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$L_{n/2}$-PINCHING THEOREM FOR SUBMANIFOLDS IN A SPHERE

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Abstract

Let M^n $(n \geq 2)$ be a n-dimensional oriented closed submanifolds with parallel mean curvature in $S^{n+p}(1)$, denote by S, the norm square of the second fundamental form of M. H is the constant mean curvature of M. We prove that if $\int_M S^{n/2} \leq A(n)$, where $A(n)$ is a positive universal constant, then M must be a totally umbilical hypersurface in the sphere S^{n+1}.

1. Introduction

Let M^n $(n \geq 2)$ be a n-dimensional oriented closed minimal submanifolds in the unit sphere in $S^{n+p}(1)$. We denote the square of the length of the second fundamental form by S. It is well known that if $S \leq \dfrac{n}{\dfrac{2-1}{p}}$ on M, then $S = 0$ and hence M is isometric to the unit sphere $S^n(1)$. Further discussions in this direction have been carried out by many other authors. It seems to be interesting to study the L_q-pinching condition for S. By using eigenvalue estimate, shen [5] proved the following

Theorem A. *Let $M^n \to S^{n+1}(1)$ be an oriented closed embedded minimal hypersurface with $Ric_M \geq 0$. If $\int_M S^{n/2} \leq C(n)$, where $C(n)$ is a positive universal constant, then M must be a totally geodesic hypersurface.*

Recently, Cai [1] proved the following

Theorem B. *Let M^n be a n-dimensional oriented closed submanifolds with parallel mean curvature and positive Ricci curvature in $S^{n+p}(1)$, denote by $(n-1)k$, the lower bound of Ricci curvature of M. Then there is a constant C depending*

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only on n, H, k such that if $\int_M S^{n/2} \le C(n,H,k)$, *then M is a umbilical hypersurface in the sphere* S^{n+1}.

In this paper, we delete the condition of Ricci curvature. We obtain the following

Main Theorem. *Let* M^n $(n \ge 2)$ *be a n-dimensional oriented closed submanifolds with parallel mean curvature in* $S^{n+p}(1)$. *If* $\int_M S^{n/2} \le A(n)$, *where* $A(n)$ *is a positive universal constant, then M must be a totally umbilical hypersurface in the sphere* S^{n+1}.

2. Preliminaries

Let M^n be a n-dimensional compact immersed in the $n+p$-dimensional unit sphere $S^{n+p}(1)$. We always take M to be oriented, and make use of the following convention on the range of indices:

$$1 \le A, B, C, \ldots \le n+p, 1 \le i, j, k, \ldots \le n, n+1 \le \alpha, \beta, \gamma, \ldots \le n+p$$

We choose a local field of orthogonal frame $e_1, e_2, \ldots, e_{n+p}$ in S^{n+p} such that restricted to M, the vectors $e_1, e_2, \ldots, e_n$ are tangent to M. Let ω_A and ω_{AB} be the field of dual frames and the connection 1-forms of $S^{n+p}(1)$ respectively. Restricting these forms to M, we have

$$\omega_{\alpha i} = \sum_j h_{ij}^{\alpha} \omega_j, \quad h_{ij}^{\alpha} = h_{ji}^{\alpha}$$

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_{\alpha} (h_{ik}^{\alpha} h_{jl}^{\alpha} - h_{il}^{\alpha} h_{jk}^{\alpha})$$

$$R_{\alpha\beta kl} = \sum_i (h_{ik}^{\alpha} h_{il}^{\beta} - h_{il}^{\alpha} h_{ik}^{\beta})$$

$$h = \sum_{\alpha, i, j} h_{ij}^{\alpha} \omega_i \otimes \omega_j \otimes e_{\alpha}, \quad \xi = \frac{1}{n} \sum_{\alpha, i} h_{ii}^{\alpha} e_{\alpha}$$

where R_{ijkl}, $R_{\alpha\beta kl}$, h and ξ are the curvature tensor, the normal curvature tensor, the second fundamental form and the mean curvature vector of M respectively.

We define

$$S = |h|^2, \quad H = |\xi|, \quad H_{\alpha} = (h_{ij}^{\alpha})_{n \times n}$$

M is called a submanifold with parallel mean curvature if ξ is parallel in the normal bundle of M. In particular, M is called minimal submanifold if $\xi \equiv 0$.

When $\xi \ne 0$, we choose $e_{n+1} = \dfrac{\xi}{H}$ such that $tr\, H_{n+1} = nH$ and $tr\, H_{\beta} = 0$, $n+2 \le \beta \le n+p$. The following lemmas and propositions will be used in the proof of our theorems.

