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GROWTH ESTIMATES FOR LOGARITHMIC DERIVATIVES OF BLASCHKE PRODUCTS AND OF FUNCTIONS IN THE NEVANLINNA CLASS

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Abstract

We prove growth estimates for logarithmic derivatives of functions in the Nevanlinna class. Blaschke products with radially restricted zero sequences will be of particular interest. Our results are sharper in a certain sense than the corresponding estimates in [2] obtained for meromorphic functions in the unit disc.

1. Introduction

We recall that the order of growth of a meromorphic function f in the unit disc $D = \{z : |z| < 1\}$ is given by

$$\rho = \rho(f) = \limsup_{r \to 1^-} \frac{\log^+ T(r, f)}{-\log(1 - r)},$$

where $\log^+ x = \max\{0, \log x\}$, and where $T(r, f)$ denotes the Nevanlinna characteristic of f.

The following result can be found in [2, Corollary 3.2].

Theorem A. *Let f be a meromorphic function in D of finite order ρ. Let $\varepsilon > 0$, and let k and j be integers satisfying $k > j \geq 0$. Assume that $f^{(j)} \not\equiv 0$. Then the following two statements hold.*

(a) *There exists a set $E_1 \subset [0, 1)$ which satisfies*

$$\int_{E_1} \frac{dr}{1 - r} < \infty, \tag{1.1}$$

such that for all $z \in D$ satisfying $|z| \notin E_1$, we have

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$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le \left(\frac{1}{1-|z|}\right)^{(\rho+2+\varepsilon)(k-j)}. \tag{1.2}$$

(b) *There exists a set* $E_2 \subset [0, 2\pi)$ *which has linear measure zero, such that if* $\theta \in [0, 2\pi)\backslash E_2$, *then there is a constant* $R = R(\theta) \in (0, 1)$ *such that for all* z *satisfying* $\arg z = \theta$ *and* $R \le |z| < 1$, *the estimate in* (1.2) *holds.*

Example 9.3 in [2] shows the sharpness of Theorem A in the following sense: There exists an analytic function f for which (1.2) cannot be replaced by "$|f'(z)/f(z)| = O(1/(1-|z|)^{\rho+2})$".

The purpose of this paper is to show that the estimate in (1.2) can be further improved in the case $j = 0$ for functions in the Nevanlinna class N. The class N consists of all meromorphic functions f in D for which $T(r, f) = O(1)$, hence $\rho(f) = 0$ for every $f \in N$.

Note that if $\{z_n\}$ denotes the sequence of all zeros and poles of a function $f \in N$, then

$$\sum_n (1 - |z_n|) < \infty. \tag{1.3}$$

For $\alpha \in (0, 1]$, we will also make use of the more restrictive condition

$$S = \sum_n (1 - |z_n|)^\alpha < \infty \tag{1.4}$$

for the zero/pole sequences $\{z_n\}$. The convergence condition (1.4) is studied, for example, in [1, 7, 8, 9, 11, 12], which typically deal with the problem of when the derivatives of a Blaschke product can belong to the Hardy spaces H^p, and hence to the Nevanlinna class N. See [3] for the basic theory of Hardy spaces. If $\{z_n\}$ is a sequence of nonzero points in D satisfying (1.4) for some $\alpha \in (0, 1]$, then the product

$$B(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z}, \tag{1.5}$$

known as the Blaschke product, represents an analytic function in D, and has zeros precisely at the points z_n.

Our first result shows that the estimates in Theorem A can be slightly improved in the case $j = 0$ for functions in the Nevanlinna class.

Theorem 1.1. *Let* $f \not\equiv 0$ *be a meromorphic function in* N. *Suppose that the sequence* $\{z_n\}$ *of all zeros and poles of* f *satisfies* (1.4) *for some* $\alpha \in (0, 1]$. *Let* $\varepsilon > 0$ *and* $k \in \mathbf{N}$. *Then the following two statements hold.*

(a) *There exists a set* $E_1 \subset [0, 1)$ *which satisfies* (1.1) *such that for all* $z \in D$ *satisfying* $|z| \notin E_1$, *we have*

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| = \begin{cases} O\left(\left(\dfrac{1}{1-|z|}\right)^{2k}\right), & 0 < \alpha < 1, \\ O\left(\left(\dfrac{1}{1-|z|}\right)^{2k}\left(\log\dfrac{1}{1-|z|}\right)^{(2+\varepsilon)k}\right), & \alpha = 1. \end{cases} \tag{1.6}$$

(b) *There exists a set* $E_2 \subset [0, 2\pi)$ *which has linear measure zero, such that if* $\theta \in [0, 2\pi)\backslash E_2$, *then there is a constant* $R = R(\theta) \in (0, 1)$ *such that for all* z *satisfying* $\arg z = \theta$ *and* $R \leq |z| < 1$, *we have*

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| = O\left(\left(\frac{1}{1-|z|}\right)^{2k}\right), \quad 0 < \alpha \leq 1. \tag{1.7}$$

The function $f(z) = \exp\left(\dfrac{1+z}{1-z}\right)$ belongs to the class N, has neither zeros nor poles, and satisfies

$$\frac{f'(z)}{f(z)} = \frac{2}{(1-z)^2}, \quad z \in D.$$

Therefore, the exponent $2k$ in (1.6), case $0 < \alpha < 1$, cannot be replaced by a smaller number. In Section 6 we discuss the sharpness of the estimates in (1.6) in the case when f has infinitely many zeros.

