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SCHWARZ-PICK INEQUALITIES FOR CONVEX DOMAINS

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Abstract

Let Ω and Π be two simply connected domains in the complex plane $\mathbf{C}$, which are not equal to the whole plane $\mathbf{C}$, and let $A(\Omega, \Pi)$ denote the set of functions $f : \Omega \to \Pi$ analytic in Ω. Define the quantities $C_n(\Omega, \Pi)$ by

$$C_n(\Omega, \Pi) := \sup_{f \in A(\Omega, \Pi)} \sup_{z \in \Omega} \frac{|f^{(n)}(z)| \lambda_\Pi(f(z))}{n!(\lambda_\Omega(z))^n}, \quad n \in \mathbf{N}$$

where λ_Ω and λ_Π are the densities of the Poincaré metric in Ω and Π, respectively. We derive sharp upper bounds for $|f^{(n)}(z)|$ $(z \in \Omega)$ and $C_n(\Omega, \Pi)$ if $2 \le n \le 8$ and Ω is a convex domain. The detailed equality condition of the estimate on $|f^{(n)}(z)|$ is also given.

1. Introduction

Let Ω and Π be two simply connected domains in the complex plane $\mathbf{C}$, which are not equal to the whole plane $\mathbf{C}$, and let $A(\Omega, \Pi)$ denote the set of functions $f : \Omega \to \Pi$ analytic in Ω. We consider the quantities $C_n(\Omega, \Pi)$ defined by

$$C_n(\Omega, \Pi) := \sup_{f \in A(\Omega, \Pi)} \sup_{z \in \Omega} \frac{|f^{(n)}(z)| \lambda_\Pi(f(z))}{n!(\lambda_\Omega(z))^n}$$

for $n \in \mathbf{N}$, where λ_Ω and λ_Π are the densities of the Poincaré metric in Ω and Π, respectively. Many papers in geometric function theory are devoted to the problem of determining $|f^{(n)}(z)|$ or $C_n(\Omega, \Pi)$ for $n \in \mathbf{N}$ (see Avkhadiev and Wirths [1, 2, 3]). Since $\lambda_\Delta = (1 - |z|^2)^{-1}$ for $z \in \Delta := \{z \in \mathbf{C} : |z| < 1\}$, the classical Schwarz-Pick lemma indicates that $C_1(\Delta, \Delta) = 1$. The generalized Schwarz-Pick lemma assures that the sharp estimate

$$|f'(z)| \le \frac{\lambda_\Omega(z)}{\lambda_\Pi(f(z))}, \quad z \in \Omega \tag{1.1}$$

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is valid for all $f \in A(\Omega, \Pi)$. This in turn shows that

$$C_1(\Omega, \Pi) = 1 \tag{1.2}$$

for any pair (Ω, Π) of simply connected domains. For $n \geq 2$, Ruscheweyh [10, 11] and Yamashita [12] showed that

$$C_n(\Delta, \Delta) = 2^{n-1}, \quad C_n(\Delta, \Lambda) = 2^{n-1}, \quad \text{and} \quad C_n(\Sigma, \Lambda) = \binom{2n-1}{n}, \tag{1.3}$$

where $\Lambda := \{z \in \mathbf{C} : \mathrm{Re}(z) > 0\}$ and $\Sigma := \mathbf{C} \backslash (-\infty, -1/4]$. Recently, Avkhadiev and Wirths [1] proved that

(i) $C_n(\Delta, \Pi) = 2^{n-1}$ for any convex domain Π and $n \geq 2$;

(ii) $C_n(\Omega, \Pi) \leq 4^{n-1}$ for all simply connected domains Ω and Π in $\mathbf{C}$ and $n \geq 2$.

Equality holds in (ii) if and only if Ω and Π are equal to Σ up to similarity.

In [1], Avkhadiev and Wirths formulated the following two conjectures.