LEMMA 1 [3]. *Assume $A_1, \dots A_p$ be symmetric matrix, then*

$$\sum_{\alpha,\beta=1}^{p} tr(A_\alpha A_\beta - A_\beta A_\alpha)^2 + \sum_{\alpha,\beta}^{p}(tr(A_\alpha A_\beta))^2 \le \frac{3}{2}\left(\sum_{\alpha=1}^{p} tr\ A_\alpha^2\right)^2$$

LEMMA 2 [4]. *Assume A, B be symmetric matrix. If $tr\ A = tr\ B = 0$ and $AB = BA$, then*

$$|tr(A^2B)| \le \frac{n-2}{\sqrt{n(n-1)}}\ tr(A^2)\sqrt{tr\ B^2}$$

PROPOSITION 1. *Let M^n be a submanifold with parallel mean curvature in $S^{n+p}(1)$. Denote $\sum_{i,j}(h_{ij}^{n+1})^2$ and $\sum_{i,j,\beta\neq n+1}(h_{ij}^{\beta})^2$ by σ and B respectively. Then*

$$\frac{1}{2}\Delta\sigma \ge \sum_{i,j,k}(h_{ijk}^{n+1})^2 + (\sigma - nH^2)\left[n + 2nH^2 - S - \frac{n(n-2)H}{\sqrt{n(n-1)}}\sqrt{S - nH^2}\right]$$

$$\frac{1}{2}\Delta B \ge \sum_{i,j,k,\beta\neq n+1}(h_{ijk}^{\beta})^2 + \left[n(1+H^2) - \left(\frac{n-2}{2\sqrt{n-1}} + 1\right)S + \left(\frac{n-2}{2\sqrt{n-1}} + 1 - \frac{3}{2}\right)B\right]B$$

Proof. By direct computation, we obtain

$$\begin{aligned}\frac{1}{2}\Delta\sigma &= \sum_{i,j,k}(h_{ijk}^{n+1})^2 + n\sigma - \sigma^2 - n^2H^2 + nH\ tr(H_{n+1})^3 - \sum_{\beta\neq n+1}[tr(H_{n+1}H_\beta)]^2 \\ &\ge \sum_{i,j,k}(h_{ijk}^{n+1})^2 + (\sigma - nH^2)\left[n + 2nH^2 - S - \frac{n(n-2)H}{\sqrt{n(n-1)}}\sqrt{S - nH^2}\right]\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\Delta B = &\sum_{i,j,k,\beta\neq n+1}(h_{ijk}^{\beta})^2 + nH\sum_{\beta\neq n+1} tr[H_{n+1}(H_\beta)^2] - \sum_{\beta\neq n+1}[tr(H_{n+1}H_\beta)]^2 + nB \\ &- \sum_{\alpha,\beta\neq n+1} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha,\beta\neq n+1}[tr(H_\alpha H_\beta)]^2\end{aligned}$$

$A = H_{n+1} - HI$, where I is the identity matrix, then $tr\ A = 0$, $AH_\alpha = H_\alpha A$. Because $tr\ A^2 = tr(H_{n+1}^2) - nH^2$, so using Lemma 1 and Lemma 2, we have

$$\begin{aligned}\frac{1}{2}\Delta B \ge &\sum_{i,j,k,\beta\neq n+1}(h_{ijk}^{\beta})^2 + nH^2\sum_{\beta\neq n+1} tr(H_\beta^2) - \frac{n-2}{2\sqrt{n-1}}\sum_{\beta\neq n+1} tr(H_\beta^2)\ tr(H_{n+1}^2) \\ &- \sum_{\beta\neq n+1} tr(H_{n+1}^2)\ tr(H_\beta^2) + nB - \frac{3}{2}\sum_{\beta\neq n+1} tr(H_\beta)^2\end{aligned}$$

$$\geq \sum_{i,j,k,\beta\neq n+1} (h_{ijk}^{\beta})^2 + \left[n(1+H^2) - \left(\frac{n-2}{2\sqrt{n-1}}+1\right)S + \left(\frac{n-2}{2\sqrt{n-1}}+1-\frac{3}{2}\right)B\right]B$$

PROPOSITION 2 [6]. *Let M^n be a closed submanifold in $S^{n+p}(1)$. Then for all $t \in R^+$ and $f \in C^1(M)$, $f \geq 0$, f satisfies*

$$\|\nabla f\|_2^2 \geq \frac{(n-2)^2}{4(n-1)^2(1+t)}\left[\frac{1}{C^2(n)}\|f\|_{2n/(n-2)}^2 - (1+H^2)\left(1+\frac{1}{t}\right)\|f\|_2^2\right]$$

where $C(n) = 2^n(1+n)^{(n-1)/n}(n-1)^{-1}\sigma_n^{-1/n}$ and $\sigma_n =$ volume of the unit ball in R^n.

