, x_0). Furthermore we see that the sum of $l_i \cap \Sigma$'s ($i \in I$ s.t. $\operatorname{codim}(l_i \cap \Sigma) = 1$) is contained in the focal set of (M, x_0). From this fact, the statement (ii) follows. Next we show the statement (iii). Clearly we have $r_\Sigma(C^\infty(\Sigma^{\mathbf{c}})^W) \subset C^\infty(\Sigma)^{W_\Sigma}$. Take an arbitrary $f \in C^\infty(\Sigma)^{W_\Sigma}$. From (5.1), we see that there exists a fundamental domain $\triangle$ of W such that $\triangle \cap \Sigma$ is contained in that of W_Σ. Clearly there exists $\hat{f} \in C^\infty(\triangle)$ such that $\hat{f}|_{\triangle \cap R(\Sigma)} = (f \circ R)|_{\triangle \cap R(\Sigma)}$ for all $R \in W$. Furthermore we can take $\hat{f}$ as one which can be extended to an element of $C^\infty(\Sigma^{\mathbf{c}})^W$. Denote by $\tilde{f}$ the extension. From $\tilde{f}|_\Sigma$, $f \in C^\infty(\Sigma)^{W_\Sigma}$ and $\tilde{f}|_{\triangle \cap R(\Sigma)} = (f \circ R)|_{\triangle \cap R(\Sigma)}$ for all $R \in W$, we have $\tilde{f}|_\Sigma = f$, that is, $f \in r_\Sigma(C^\infty(\Sigma^{\mathbf{c}})^W)$. From the arbitrariness of f, we have $r_\Sigma(C^\infty(\Sigma^{\mathbf{c}})^W) = C^\infty(\Sigma)^{W_\Sigma}$. q.e.d.

Now we prove Theorem B in terms of Theorem A and this lemma.

Proof of Theorem B. Let $M^{\mathbf{c}}$ be the extrinsic complexification of M. Let $\phi^{\mathbf{c}} : H^0([0,1], \mathfrak{g}^{\mathbf{c}}) \to G^{\mathbf{c}}$ be the parallel transport map for $G^{\mathbf{c}}$ and $\pi^{\mathbf{c}} : G^{\mathbf{c}} \to G^{\mathbf{c}}/K^{\mathbf{c}}$ be the natural projection. Set $\tilde{\phi}^{\mathbf{c}} := \pi^{\mathbf{c}} \circ \phi^{\mathbf{c}}$ and $\tilde{M}^{\mathbf{c}} := \tilde{\phi}^{\mathbf{c}-1}(M^{\mathbf{c}})$, which is a proper anti-Kaehlerian isoparametric submanifold in $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ (see [K2]). Since this foliation $\mathfrak{F}^{\mathbf{c}}$ in the statement is defined on the whole of $G^{\mathbf{c}}/K^{\mathbf{c}}$, $\tilde{\mathfrak{F}}^{\mathbf{c}}$ is defined on the whole of $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$. According to (ii) of Lemma 5.2, the image of the restriction map $r : C^\infty(G/K)^{\mathfrak{F}} \to C^\infty(\Sigma)$ to Σ is contained in $C^\infty(\Sigma)^{W_\Sigma}$. Since all leaves of $\mathfrak{F}$ intersect with Σ, r is injective. In the sequel, we shall show that $r(C^\infty(G/K)^{\mathfrak{F}}) = C^\infty(\Sigma)^{W_\Sigma}$. Let $\Sigma^{\mathbf{c}}$ be the extrinsic complexification of Σ and $\tilde{\Sigma}^{\mathbf{c}}$ be the horizontal lift of $\Sigma^{\mathbf{c}}$ to some point of $\tilde{M}^{\mathbf{c}}$. The group W acts on both $\Sigma^{\mathbf{c}}$ and $\tilde{\Sigma}^{\mathbf{c}}$. Take an arbitrary $f \in C^\infty(\Sigma)^{W_\Sigma}$. According to (iii) of Lemma 5.2, there exists $\tilde{f} \in C^{\infty}(\Sigma^{\mathbf{c}})^{W}$ with $\tilde{f}|_{\Sigma} = f$. Under the identification of $\Sigma^{\mathbf{c}}$ with $\tilde{\Sigma}^{\mathbf{c}}$, we regard $\tilde{f}$ as an element of $C^{\infty}(\tilde{\Sigma}^{\mathbf{c}})^{W}$. According to Theorem A, $\tilde{f}$ extends to an element of $C^{\infty}(H^0([0,1], \mathfrak{g}^{\mathbf{c}}))^{\tilde{\mathfrak{F}}^{\mathbf{c}}}$, which we denote by F. Since $G^{\mathbf{c}}$ is simply connected and hence each fibre of $\tilde{\phi}^{\mathbf{c}}$ is connected, each leaf of $\tilde{\mathfrak{F}}^{\mathbf{c}}$ is the inverse image of a leaf of $\mathfrak{F}^{\mathbf{c}}$ by $\tilde{\phi}^{\mathbf{c}}$. Hence we see that there uniquely exists $\bar{F} \in C^{\infty}(G^{\mathbf{c}}/K^{\mathbf{c}})^{\mathfrak{F}^{\mathbf{c}}}$ with $\bar{F} \circ \tilde{\phi}^{\mathbf{c}} = F$. Easily we can show $\bar{F}|_{G/K} \in C^{\infty}(G/K)^{\mathfrak{F}}$ and $\bar{F}|_{\Sigma} = f$. That is, we have $f = r(\bar{F}|_{G/K}) \in r(C^{\infty}(G/K)^{\mathfrak{F}})$. From the arbtrariness of f, we have $r(C^{\infty}(G/K)^{\mathfrak{F}}) = C^{\infty}(\Sigma)^{W_{\Sigma}}$. This completes the proof. q.e.d.

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