Conjecture 1. $C_n(\Omega, \Pi) \geq 2^{n-1}$ *for all simply connected domains* Ω *and* Π *in* $\mathbf{C}$.

Conjecture 2. $C_n(\Omega, \Pi) = 2^{n-1}$ *if and only if* Ω *and* Π *are convex.*

Motivated by the above conjectures, in the present paper we shall generalize the above known results by giving the sharp upper bounds for $|f^{(n)}(z)|$ $(z \in \Omega)$ and $C_n(\Omega, \Pi)$ in the case when $2 \leq n \leq 8$ and Ω is convex. The detailed equality condition of the estimate on $|f^{(n)}(z)|$ is also given.

2. Main theorems and their consequences

Theorem 1. *Let* Ω *and* Π *be two convex domains in* $\mathbf{C}$. *If* $f(z) \in A(\Omega, \Pi)$, *then*

$$|f^{(n)}(z)| \leq n! 2^{n-1} \frac{(\lambda_\Omega(z))^n}{\lambda_\Pi(f(z))}, \quad z \in \Omega \tag{2.1}$$

holds for $2 \leq n \leq 8$. *Equality holds in* (2.1) *at a point* $z = z_0 \in \Omega$ *if and only if the following conditions are satisfied*: (1) Ω *and* Π *are both half-planes*, (2) f *is a Möbius transformation mapping* Ω *onto* Π *with* $f(\infty) \neq \infty$, *and* (3) *the line segment joining* z_0 *and* $f^{-1}(\infty)$ *is perpendicular to* $\partial\Omega$.

Proof. For the convex domains Ω and Π, let $w = f(z) \in A(\Omega, \Pi)$, $z_0 \in \Omega$ and $w_0 = f(z_0) \in \Pi$. Denote by Φ_Ω (resp. Φ_Π) the conformal map of Δ onto Ω (resp. Π) with $\Phi_\Omega(0) = z_0$ (resp. $\Phi_\Pi(0) = w_0$) and $\Phi'_\Omega(0) = 1/\lambda_\Omega(z_0) > 0$ (resp. $\Phi'_\Pi(0) = 1/\lambda_\Pi(w_0) > 0$). Then the functions

(2.2) $$z = h_\Omega(\zeta) := \frac{\Phi_\Omega(\zeta) - \Phi_\Omega(0)}{\Phi'_\Omega(0)}, \quad \zeta \in \Delta$$

(2.3) $$w = h_\Pi(\zeta) := \frac{\Phi_\Pi(\zeta) - \Phi_\Pi(0)}{\Phi'_\Pi(0)}, \quad \zeta \in \Delta$$

are normalized convex functions. Let

(2.4) $$(h_\Omega^{-1}(z))^k = \sum_{n=k}^{\infty} B_{n,k} z^n, \quad k \in \mathbf{N}.$$

Then $B_{k,k} = 1$ $(k \in \mathbf{N})$ and

(2.5) $$(\Phi_\Omega^{-1}(z))^k = \sum_{n=k}^{\infty} \frac{B_{n,k}}{(\Phi'_\Omega(0))^n}(z - z_0)^n, \quad k \in \mathbf{N}.$$

Consider the function $g \in A(\Delta, \Pi)$ defined by

(2.6) $$g(\zeta) := f(\Phi_\Omega(\zeta)) = \sum_{k=0}^{\infty} a_k \zeta^k, \quad \zeta \in \Delta.$$

We have

(2.7) $$f(z) - f(z_0) = \sum_{k=1}^{\infty} a_k (\Phi_\Omega^{-1}(z))^k, \quad z \in \Omega$$

or from (2.5)

(2.8) $$\sum_{n=1}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = \sum_{k=1}^{\infty} a_k \sum_{n=k}^{\infty} \frac{B_{n,k}}{(\Phi'_\Omega(0))^n}(z - z_0)^n$$
$$= \sum_{n=1}^{\infty} \left\{ \sum_{k=1}^{n} a_k \frac{B_{n,k}}{(\Phi'_\Omega(0))^n} \right\} (z - z_0)^n,$$

which yields

(2.9) $$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{(\Phi'_\Omega(0))^n} \sum_{k=1}^{n} a_k B_{n,k}.$$

We shall estimate $|a_k|$ and $|B_{n,k}|$ respectively.