Note that if $f \in N$, then the function f' need not belong to N. See [1] for a counterexample, where a Blaschke product B is constructed such that $B' \notin N$. This is roughly the reason why we have to assume $j = 0$ in Theorem 1.1. These observations also suggest that we should study the logarithmic derivatives of Blaschke products in the case $j = 0$.

Theorem 1.2. *Let* B *be a Blaschke product with zeros* $\{z_n\}$, $z_n \neq 0$, *such that* (1.4) *holds for some* $\alpha \in (0, 1]$, *and let* $\varepsilon > 0$ *and* $k \in \mathbf{N}$. *Then the following two statements hold.*

(a) *There exists a set* $E_1 \subset [0, 1)$ *which satisfies* (1.1) *such that for all* $z \in D$ *satisfying* $|z| \notin E_1$, *we have*

$$\left|\frac{B^{(k)}(z)}{B(z)}\right| = O\left(\left(\frac{1}{1-|z|}\right)^{(1+\alpha)k}\left(\log\frac{1}{1-|z|}\right)^{(2+\varepsilon)k}\right). \tag{1.8}$$

(b) *There exists a set* $E_2 \subset [0, 2\pi)$ *which has linear measure zero, such that if* $\theta \in [0, 2\pi)\backslash E_2$, *then, for all* z *satisfying* $\arg z = \theta$ *and* $|z| \to 1^-$, *we have*

$$\left|\frac{B^{(k)}(z)}{B(z)}\right| = o\left(\left(\frac{1}{1-|z|}\right)^{(1+\alpha)k}\right). \tag{1.9}$$

For completeness, we next restate [9, Theorem 1], which can be considered as an integrated analogue of Theorem 1.2.

Theorem B. *Let* $k \in \mathbf{N}$, *and let* B *be a Blaschke product with zeros* $\{z_n\}$, $z_n \neq 0$, *such that* (1.4) *holds for some* $\alpha \in \left(0, \frac{1}{k+1}\right)$. *Then, if* $m = \frac{1-\alpha}{k}$, *there is a constant* $C = C(\alpha, k) > 0$ *such that*

$$\int_0^{2\pi} \left| \frac{B^{(k)}(re^{i\theta})}{B(re^{i\theta})} \right|^m d\theta \leq CS, \quad \frac{1}{2} < r < 1.$$

In particular, $B^{(k)} \in H^p$ *for each* $p \in (0, m]$.

For further motivation, we note that growth estimates for logarithmic derivatives of Blaschke products have been applied in the theory of complex differential equations, see [5, 6].

This paper is organized as follows. Theorem 1.1 is proved in Sections 2 and 3, while Theorem 1.2 is proved in Sections 4 and 5. In Section 6 we will construct a Blaschke product illustrating the sharpness of our estimates. The proofs of Theorems 1.1(a) and 1.2(a) rely heavily on the reasoning in [2], hence on the well-known Cartan's lemma [10, pp. 19–21]. We prove Theorems 1.1(b) and 1.2(b) using a method which does not rely on Cartan's lemma. Unfortunately, this method does not seem to work for proving Theorems 1.1(a) and 1.2(a).

In all of the proofs we may suppose that the points z_n are listed according to multiplicities and ordered by increasing moduli.

2. Proof of Theorem 1.1(a)

First, we observe that (1.4) enables us to estimate $n(r)$—the number of the points z_n lying in the disc $\{z : |z| \leq r\}$.

Lemma 2.1. *Let* $\{z_n\}$ *be a sequence of nonzero points in* D *such that* (1.4) *holds for some* $\alpha \in (0, 1]$. *Then*

$$n(r) \leq \frac{S}{(1-r)^{\alpha}}, \quad 0 \leq r < 1.$$

Proof. By (1.4), we have

$$S \geq \sum_{0<|z_n|<r} (1-|z_n|)^{\alpha} \geq \sum_{0<|z_n|<r} (1-r)^{\alpha} \geq (1-r)^{\alpha} n(r),$$

where $r \in [0, 1)$ is arbitrary. □

Second, we denote $R = \frac{1+|z|}{2}$, and deduce the following estimate by the proof of [2, Lemma 5.2]: There exist constants $R_0 \in (0, 1)$ and $C > 0$ such that for any z satisfying $R_0 < |z| < 1$ and $f(z) \neq 0, \infty$, we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \le C\left(\left(\frac{1}{1-|z|}\right)^2 + \sum_{|z_n|<R}\frac{1}{|z-z_n|} + \frac{n(R)}{1-r}\right)^k. \tag{2.1}$$

In what follows, the value of the constant C may not be the same at each occurrence. By Lemma 2.1 and the assumption on the sequence $\{z_n\}$, the estimate in (2.1) reduces into

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \le C\left(\left(\frac{1}{1-|z|}\right)^2 + \sum_{|z_n|<R}\frac{1}{|z-z_n|}\right)^k. \tag{2.2}$$

Let $\{D_j\}$, $D_j = \{z : |z-c_j| < r_j\}$, be the sequence of discs as defined in the proof of [2, Theorem 3.1]. Then every disc D_j is contained in the punctured unit disc $D\setminus\{0\}$, and the union $\bigcup_j D_j$ contains the points in the sequence $\{z_n\}$. Now, for some $R_1 \in [R_0, 1)$ and for all $\varepsilon > 0$, we have

$$\sum_{|z_n|<R}\frac{1}{|z-z_n|} \le 2\frac{n(R)}{1-r}\left(\log\frac{1}{1-r}\right)^{1+\varepsilon}\log n(R), \tag{2.3}$$

provided that $z \notin \bigcup_j D_j$ and $R_1 \le |z| < 1$, see formula (6.7) in the proof of [2, Theorem 3.1]. Again by Lemma 2.1 and the assumption on the sequence $\{z_n\}$, the estimate in (2.3) reduces into

$$\sum_{|z_n|<R}\frac{1}{|z-z_n|} = \begin{cases} O\left(\left(\frac{1}{1-|z|}\right)^2\right), & 0<\alpha<1 \\ O\left(\left(\frac{1}{1-|z|}\right)^2\left(\log\frac{1}{1-|z|}\right)^{2+\varepsilon}\right), & \alpha = 1, \end{cases} \tag{2.4}$$

where $z \notin \bigcup_j D_j$ and $R_1 \le |z| < 1$.