3. Proof the Main Theorem

MAIN THEOREM. *let M^n be a n-dimensional oriented closed submaniflods with parallel mean curvature vector in $S^{n+p}(1)$. If $\int_M S^{n/2} \leq A(n)$, where $A(n)$ is a positive universal constant, then M must be a totally umbilical hypersurface in $S^{n+1}(1)$. Where*

$$A(n) = \begin{cases} \dfrac{2n\sqrt{n-1}(n+2)(n-2)^2}{C^2(n)[(n-2)+2\sqrt{n-1}][(n+2)(n-2)^2+4n^2(n-1)^2]} & n \geq 4 \\ \dfrac{2n(n+2)(n-2)^2}{3C^2(n)[(n+2)(n-2)^2+4n^2(n-1)^2]} & n \leq 3 \end{cases}$$

Proof. When $n \geq 4$, $\dfrac{n-2}{2\sqrt{n-1}}+1 > \dfrac{3}{2}$, so

$$\frac{1}{2}\Delta B \geq \sum_{i,j,k,\beta\neq n+1} (h_{ijk}^{\beta})^2 + \left[n(1+H^2) - \left(\frac{n-2}{2\sqrt{n-1}}+1\right)S\right]B$$

$$\geq \frac{n+2}{n}|\nabla\sqrt{B}|^2 + \left[n(1+H^2) - \left(\frac{n-2}{2\sqrt{n-1}}+1\right)S\right]B$$

$$0 = \int_M \Delta B \geq \frac{(n+2)(n-2)^2}{4n(n-1)^2(1+t)}\left[\frac{1}{C^2(n)}\|\sqrt{B}\|_{2n/(n-2)}^2 - (H^2+1)\left(1+\frac{1}{t}\right)\|\sqrt{B}\|_2^2\right] + n(1+H^2)\|B\|_1 - \int_M \left(\frac{n-2}{2\sqrt{n-1}}+1\right)SB$$

$$\geq \frac{(n+2)(n-2)^2}{4n(n-1)^2(1+t)}\left[\frac{1}{C^2(n)}\|B\|_{n/(n-2)} - (H^2+1)\left(1+\frac{1}{t}\right)\|B\|_1\right]$$

$$+ n(1+H^2)\|B\|_1 - \left(\frac{n-2}{2\sqrt{n-1}}+1\right)\|B\|_{n/(n-2)}\|S\|_{n/2}$$

$$= \left[\frac{(n+2)(n-2)^2}{4n(n-1)^2(1+t)}\frac{1}{C^2(n)} - \left(\frac{n-2}{2\sqrt{n-1}}+1\right)\|S\|_{n/2}\right]\|B\|_{n/(n-2)}$$

$$+ \left[n(1+H^2) - \frac{(n+2)(n-2)^2}{4n(n-1)^2(1+t)}(H^2+1)\left(1+\frac{1}{t}\right)\right]\|B\|_1$$

Let $t = \dfrac{(n+2)(n-2)^2}{4n(n-1)^2}$, using condition $\|S\|_{n/2} \leq A(n)$, we have $B = 0$. When $n \leq 3$, $\dfrac{n-2}{\sqrt{n(n-1)}} + 1 < \dfrac{3}{2}$, so

$$\frac{1}{2}\Delta B \geq \sum_{i,j,k,\beta\neq n+1}(h_{ijk}^{\beta})^2 + \left[n(1+H^2) - \frac{3}{2}S\right]B$$

using the same method, we also have $B = 0$.

Then we consider

$$\frac{1}{2}\Delta\sigma \geq \sum_{i,j,k}(h_{ijk}^{n+1})^2 + (\sigma - nH^2)\left[n + 2nH^2 - S - \frac{n(n-2)H}{\sqrt{n(n-1)}}\sqrt{S-nH^2}\right]$$

$$\geq \frac{n+2}{n}|\nabla\sqrt{\sigma - nH^2}|^2 + (\sigma - nH^2)\left[n + 2nH^2 - S - \frac{(n-2)}{2\sqrt{(n-1)}}S\right]$$

discussing as the above, we have $\sigma = nG^2$. Because $B = 0$ and $\sigma = nH^2$, we know M must be a totally umbilical hypersurface in $S^{n+1}(1)$.

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