First note that $g(\Delta) \subset \Pi$. The function

$$\frac{g(\zeta) - g(0)}{\Phi'_\Pi(0)} = \sum_{k=1}^{\infty} \lambda_\Pi(w_0) a_k \zeta^k$$

defined in accordance with (2.6) is subordinate to the convex function $h_\Pi(\zeta)$. The subordination principle and Theorem 6.4 on page 195 in [6] imply

(2.10) $$\lambda_\Pi(w_0)|a_k| \le 1, \quad k \in \mathbf{N}.$$

Equality in (2.10) holds for all $k = 1, 2, \dots, n$ with $n \geq 2$ if and only if

$$g(\zeta) = \gamma\Phi_\Pi'(0)\frac{\zeta}{1-\delta\zeta} + g(0), \quad \gamma, \delta \in \mathbf{C},\ |\gamma| = |\delta| = 1,\ \zeta \in \Delta. \tag{2.11}$$

This shows that f is a conformal map and Π is equal to Λ up to similarity.

Next we estimate $|B_{n,k}|$. Let $B_{n,1} = B_n$ $(n \in \mathbf{N})$. Then we can write $(h_\Omega^{-1}(z))^k$ as follows,

$$(h_\Omega^{-1}(z))^k = z^k \sum_{j=0}^{k} \binom{k}{j} \left(\sum_{n=2}^{\infty} B_n z^{n-1} \right)^j = \sum_{n=k}^{\infty} B_{n,k} z^n,$$

which yields

$$\begin{aligned}
B_{k+1,k} &= \binom{k}{1} B_2, \quad B_{k+2,k} = \binom{k}{1} B_3 + \binom{k}{2} B_2^2, \\
B_{k+3,k} &= \binom{k}{1} B_4 + 2\binom{k}{2} B_2 B_3 + \binom{k}{3} B_2^3, \\
B_{k+4,k} &= \binom{k}{1} B_5 + \binom{k}{2} (2B_2B_4 + B_3^2) + 3\binom{k}{3} B_2^2 B_3 + \binom{k}{4} B_2^4, \\
B_{k+5,k} &= \binom{k}{1} B_6 + 2\binom{k}{2} (B_2B_5 + B_3B_4) + 3\binom{k}{3} (B_2^2 B_4 + B_3^2 B_2) \\
&\quad + 4\binom{k}{4} B_2^3 B_3 + \binom{k}{5} B_2^5, \\
B_{k+6,k} &= \binom{k}{1} B_7 + \binom{k}{2} (2B_2B_6 + B_4^2 + 2B_3B_5) \\
&\quad + \binom{k}{3} (B_3^3 + 3B_2^2B_5 + 6B_2B_3B_4) + 2\binom{k}{4} (2B_2^3B_4 + 3B_2^2B_3^2) \\
&\quad + 5\binom{k}{5} B_2^4 B_3 + \binom{k}{6} B_2^6, \\
B_{k+7,k} &= \binom{k}{1} B_8 + 2\binom{k}{2} (B_2B_7 + B_3B_6 + B_4B_5) \\
&\quad + 3\binom{k}{3} (B_2^2B_6 + 2B_2B_3B_5 + B_2B_4^2 + B_3^2B_4) \\
&\quad + 4\binom{k}{4} (B_2^3B_5 + 3B_2^2B_3B_4 + B_2B_3^3) \\
&\quad + 5\binom{k}{5} (B_2^4B_4 + 2B_2^3B_3^2) + 6\binom{k}{6} B_2^5B_3 + \binom{k}{7} B_2^7,
\end{aligned} \tag{2.12}$$

where $\binom{k}{j}$ are the binomial coefficients.