Putting (2.2) and (2.4) together, we have shown that (1.6) holds, provided that $z \notin \bigcup_j D_j$ and $R_1 \le |z| < 1$. Thus it remains to show that the union $\bigcup_j D_j$ leads to the exceptional set E_1, as indicated in Theorem 1.1(a).

We define a set $E \subset [0,1)$ as a union of closed intervals:

$$E = \bigcup_{j=1}^{\infty}[|c_j| - r_j, |c_j| + r_j].$$

By formula (6.8) in [2], we obtain

$$\sum_{j=1}^{\infty}\frac{r_j}{1-|c_j|} < \infty.$$

Therefore, by the Limit Comparison Test, it follows that

$$\int_E \frac{dr}{1-r} \le \sum_{j=1}^{\infty}\int_{|c_j|-r_j}^{|c_j|+r_j}\frac{dr}{1-r} \le \sum_{j=1}^{\infty}\frac{2r_j}{1-|c_j|-r_j} < \infty.$$

Finally, we define $E_1 = E \cup [0, R_1]$, and conclude that the set E_1 so defined satisfies (1.1).

The proof of Theorem 1.1(a) is now completed.

3. Proof of Theorem 1.1(b)

We begin by surrounding each point z_n by a Euclidean disc

$$K_n = K_n(z_n, r_n) = \{z : |z_n - z| \le r_n\},$$

where $r_n = a(1 - |z_n|)$ and $a = \min\left\{\frac{1}{2}, \frac{|z_1|}{2(1 - |z_1|)}\right\} > 0$. For each n, we get

$$|z_n| + r_n \le |z_n| + \frac{1}{2}(1 - |z_n|) = \frac{1}{2}(1 + |z_n|) < 1$$

and

$$|z_n| - r_n \ge |z_n| - \frac{|z_1|}{2(1 - |z_1|)}(1 - |z_n|) \ge |z_n| - \frac{|z_1|}{2} > 0,$$

so that each of the discs K_n is properly contained in D, and none of them contains the origin.

Let ϕ_n be the angle that the disc K_n subtends at the origin. We have

$$\sum_{n=1}^{\infty} \frac{r_n}{|z_n|} \le \frac{1}{2|z_1|} \sum_{n=1}^{\infty} (1 - |z_n|) < \infty.$$

Further, there exists an $N \in \mathbf{N}$ such that if $n \ge N$, we have

$$\sin^{-1}\left(\frac{r_n}{|z_n|}\right) \le 2\frac{r_n}{|z_n|},$$

so that

$$\sum_{n=N}^{\infty} \phi_n = 2\sum_{n=N}^{\infty} \sin^{-1}\left(\frac{r_n}{|z_n|}\right) < \infty.$$

Hence, for every $\varepsilon > 0$, there exists an $M = M(\varepsilon) \in \mathbf{N}$, $M \ge N$, such that

$$\sum_{n=M}^{\infty} \phi_n < \varepsilon. \tag{3.1}$$

For the rest of the proof we suppose that $z \in D$ is fixed such that $z \notin \bigcup_{n=1}^{\infty} K_n$. Denote $R = \frac{1 + |z|}{2}$. Lemma 2.1 now yields

$$\sum_{|z_n| < R} \frac{1}{|z - z_n|} \le \sum_{|z_n| < R} \frac{1}{r_n} \le \frac{n(R)}{a(1 - R)} \le C\left(\frac{1}{1 - |z|}\right)^{1+\alpha} \tag{3.2}$$

for some constant $C > 0$ independent of R. The estimate in (1.7) follows by (2.2), (3.1) and (3.2).

The proof of Theorem 1.1(b) is now completed.

4. Proof of Theorem 1.2(a)

The following two auxiliary results are restatements of [8, Lemma 2] and [8, Lemma 3], respectively.

Lemma C. *Let $J \in \mathbf{N}$, and let $\{z_n\}$ be a sequence of nonzero points in D satisfying* (1.4) *for some $\alpha \in (0, 1]$. Then, for any number p satisfying $p \geq \alpha$ and $Jp + \alpha > 1$, we have*

$$\sum_{n=1}^{\infty} \frac{(1 - |z_n|)^p}{(1 - |z_n z|)^{(J+1)p-1}} = o\left(\left(\frac{1}{1 - |z|}\right)^{Jp+\alpha-1}\right), \quad |z| \to 1^-.$$

Lemma D. *Let B be a Blaschke product with zeros $\{z_n\}$, $z_n \neq 0$. Then, for any $J \in \mathbf{N} \cup \{0\}$, we have*

$$B^{(J+1)}(z) = \sum_{m=0}^{J} \sum_{n=1}^{\infty} \frac{J!(m+1)}{(J-m)!} B_n^{(J-m)}(z) \frac{(\bar{z}_n)^{m+1}(|z_n|^2 - 1)}{|z_n|(1 - \bar{z}_n z)^{m+2}},$$

where

$$B_n(z) = \prod_{j \neq n} \frac{|z_j|}{z_j} \frac{z_j - z}{1 - \bar{z}_j z}. \tag{4.1}$$

Let $\{D_j\}$, $D_j = \{z : |z - c_j| < r_j\}$, be the sequence of discs as in the proof of Theorem 1.1(a). For the rest of the proof, we suppose that $z \in D$ is fixed such that $z \notin \bigcup_j D_j$, and that $|z|$ is sufficiently close to 1. By the proof of Theorem 1.1(a), we know that this assumption leads to an exceptional set E_1 satisfying (1.1). Further, $C > 0$ denotes a constant (independent of n) the value of which may not be the same at each occurrence. The proof is by induction.