For the convex function $z = h_\Omega(\zeta)$ of (2.2), we have

$$|B_n| \le 1 \quad (n = 2, 3, \dots, 8), \tag{2.13}$$

and this bound is sharp. See Libera and Zlotkiewicz [8, 9] and Campschroer [4]. Hence we deduce from the above expression (2.12) that

$$|B_{k+1,k}| \le \binom{k}{1}, \quad |B_{k+2,k}| \le \binom{k}{1} + \binom{k}{2} = \binom{k+1}{k-1},$$

$$|B_{k+3,k}| \le \binom{k}{1} + 2\binom{k}{2} + \binom{k}{3} = \binom{k+2}{k-1},$$

$$|B_{k+4,k}| \le \binom{k}{1} + 3\binom{k}{2} + 3\binom{k}{3} + \binom{k}{4} = \binom{k+3}{k-1},$$

$$|B_{k+5,k}| \le \binom{k}{1} + 4\binom{k}{2} + 6\binom{k}{3} + 4\binom{k}{4} + \binom{k}{5} = \binom{k+4}{k-1},$$

$$|B_{k+6,k}| \le \binom{k}{1} + 5\binom{k}{2} + 10\binom{k}{3} + 10\binom{k}{4} + 5\binom{k}{5} + \binom{k}{6} = \binom{k+5}{k-1},$$

$$|B_{k+7,k}| \le \binom{k}{1} + 6\binom{k}{2} + 15\binom{k}{3} + 20\binom{k}{4} + 15\binom{k}{5} + 6\binom{k}{6} + \binom{k}{7} = \binom{k+6}{k-1},$$

that is,

$$|B_{n,k}| \le \binom{n-1}{k-1}, \quad (n = k, k+1, \dots, k+7). \tag{2.14}$$

Equality in (2.14) holds if and only if

$$z = h_\Omega(\zeta) = \frac{\zeta}{1 - \varepsilon\zeta}, \quad \varepsilon \in \mathbf{C},\ |\varepsilon| = 1,\ \zeta \in \Delta. \tag{2.15}$$

This shows that Ω is equal to Λ up to similarity.

It follows from (2.9), (2.10) and (2.14) that for $2 \le n \le 8$,

$$\begin{aligned} \frac{|f^{(n)}(z_0)|}{n!} &\le (\lambda_\Omega(z_0))^n \sum_{k=1}^{n} |a_k|\,|B_{n,k}| \\ &\le \frac{(\lambda_\Omega(z_0))^n}{\lambda_\Pi(f(z_0))} \sum_{k=1}^{n} \binom{n-1}{k-1} \\ &= \frac{(\lambda_\Omega(z_0))^n}{\lambda_\Pi(f(z_0))} 2^{n-1}, \end{aligned} \tag{2.16}$$

which yields (2.1) for $2 \le n \le 8$.

Finally we shall deal with the equality condition of the estimate (2.1) in detail.

If the equality holds in (2.1) at a point $z = z_0 \in \Omega$ and for an n with $2 \le n \le 8$, then (2.11) and (2.15) hold, which yield

$$\Omega = \{z \in \mathbf{C} : \mathrm{Re}(Qz + R) > -1/2\}, \tag{2.17}$$

$$\Pi = \{w \in \mathbf{C} : \mathrm{Re}(\tilde{Q}w + \tilde{R}) > -1/2\}$$

with the complex constants $Q \ne 0$, $\tilde{Q} \ne 0$, R and $\tilde{R}$ given by

$$Q = \frac{\varepsilon}{\Phi'_\Omega(0)}, \quad R = -\frac{\varepsilon\Phi_\Omega(0)}{\Phi'_\Omega(0)}, \quad \tilde{Q} = \frac{\delta}{\gamma\Phi'_\Pi(0)}, \quad \tilde{R} = -\frac{\delta f(z_0)}{\gamma\Phi'_\Pi(0)}. \tag{2.18}$$

This first shows that Ω and Π are both half-planes.

Next, it follows from (2.9) that

$$f^{(n)}(z_0) = n!\frac{(\lambda_\Omega(z_0))^n}{\lambda_\Pi(f(z_0))}\sum_{k=1}^{n}\gamma\delta^{k-1}\binom{n-1}{k-1}(-\varepsilon)^{n-k} \tag{2.19}$$

with $|f^{(n)}(z_0)| = n!2^{n-1}(\lambda_\Omega(z_0))^n/\lambda_\Pi(f(z_0))$ and $2 \le n \le 8$. Hence for $2 \le n \le 8$

$$2^{n-1} = \left|\sum_{k=1}^{n}\binom{n-1}{k-1}(-\delta\bar{\varepsilon})^k\right| = |(1-\delta\bar{\varepsilon})^{n-1}|$$

which gives $\delta = -\varepsilon$. Consequently, for $z \in \Omega$, we obtain from (2.11), (2.15) and (2.18) that

$$f(z) = g \circ \Phi_\Omega^{-1}(z) = \frac{-(R+Qz)}{\tilde{Q}(1+2R+2Qz)} - \frac{\tilde{R}}{\tilde{Q}}, \tag{2.20}$$

which shows that f is a Möbius transformation mapping Ω onto Π with $f(\infty) \ne \infty$.

For the function f of (2.20), which is given in $A(\Omega, \Pi)$, where Ω and Π are given by (2.17), we have

$$f^{(n)}(z) = \frac{(-1)^n n! 2^{n-1} Q^n}{\tilde{Q}(1+2R+2Qz)^{n+1}}, \quad \lambda_\Omega(z) = \frac{|Q|}{\mathrm{Re}(1+2R+2Qz)}, \tag{2.21}$$

$$\lambda_\Pi(f(z)) = \frac{|\tilde{Q}|}{\mathrm{Re}(1+2\tilde{R}+2\tilde{Q}f(z))} = \frac{|\tilde{Q}|\,|1+2R+2Qz|^2}{\mathrm{Re}(1+2R+2Qz)}.$$

Hence we see from (2.21) that

$$|f^{(n)}(z)| = n!2^{n-1}\frac{(\lambda_\Omega(z))^n}{\lambda_\Pi(f(z))}$$

if and only if $|1+2R+2Qz| = \mathrm{Re}(1+2R+2Qz)$ or $\mathrm{Im}(1+2R+2Qz) = 0$. That is, for the function $f \in A(\Omega, \Pi)$ of (2.20) and for the Ω and Π of (2.17), the equality holds in (2.1) at each point of the half-line

(2.22) $$L = \{z \in \Omega : \mathrm{Im}(1 + 2R + 2Qz) = 0\}$$

and for each $n \geq 2$, whereas the inequality (2.1) is strict at each point of $\Omega \backslash L$ and for each $n \geq 2$. Geometrically, the half-line L intersects $\partial\Omega = \{z \in \mathbf{C} : \mathrm{Re}(1 + 2R + 2Qz) = 0\}$ perpendicularly at the point $f^{-1}(\infty)$, this shows that if the equality holds in (2.1) at a point $z = z_0 \in \Omega$, then the line segment joining z_0 and $f^{-1}(\infty)$ is perpendicular to $\partial\Omega$. Thus the conditions (1), (2) and (3) are fulfilled.

Conversely, under these conditions, the above discussion shows that the equality holds in (2.1) at the point $z = z_0 \in \Omega$. This completes the proof of Theorem 1. □

The above proof of Theorem 1 also yields the following.