The case $k = 1$. Lemma D (or a direct computation) yields

$$\frac{B'(z)}{B(z)} = \sum_{n=1}^{\infty} \frac{|z_n|^2 - 1}{(1 - \bar{z}_n z)(z_n - z)}. \tag{4.2}$$

Denote $R = \dfrac{1 + |z|}{2}$. Then (4.2) implies

$$\left|\frac{B'(z)}{B(z)}\right| \leq \sum_{|z_n| < R} \frac{1 - |z_n|^2}{|1 - \bar{z}_n z|\,|z_n - z|} + \sum_{|z_n| \geq R} \frac{1 - |z_n|^2}{|1 - \bar{z}_n z|\,|z_n - z|} = S_1 + S_2, \tag{4.3}$$

where S_1 may be empty. We proceed to show that both of the series S_1 and S_2 converge uniformly for all $z \in D$ such that $|z| \notin E_1$.

Estimating S_2 in (4.3) is easy:

$$\begin{aligned} S_2 &\le \frac{2}{R-|z|} \sum_{|z_n|\ge R} \frac{(1-|z_n|)^{\alpha+1-\alpha}}{|1-\bar{z}_n z|^{\alpha+1-\alpha}} \\ &\le \frac{4}{(1-|z|)^{1+\alpha}} \sum_{|z_n|\ge R} (1-|z_n|)^{\alpha} \le C\left(\frac{1}{1-|z|}\right)^{1+\alpha}, \end{aligned} \tag{4.4}$$

where we have used the fact that $\alpha \le 1$. To estimate S_1 in (4.3) (assuming that it is non-empty), we use Lemma 2.1 in inequality (2.3) to conclude, for an arbitrary $\varepsilon > 0$, that

$$S_1 \le \sum_{|z_n|<R} \frac{2}{|z-z_n|} \le C\left(\frac{1}{1-|z|}\right)^{1+\alpha}\left(\log\frac{1}{1-|z|}\right)^{2+\varepsilon}, \quad |z| \notin E_1. \tag{4.5}$$

By combining (4.3), (4.4) and (4.5), it follows that

$$\left|\frac{B'(z)}{B(z)}\right| \le C\left(\frac{1}{1-|z|}\right)^{1+\alpha}\left(\log\frac{1}{1-|z|}\right)^{2+\varepsilon}, \quad |z| \notin E_1. \tag{4.6}$$

This proves Theorem 1.2(a) in the case $k = 1$.

Remark. Note that the estimate in (4.6) holds for any Blaschke product whose zeros satisfy (1.4), but possibly for a different exceptional set. In particular, the estimate in (4.6) holds for the products $B_n(z)$ defined in (4.1), for the same exceptional set E_1 as above, since each $B_n(z)$ has the same zeros as $B(z)$ except for one (or for one multiplicity in case of a multiple zero).

Induction assumption. Suppose then that the estimate

$$\left|\frac{B^{(k)}(z)}{B(z)}\right| \le C\left(\left(\frac{1}{1-|z|}\right)^{1+\alpha}\left(\log\frac{1}{1-|z|}\right)^{2+\varepsilon}\right)^{k}, \quad |z| \notin E_1, \tag{4.7}$$

holds for every $k = 1, \ldots, J$.

The case $k = J+1$. We make use of Lemma D:

$$\frac{B^{(J+1)}(z)}{B(z)} = \sum_{m=0}^{J}\sum_{n=1}^{\infty} \frac{J!(m+1)}{(J-m)!} \frac{B_n^{(J-m)}(z)}{B_n(z)} \frac{(\bar{z}_n)^{m+1} z_n}{|z_n|^2} \frac{|z_n|^2-1}{(1-\bar{z}_n z)^{m+1}(z_n-z)}.$$

It follows that

$$\left|\frac{B^{(J+1)}(z)}{B(z)}\right| \le \sum_{m=0}^{J}\sum_{n=1}^{\infty} \frac{J!(m+1)}{(J-m)!}\left|\frac{B_n^{(J-m)}(z)}{B_n(z)}\right| \frac{1-|z_n|^2}{|1-\bar{z}_n z|^{m+1}|z-z_n|}. \tag{4.8}$$

By Remark following formula (4.6), the induction assumption (4.7) applies to each of the products $B_n(z)$, and so, for every $m = 0, \dots, J-1$, we have

$$\left|\frac{B_n^{(J-m)}(z)}{B_n(z)}\right| \le C\left(\left(\frac{1}{1-|z|}\right)^{1+\alpha}\left(\log\frac{1}{1-|z|}\right)^{2+\varepsilon}\right)^{J-m}, \quad |z| \notin E_1. \tag{4.9}$$

Similarly as in proving (4.4) and (4.5), we deduce that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1-|z_n|^2}{|1-\bar{z}_n z|^{m+1}|z-z_n|} &\le C\left(\frac{1}{1-|z|}\right)^{m}\sum_{n=1}^{\infty}\frac{1-|z_n|^2}{|1-\bar{z}_n z|\,|z-z_n|} \\ &\le C\left(\frac{1}{1-|z|}\right)^{m+1+\alpha}\left(\log\frac{1}{1-|z|}\right)^{2+\varepsilon},\end{aligned} \tag{4.10}$$

provided that $|z| \notin E_1$.