Theorem 2. *Let* Ω *be a convex domain and* Π *be a simply connected domain in* $\mathbf{C}$. *If* $f(z) \in A(\Omega, \Pi)$, *then*

(2.23) $$|f^{(n)}(z)| \leq (n+1)!2^{n-2} \frac{(\lambda_\Omega(z))^n}{\lambda_\Pi(f(z))}, \quad z \in \Omega$$

holds for $2 \leq n \leq 8$. *Equality holds in* (2.23) *at a point* $z = z_0 \in \Omega$ *if and only if the following conditions are satisfied*: (1) Ω *is a half-plane and* Π *is the complex plane slit along a ray* S, (2) f *is a conformal mapping of* Ω *onto* Π *which sends* ∞ *to the tip of the ray* S, *and* (3) *the line segment joining* z_0 *and* $f^{-1}(\infty)$ *is perpendicular to* $\partial\Omega$.

Proof. With the same notation as above, we see that if Π is a simply connected domain, then the function $w = h_\Pi(\zeta)$ defined by (2.3) is a normalized univalent function. Since de Branges' celebrated proof of the Bieberbach conjecture implies the Rogosinski conjecture (see [5] [6, p. 196]), this leads to the inequalities

(2.24) $$\lambda_\Pi(w_0)|a_k| \leq k, \quad k \in \mathbf{N}$$

instead of (2.10).

Equality in (2.24) holds for all $k = 1, 2, \dots, n$ with $n \geq 2$ if and only if

(2.25) $$g(\zeta) = \alpha\Phi_\Pi'(0)\frac{\zeta}{(1-\beta\zeta)^2} + g(0), \quad \alpha, \beta \in \mathbf{C},\ |\alpha| = |\beta| = 1,\ \zeta \in \Delta.$$

This shows that f is a conformal map and Π is equal to Σ up to similarity.

It follows from (2.9), (2.14) and (2.24) that for $2 \leq n \leq 8$,

(2.26) $$\begin{aligned}\frac{|f^{(n)}(z_0)|}{n!} &\leq (\lambda_\Omega(z_0))^n \sum_{k=1}^{n} |a_k|\,|B_{n,k}| \\ &\leq \frac{(\lambda_\Omega(z_0))^n}{\lambda_\Pi(f(z_0))} \sum_{k=1}^{n} k\binom{n-1}{k-1} \\ &= \frac{(\lambda_\Omega(z_0))^n}{\lambda_\Pi(f(z_0))}(n+1)2^{n-2},\end{aligned}$$

which yields (2.23) for $2 \leq n \leq 8$.

The equality in this case gives (2.15) and (2.25). Hence the same discussion yields

$$\Omega = \{z \in \mathbf{C} : \mathrm{Re}(Qz + R) > -1/2\}, \tag{2.27}$$
$$\Pi = \{w \in \mathbf{C} : \tilde{Q}w + \tilde{R} \in \Sigma\},$$

and

$$f(z) = g \circ \Phi_\Omega^{-1}(z) = \frac{-(R + Qz)(1 + R + Qz)}{\tilde{Q}(1 + 2R + 2Qz)^2} - \frac{\tilde{R}}{\tilde{Q}}, \tag{2.28}$$

where

$$Q = \frac{\varepsilon}{\Phi'_\Omega(0)}, \quad R = -\frac{\varepsilon\Phi_\Omega(0)}{\Phi'_\Omega(0)}, \quad \tilde{Q} = \frac{\beta}{\alpha\Phi'_\Pi(0)}, \quad \tilde{R} = -\frac{\beta g(0)}{\alpha\Phi'_\Pi(0)}, \quad \beta = -\varepsilon.$$

This shows that (1) Ω is a half-plane and Π is the complex plane slit along a ray S, and (2) f is a conformal mapping of Ω onto Π which sends ∞ to the tip of the ray S.