By combining (4.8), (4.9) and (4.10), it follows that

$$\left|\frac{B^{(J+1)}(z)}{B(z)}\right| \le C\left(\left(\frac{1}{1-|z|}\right)^{1+\alpha}\left(\log\frac{1}{1-|z|}\right)^{2+\varepsilon}\right)^{J+1}, \quad |z| \notin E_1.$$

This proves Theorem 1.2(a) in the case $k = J+1$.

The proof of Theorem 1.2(a) is now completed.

5. Proof of Theorem 1.2(b)

This proof is by induction and combines the ideas used in proving Theorems 1.1(b) and 1.2(a), see Sections 3 and 4, respectively.

Let $\{K_n\}$ be the sequence of discs as in the proof of Theorem 1.1(b). We suppose that $z \in D\backslash U$ is fixed, where $U = \bigcup_{n=1}^{\infty} K_n$. By the proof of Theorem 1.1(b), we know that this assumption leads to an exceptional set E_2 of linear measure zero. As earlier, we denote $R = \dfrac{1+|z|}{2}$, and $C > 0$ stands for a constant (independent of n) the value of which may not be the same at each occurrence.

The case $k = 1$. By (4.3), it follows that

$$\begin{aligned}\left|\frac{B'(z)}{B(z)}\right| &\le \sum_{|z_n|<R}\frac{1-|z_n|^2}{|1-\bar{z}_n z| r_n} + \sum_{|z_n|\ge R}\frac{1-|z_n|^2}{|1-\bar{z}_n z|(|z_n|-|z|)} \\ &\le \frac{2}{a(1-R)}\sum_{|z_n|<R}\frac{1-|z_n|}{1-|z_n z|} + \frac{2}{R-|z|}\sum_{|z_n|\ge R}\frac{1-|z_n|}{1-|z_n z|} \\ &\le \frac{C}{1-|z|}\sum_{n=1}^{\infty}\frac{1-|z_n|}{1-|z_n z|}, \quad z \in D\backslash U.\end{aligned} \tag{5.1}$$

Lemma C, with $p = J = 1$, yields

$$\sum_{n=1}^{\infty}\frac{1-|z_n|}{1-|z_n z|}=o\left(\left(\frac{1}{1-|z|}\right)^{1+\alpha}\right),\quad |z|\to 1^-. \tag{5.2}$$

Now, (3.1), (5.1) and (5.2) prove Theorem 1.2(b) in the case $k=1$.

Induction assumption. Suppose then that

$$\left|\frac{B^{(k)}(z)}{B(z)}\right|=o\left(\left(\frac{1}{1-|z|}\right)^{(1+\alpha)k}\right),\quad |z|\to 1^-,\ z\in D\backslash U, \tag{5.3}$$

holds for every $k=1,\dots,J$. So, by (3.1), we assume that the assertion of Theorem 1.2(b) holds for $k=1,\dots,J$.

The case $k=J+1$. Applying Lemma C (with $p=1$ and $J=m+1$), we get, in the spirit of the case $k=1$, that the estimate

$$\sum_{n=1}^{\infty}\frac{1-|z_n|^2}{|1-\bar{z}_n z|^{m+1}|z_n-z|}=o\left(\left(\frac{1}{1-|z|}\right)^{m+1+\alpha}\right),\quad |z|\to 1^-, \tag{5.4}$$

holds for every $m=0,\dots,J$, provided that $z\in D\backslash U$. Just as in Section 4, we note that the estimate in (5.3) holds for the products $B_n(z)$ as well. Therefore, by (4.8), (5.3) and (5.4), we conclude that

$$\left|\frac{B^{(J+1)}(z)}{B(z)}\right|=o\left(\left(\frac{1}{1-|z|}\right)^{(1+\alpha)(J+1)}\right),\quad |z|\to 1^-,\ z\in D\backslash U.$$

This, together with (3.1), proves Theorem 1.2(b) in the case $k=J+1$.

The proof of Theorem 1.2(b) is now completed.

6. Discussion on sharpness

We illustrate the sharpness of Theorem 1.2(a) (and of Theorem 1.1(a)) by constructing a suitable Blaschke product with infinitely many zeros. An analogous example for entire functions is constructed in [4, p. 103]. However, the reasoning in the unit disc seems to be more involved than the corresponding reasoning in the complex plane.

Let $0<\alpha<1$, and define a constant

$$N=N(\alpha)=\left[\frac{1+\alpha}{1-\alpha}\right]+1\geq 2,$$

where $[x]$ denotes the largest integer not exceeding x. Further, define

$$z_n=1-\left(\frac{1}{n}\right)^{1/\alpha},\quad n\geq N.$$

For every $\varepsilon>0$, we clearly have

$$\sum_{n=N}^{\infty}(1-z_n)^{\alpha}=\infty \quad \text{and} \quad \sum_{n=N}^{\infty}(1-z_n)^{\alpha+\varepsilon}<\infty.$$

The Blaschke product having zeros at the points z_n, $n \geq N$, is simply

$$B(z)=\prod_{n=N}^{\infty}\frac{z_n-z}{1-z_n z}. \tag{6.1}$$

Theorem 1.2(a) implies, for any $\varepsilon > 0$, that there is an exceptional set $E_1 \subset [0,1)$ satisfying (1.1) such that

$$\left|\frac{B'(z)}{B(z)}\right| = O\left(\left(\frac{1}{1-|z|}\right)^{1+\alpha+\varepsilon}\right), \quad |z| \notin E_1. \tag{6.2}$$

To illustrate the sharpness of (6.2), we prove the following result.

Proposition 6.1. *Let B be the Blaschke product defined in* (6.1), *and let E_1 be the exceptional set in* (6.2). *Then there exist a set $F_1 \in [0,1)$, satisfying $\int_{F_1}\frac{dr}{1-r}=\infty$, and a constant $C=C(\alpha)>0$ such that*

$$\left|\frac{B'(x)}{B(x)}\right| \geq \frac{C}{(1-x)^{1+\alpha}}\log\frac{1}{1-x}, \quad x \in F_1 \backslash E_1. \tag{6.3}$$

The remaining part of the present section is devoted to proving Proposition 6.1. To begin with, we define the sequences

$$\beta_n=(1-z_n)^{1+\alpha}=\left(\frac{1}{n}\right)^{1+1/\alpha}, \quad n \geq N,$$

and

$$\gamma_n=\frac{(1-z_n)^{1+\alpha}}{\alpha\delta\log\dfrac{1}{1-z_n}}=\frac{\beta_n}{\delta\log n} \quad (\leq \beta_n), \quad n \geq N, \tag{6.4}$$

where $\delta \geq \dfrac{1}{\log 2}$ is a constant to be fixed later on. Also, we define an auxiliary function

$$g(x)=\frac{x}{x-1}\left(\frac{x}{x+1}\right)^{1/\alpha}, \quad x>1.$$

Obviously, g is differentiable and $\lim_{x\to\infty} g(x)=1$. Further,

$$g'(x)=x^{1/\alpha}(x+1)^{1/\alpha}\frac{\left(1+\dfrac{1}{\alpha}\right)(x-1)-x\left(1+\dfrac{1}{\alpha}\cdot\dfrac{x-1}{x+1}\right)}{(x-1)^2(x+1)^{2/\alpha}},$$

so that $g'(x)>0$ if and only if

$$\left(1+\frac{1}{\alpha}\right)(x-1) > x\left(1+\frac{1}{\alpha}\cdot\frac{x-1}{x+1}\right),$$

which is equivalent to the statement $x > \dfrac{1+\alpha}{1-\alpha}$. Hence, we conclude that $g(x) < 1$ for $N \le x < \infty$. This property of g will be used to prove the inequalities

$$0 < z_N < z_N + \beta_N < z_n < z_n + \beta_n < z_{n+1} < \cdots < 1, \quad n > N. \tag{6.5}$$

Namely, we have

$$\frac{n}{n-1}\left(\frac{n}{n+1}\right)^{1/\alpha} = g(n) < 1, \quad n \ge N. \tag{6.6}$$

Multiplying both sides of (6.6) by $\dfrac{n-1}{n}\left(\dfrac{1}{n}\right)^{1/\alpha}$, we obtain

$$\left(\frac{1}{n+1}\right)^{1/\alpha} < \left(\frac{1}{n}\right)^{1/\alpha}\left(1-\frac{1}{n}\right), \quad n \ge N,$$

so that

$$1-\left(\frac{1}{n}\right)^{1/\alpha} + \left(\frac{1}{n}\right)^{1+1/\alpha} < 1-\left(\frac{1}{n+1}\right)^{1/\alpha}, \quad n \ge N.$$

But this is equivalent to $z_n + \beta_n < z_{n+1}$, $n \ge N$. Note that all the other inequalities in (6.5) are trivial. Further, since $\gamma_n \le \beta_n$ for $n \ge N$, the inequalities (6.5) give us

$$0 < z_N < z_N + \gamma_N < z_n < z_n + \gamma_n < z_{n+1} < \cdots < 1, \quad n > N. \tag{6.7}$$

Next, we consider the union of open intervals

$$F = \bigcup_{n=N}^{\infty} (z_n, z_n + \gamma_n).$$

By (6.7), F is a subset of $[0,1)$, and the intervals $(z_n, z_n + \gamma_n)$ are pairwise disjoint. Further,

$$\int_F \frac{dr}{1-r} = \sum_{n=N}^{\infty} \int_{z_n}^{z_n+\gamma_n} \frac{dr}{1-r} \ge \sum_{n=N}^{\infty} \frac{\gamma_n}{1-z_n} = \frac{1}{\delta}\sum_{n=N}^{\infty} \frac{1}{n \log n} = \infty,$$

no matter how we choose the constant $\delta \ge \dfrac{1}{\log 2}$.

A simple computation results in

$$\frac{B'(z)}{B(z)} = \sum_{n=N}^{\infty} \frac{z_n^2 - 1}{(1-z_n z)(z-z_n)}.$$

Suppose then that $z = x \in F$. Then there exists a unique integer $k \geq N$ such that $z_k < x < z_k + \gamma_k$. We have

$$\left|\frac{B'(x)}{B(x)}\right| \geq \left|\frac{z_k^2 - 1}{(1 - z_k x)(x - z_k)}\right| - \left|\sum_{n \neq k} \frac{z_n^2 - 1}{(1 - z_n x)(x - z_n)}\right| \tag{6.8}$$
$$\geq \frac{1 - z_k^2}{(1 - z_k x)(x - z_k)} - \sum_{n=N}^{k-1} \frac{1 - z_n^2}{(1 - z_n x)(x - z_n)}$$
$$- \sum_{n=k+1}^{\infty} \frac{1 - z_n^2}{(1 - z_n x)(z_n - x)}.$$

With the notation fixed above, we obtain the following two lemmas, which are valid for all k large enough. By this we mean that x is close to 1, yet $x \in F$ and $z_k < x < z_k + \gamma_k$. The proofs of the lemmas will be given at the end of this section.