For such $f \in A(\Omega, \Pi)$ of (2.28) and for such Ω and Π of (2.27),

$$f^{(n)}(z) = \frac{(-1)^n(n+1)!2^{n-2}Q^n}{\tilde{Q}(1 + 2R + 2Qz)^{n+2}}, \quad \lambda_\Pi(f(z)) = \frac{|\tilde{Q}|\,|1 + 2R + 2Qz|^3}{\mathrm{Re}(1 + 2R + 2Qz)}, \tag{2.29}$$

and $\lambda_\Omega(z)$ is unchanged as in (2.21). Then the equality holds in (2.23) at each point of the half-line L given by (2.22) and for each $n \geq 2$, whereas the inequality (2.23) is strict at each point of $\Omega\backslash L$ and for each $n \geq 2$.

This gives the desired conclusion in the same manner as Theorem 1. □

Note that, it follows from the above discussion on the equality condition that one can give many concrete examples. For instance, let $\Omega = \Lambda$ and let Π be the complex plane slit along a ray S_{θ_0}, where $S_{\theta_0} = \{te^{i\theta_0} : -\infty < t \leq -1/4\}$ and $\theta_0 \in \mathbf{R}$. The function $f : \Omega \to \Pi$ defined by $f(z) = e^{i\theta_0}(z^{-2} - 4)/16$ is a conformal mapping of Ω onto Π which sends ∞ to the tip $-e^{i\theta_0}/4$ of the ray S_{θ_0}. For this function, the equality holds in (2.23) for $z = x > 0$ and $n \geq 2$. However, the function $f_0 : \Omega \to \Pi$ defined by $f_0(z) = e^{i\theta_0}(z^2 - 1/4)$ is also a conformal mapping of Ω onto Π but sends ∞ to ∞. For this function $f_0(z)$, the inequality (2.23) is always strict for $z \in \Omega$ and $n \geq 2$.

From Theorems 1 and 2, we have the following corollaries which give partial solution to the conjecture of Avkhadiev-Wirths.

Corollary 1. *For any convex domains Ω and Π in $\mathbf{C}$, the assertion*

$$C_n(\Omega, \Pi) \leq 2^{n-1} \tag{2.30}$$

is valid for $2 \leq n \leq 8$.

Corollary 2. *For any convex domain Ω and any simply connected domain Π in $\mathbf{C}$, the assertion*

$$C_n(\Omega, \Pi) \le (n+1)2^{n-2} \tag{2.31}$$

is valid for $2 \le n \le 8$.

For any simply connected domains Ω and Π in $\mathbf{C}$ whose boundaries contain sectorial accessible analytic arcs, Avkhadiev and Wirths [1, Theorem 3] proved that $C_n(\Omega, \Pi) \ge 2^{n-1}$ for $n \ge 2$. This combines with Corollary 1 gives the following.

Corollary 3. *For any convex domains* Ω *and* Π *in* $\mathbf{C}$ *whose boundaries contain sectorial accessible analytic arcs, the assertion*

$$C_n(\Omega, \Pi) = 2^{n-1} \tag{2.32}$$

is valid for $2 \le n \le 8$.

3. Concluding remarks

The estimates $|B_n| \le 1$ in (2.13) provide the best way to deduce (2.14) from (2.12). However, the counterexample

$$z = h(\zeta) := \frac{5}{9}\left\{\left(\frac{1+\zeta}{1-\zeta}\right)^{9/10} - 1\right\}, \quad \zeta \in \Delta \tag{3.1}$$

with

$$\zeta = h^{-1}(z) = \sum_{n=1}^{\infty} B_n z^n$$

given by Kirwan and Schober [7] illustrates that (2.13) is not true for $n \ge 10$. Hence we cannot obtain (2.14) from (2.13) in the case when $n \ge 10$, and (2.14) is false in the case when $k = 1$ and $n \ge 10$. Since $B_{n,k}$ is a polynomial of $B_2, B_3, \dots, B_{n-k+1}$ with positive coefficients (see (2.12)), we can write

$$\begin{aligned} B_{k+j,k} &= p(B_2, B_3, \dots, B_{j+1}) \\ &= \binom{k}{1} B_{j+1} + 2\binom{k}{2} B_2 B_j + \tilde{p}(B_2, B_3, \dots, B_{j-1}) \end{aligned} \tag{3.2}$$