Lemma 6.2. *For all k large enough, we have*

$$\sum_{n=k+1}^{\infty} \frac{1 - z_n^2}{(1 - z_n x)(z_n - x)} \leq \frac{C_1}{(1 - z_k)^{1+\alpha}} \log \frac{1}{1 - z_k},$$

where $C_1 = C_1(\alpha) > 0$ is a constant independent of k.

Lemma 6.3. *For all k large enough, we have*

$$\sum_{n=N}^{k-1} \frac{1 - z_n^2}{(1 - z_n x)(x - z_n)} \leq \frac{4\alpha^2}{(1 - z_k)^{1+\alpha}} \log \frac{1}{1 - z_k}.$$

We will also make use of the next result, which follows directly from the assumption $z_k < x < z_k + \gamma_k$ and the definition of the points γ_k.

Lemma 6.4. *For any $k \geq N$, we have*

$$\frac{1 - z_k^2}{(1 - z_k x)(x - z_k)} \geq \frac{\alpha\delta}{(1 - z_k)^{1+\alpha}} \log \frac{1}{1 - z_k},$$

where $\delta \geq \dfrac{1}{\log 2}$ is the constant from (6.4).

Suppose for a moment that the assertions in Lemmas 6.2 and 6.3 hold for all $k \geq M \geq N$. Choose the constant $\delta \geq \dfrac{1}{\log 2}$ such that

$$C_0 = \alpha\delta - C_1 - 4\alpha^2 > 0.$$

Define

$$F_1 = \bigcup_{n=M}^{\infty} (z_n, z_n + \gamma_n).$$

Clearly, $F_1 \subset F$ and $\int_{F_1} \frac{dr}{1-r} = \infty$. Suppose that $z = x \in F_1 \backslash E_1$, where E_1 is the exceptional set in (6.2). Then, by (6.8), Lemmas 6.2–6.4, and the inequalities $z_k < x < z_k + \gamma_k$, we conclude that

$$\left|\frac{B'(x)}{B(x)}\right| \geq \frac{C_0}{(1-z_k)^{1+\alpha}} \log \frac{1}{1-z_k} \geq \frac{C_0 2^{-(1+\alpha)}}{(1-x)^{1+\alpha}} \log \frac{2^{-1}}{1-x}. \tag{6.9}$$

The estimate in (6.3) follows from (6.9).

To conclude Proposition 6.1, it remains to prove Lemmas 6.2 and 6.3.

Proof of Lemma 6.2. For any $n \geq k+1$, we have

$$\begin{aligned} \frac{1-z_n^2}{(1-z_n x)(z_n - x)} &\leq \frac{2}{1-(z_k+\gamma_k)} \cdot \frac{1-z_n}{z_n - (z_k+\gamma_k)} \\ &\leq \frac{2}{1-(z_k+\beta_k)} \cdot \frac{1-z_n}{z_n-(z_k+\beta_k)} \\ &= \frac{k}{k-1} \cdot \frac{2}{1-z_k} \cdot \frac{k^{1+1/\alpha}}{n^{1/\alpha}(k-1) - k^{1+1/\alpha}} \\ &\leq \frac{4k^{1+1/\alpha}}{1-z_k} \cdot \frac{1}{n^{1/\alpha}(k-1)-k^{1+1/\alpha}}. \end{aligned} \tag{6.10}$$

Note that inequality (6.6) guarantees that

$$n^{1/\alpha}(k-1) - k^{1+1/\alpha} \geq (k+1)^{1/\alpha}(k-1) - k^{1+1/\alpha} > 0, \quad n \geq k+1. \tag{6.11}$$

In fact, by L'Hopital's rule,

$$\lim_{k\to\infty} k\left(1 - \left(\frac{k}{k+1}\right)^{1/\alpha}\right) = \frac{1}{\alpha}, \tag{6.12}$$

so that

$$\lim_{k\to\infty} \frac{(k+1)^{1/\alpha}}{(k+1)^{1/\alpha}(k-1) - k^{1+1/\alpha}} = \lim_{k\to\infty} \frac{1}{k\left(1-\left(\frac{k}{k+1}\right)^{1/\alpha}\right) - 1} = \frac{\alpha}{1-\alpha}. \tag{6.13}$$

We will also make use of the fact that

$$\lim_{k\to\infty} \frac{k-1}{(k+1)^{1/\alpha}(k-1) - k^{1+1/\alpha}} \cdot \frac{(k+1)^{1/\alpha}}{k-1} = \frac{\alpha}{1-\alpha}, \tag{6.14}$$

which clearly follows from (6.13).

Now, by using (6.14) and integrating by parts, we deduce that

$$
\begin{aligned}
\int_{k+1}^{\infty} \frac{dt}{t^{1/\alpha}(k-1)-k^{1+1/\alpha}} &= \int_{k+1}^{\infty} t^{1-1/\alpha} \cdot \frac{t^{1/\alpha-1}}{t^{1/\alpha}(k-1)-k^{1+1/\alpha}}\, dt \\
&= \frac{\alpha}{k-1}\left[t^{1-1/\alpha} \log\left(\frac{t^{1/\alpha}(k-1)-k^{1+1/\alpha}}{k-1}\right)\right]_{k+1}^{\infty} \\
&\quad + \frac{1-\alpha}{k-1}\int_{k+1}^{\infty} t^{-1/\alpha} \log\left(\frac{t^{1/\alpha}(k-1)-k^{1+1/\alpha}}{k-1}\right) dt \\
&\le \frac{k+1}{k-1}\,\frac{\alpha}{(k+1)^{1/\alpha}}\, \log\left(\frac{k-1}{(k+1)^{1/\alpha}(k-1)-k^{1+1/\alpha}}\right) \\
&\quad + \frac{1-\alpha}{k-1}\int_{k+1}^{\infty} t^{-1/\alpha} \log(t^{1/\alpha})\, dt \\
&\le \frac{k+1}{k-1}\,\frac{\alpha}{(k+1)^{1/\alpha}}\, \log\left(\frac{2\alpha}{1-\alpha}\cdot\frac{k-1}{(k+1)^{1/\alpha}}\right) \\
&\quad - \frac{1}{k-1}\left[t^{1-1/\alpha}\left(\frac{\alpha}{1-\alpha}+\log t\right)\right]_{k+1}^{\infty} \\
&\le \frac{2}{(k+1)^{1/\alpha}}\left(\frac{\alpha}{1-\alpha}+\log(k+1)\right),
\end{aligned}
$$

which holds for all k large enough. Hence, by using (6.10), (6.11), and (6.13), we obtain

$$
\begin{aligned}
&\sum_{n=k+1}^{\infty} \frac{1-z_n^2}{(1-z_n x)(z_n-x)} \\
&\quad \le \frac{4k^{1+1/\alpha}}{1-z_k}\left(\frac{1}{(k+1)^{1/\alpha}(k-1)-k^{1+1/\alpha}} + \int_{k+1}^{\infty} \frac{dt}{t^{1/\alpha}(k-1)-k^{1+1/\alpha}}\right) \\
&\quad \le \frac{4k^{1+1/\alpha}}{1-z_k}\left(\frac{\alpha}{1-\alpha}\cdot\frac{2}{(k+1)^{1/\alpha}} + \frac{2}{(k+1)^{1/\alpha}}\left(\frac{\alpha}{1-\alpha}+\log(k+1)\right)\right) \\
&\quad \le \frac{4k}{1-z_k}\left(\frac{4\alpha}{1-\alpha}+2\log(k+1)\right),
\end{aligned}
$$

which holds for all k large enough. The assertion now follows by using the fact that $k = \dfrac{1}{(1-z_k)^{\alpha}}$ for every $k \ge N$. □

Proof of Lemma 6.3. We assume that $k \ge N+1$, for otherwise there is nothing to prove. Then, for any $n \in \{N, \dots, k-1\}$, we have

$$\begin{aligned}\frac{1-z_n^2}{(1-z_nx)(x-z_n)} &\le 2\frac{1-z_n}{(1-z_n(z_k+\beta_k))(z_k-z_n)}\\ &= 2k^{2/\alpha}\frac{n^{1/\alpha}}{\left((n^{1/\alpha}-1)\dfrac{k-1}{k}+k^{1/\alpha}\right)(k^{1/\alpha}-n^{1/\alpha})}\\ &\le 2k^{1/\alpha}\frac{n^{1/\alpha}}{k^{1/\alpha}-n^{1/\alpha}}.\end{aligned}$$

Define a continuous function

$$h(x)=\frac{x^{1/\alpha}}{k^{1/\alpha}-x^{1/\alpha}},\quad 1\le x<k^{1/\alpha}.$$

Since the function h is strictly increasing, we get

$$h(n)<2\int_n^{n+1/2}h(t)\,dt,\quad 1\le n<n+\frac{1}{2}<k^{1/\alpha}.$$

Therefore, it follows that

$$\begin{aligned}\text{(6.15)}\qquad \sum_{n=N}^{k-1}\frac{1-z_n^2}{(1-z_nx)(x-z_n)} &\le 2k^{1/\alpha}\sum_{n=N}^{k-1}\frac{n^{1/\alpha}}{k^{1/\alpha}-n^{1/\alpha}}\\ &\le 4k^{1/\alpha}\int_1^{k-1/2}\frac{t^{1/\alpha}}{k^{1/\alpha}-t^{1/\alpha}}\,dt\\ &\le 4k^{1/\alpha}\left(k-\frac{1}{2}\right)\int_1^{k-1/2}\frac{t^{1/\alpha-1}}{k^{1/\alpha}-t^{1/\alpha}}\,dt\\ &= 4\alpha k^{1/\alpha}\left(k-\frac{1}{2}\right)\log\left(\frac{k^{1/\alpha}-1}{k^{1/\alpha}-\left(k-\frac{1}{2}\right)^{1/\alpha}}\right),\end{aligned}$$

which holds for any $k\ge N+1$. Similarly as in (6.12), we get

$$\text{(6.16)}\qquad \lim_{k\to\infty}\frac{k^{1/\alpha}-1}{k\left(k^{1/\alpha}-\left(k-\frac{1}{2}\right)^{1/\alpha}\right)}=\lim_{k\to\infty}\frac{1-k^{-1/\alpha}}{k\left(1-\left(\dfrac{k-\frac{1}{2}}{k}\right)^{1/\alpha}\right)}=\alpha.$$

Now, by (6.15) and (6.16), we deduce that

$$\sum_{n=N}^{k-1}\frac{1-z_n^2}{(1-z_nx)(x-z_n)}\le 4\alpha k^{1+1/\alpha}\log k,$$

which holds for all k large enough. Finally, since $k=\dfrac{1}{(1-z_k)^{\alpha}}$ for every $k\ge N$, we conclude the assertion. □

Remark. The discussion above holds in the case when $0 < \alpha < 1$. We note that if $\alpha \notin (0,1)$, then the sequence $\{z_n\}$ is not even a Blaschke sequence.

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