for each given $j = 3, 4, \dots,$ where $p(x_1, x_2, \dots, x_j)$ and $\tilde{p}(x_1, x_2, \dots, x_{j-2})$ are two multivariable polynomials with positive coefficients depending only on k. The proved inequality (2.14) is just

$$|B_{k+j,k}| \le p(1, 1, \dots, 1) \quad \text{for } j = 1, 2, \dots, 7.$$

This inequality will not hold anymore for $j \ge 9$. However, we see from (3.2) that one can deal with the remanent case by applying the following observation. That is, to prove the inequality

$$\left|\binom{k}{1} B_{j+1} + 2\binom{k}{2} B_2 B_j\right| \le \binom{k}{1} + 2\binom{k}{2}$$

instead of proving $|B_j| \leq 1$ and $|B_{j+1}| \leq 1$. For example, in the case when $j = 7$, we can obtain (2.14) from (2.12) by applying

$$\left|\binom{k}{1}B_8 + 2\binom{k}{2}B_2B_7\right| \leq \binom{k}{1} + 2\binom{k}{2} \tag{3.3}$$

and $|B_n| \leq 1$ $(n = 2, 3, \dots, 6)$. That is, the inequalities $|B_8| \leq 1$ and $|B_7| \leq 1$ can be replaced by (3.3) in the process of getting

$$|B_{k+7,k}| \leq \binom{k+6}{k-1}.$$

Whether the inequality $|B_9| \leq 1$ is true remains a question for at least two decades. The above observation may give us a way independent of the unknown inequality $|B_9| \leq 1$ to deal with (2.14) in the case when $n = k + 8$.

It should be pointed out that (2.14) does not hold for each k and $n \geq 6 + 4k$. To see this, we first note that for the function (3.1),

$$\zeta = h^{-1}(z) = \frac{(9z/5+1)^{10/9} - 1}{(9z/5+1)^{10/9} + 1} = \sum_{n=1}^{\infty} B_n z^n,$$

which yields $B_1 = 1$ and

$$B_n = \frac{1}{2}\{p_n - p_1B_{n-1} - p_2B_{n-2} - \cdots - p_{n-1}B_1\}, \quad (n \geq 2) \tag{3.4}$$

where $(1 + 9z/5)^{10/9} = 1 + \sum_{n=1}^{\infty} p_n z^n$ and

$$p_n = \frac{1}{5^n n!}\prod_{k=0}^{n-1}(10 - 9k), \quad (n \geq 1). \tag{3.5}$$

We then see that

$$\sum_{n=k+1}^{\infty} B_{n,k+1}z^n = \left(\sum_{n=k}^{\infty} B_{n,k}z^n\right)\left(\sum_{n=1}^{\infty} B_n z^n\right) \tag{3.6}$$

which gives

$$B_{n,k+1} = \sum_{j=k}^{n-1} B_{j,k}B_{n-j}, \quad (n \geq k+1, k = 1, 2, \dots) \tag{3.7}$$

Hence, with some computation, one can obtain from (3.4), (3.5) and (3.7) that

$$|B_{n,k}| > \binom{n-1}{k-1} \tag{3.8}$$

holds for each $k = 1, 2, \dots$ and $n \geq 6 + 4k$.

Even though the inequality (2.14) is not true in general, all we need is that it holds on average, namely that

$$\sum_{k=1}^{n} k|B_{n,k}| \leq \sum_{k=1}^{n} k\binom{n-1}{k-1} = (n+1)2^{n-2} \tag{3.9}$$

and

$$\sum_{k=1}^{n} |B_{n,k}| \leq \sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1} \tag{3.10}$$

hold for all $n \geq 2$. We have proved the inequalities (3.9) and (3.10) for $2 \leq n \leq 8$. We thus conjecture that (3.9) and (3.10) as well as Theorems 1 and 2 should be true for all $n \geq 2$.

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