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BOUNDS IN CAPACITY INEQUALITIES FOR TWO SHEETED SPHERES

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Abstract

Take a pair of two disjoint nonpolar compact subsets A and B of the complex plane $\mathbf{C} = \hat{\mathbf{C}} \backslash \{\infty\}$, the complex sphere less the point at infinity, with connected complement $\hat{\mathbf{C}} \backslash (A \cup B)$ and a simple arc γ in $\hat{\mathbf{C}} \backslash (A \cup B)$. We form the two sheeted covering surface $\hat{\mathbf{C}}_\gamma$ of $\hat{\mathbf{C}}$ by pasting $\hat{\mathbf{C}} \backslash \gamma$ with another copy $\hat{\mathbf{C}} \backslash \gamma$ crosswise along γ. Embed A and B in $\hat{\mathbf{C}}_\gamma$ either in the same sheet or in the different sheets and consider the variational 2-capacity $\operatorname{cap}(A, \hat{\mathbf{C}}_\gamma \backslash B)$ of A contained in the open subset $\hat{\mathbf{C}}_\gamma \backslash B$ of $\hat{\mathbf{C}}_\gamma$. Concerning the relation between the above capacity and the variational 2-capacity $\operatorname{cap}(A, \hat{\mathbf{C}} \backslash B)$ of A contained in the open subset $\hat{\mathbf{C}} \backslash B$ of $\hat{\mathbf{C}}$, we will establish the following capacity inequality for the two sheeted cover and its base:

$$0 < \operatorname{cap}(A, \hat{\mathbf{C}}_\gamma \backslash B) < 2 \cdot \operatorname{cap}(A, \hat{\mathbf{C}} \backslash B),$$

where the bound 2 in the above inequality is the best possible in the sense that, for any $0 < \tau < 2$, there is a triple of A, B, and γ such that $\operatorname{cap}(A, \hat{\mathbf{C}}_\gamma \backslash B) > \tau \cdot \operatorname{cap}(A, \hat{\mathbf{C}} \backslash B)$, where A and B may in the same sheet or in the different sheets.

1. Introduction

We have been using the notation $(R \backslash \gamma) \mathbin{\cup\!\!\!\!\times}_\gamma (S \backslash \gamma)$ for the Riemann surface constructed from the two Riemann surfaces R and S and a common simple arc γ in R and S in the sense that there is a parametric disc $V : |z| < 1$ and a simple arc γ in V included both in R and S by pasting $R \backslash \gamma$ and $S \backslash \gamma$ crosswise along γ ([11]). The arc γ is referred to as the *pasting arc* for $(R \backslash \gamma) \mathbin{\cup\!\!\!\!\times}_\gamma (S \backslash \gamma)$. We denote by $\hat{\mathbf{C}}$ the complex sphere (the Riemann sphere) and by $\mathbf{C}$ the complex plane $\hat{\mathbf{C}} \backslash \{\infty\}$ where ∞ is the point at infinity of $\mathbf{C}$. Then $\hat{\mathbf{C}}_\gamma := (\hat{\mathbf{C}} \backslash \gamma) \mathbin{\cup\!\!\!\!\times}_\gamma (\hat{\mathbf{C}} \backslash \gamma)$ or more precisely $(\hat{\mathbf{C}}_\gamma, \hat{\mathbf{C}}, \pi_\gamma)$ is a covering surface of $\hat{\mathbf{C}}$ with the natural projection $\pi = \pi_\gamma$, where γ is a simple arc in $\hat{\mathbf{C}}$. For definiteness we let $\hat{\mathbf{C}}_j$ $(j = 1, 2)$ be two copies of $\hat{\mathbf{C}}$ (so that $\hat{\mathbf{C}}_1 = \hat{\mathbf{C}}_2 = \hat{\mathbf{C}}$) and γ be a simple arc in $\hat{\mathbf{C}}$ and set

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$$\hat{\mathbf{C}}_\gamma := (\hat{\mathbf{C}}\backslash\gamma) \uplus_\gamma (\hat{\mathbf{C}}\backslash\gamma) = (\hat{\mathbf{C}}_1\backslash\gamma) \uplus_\gamma (\hat{\mathbf{C}}_2\backslash\gamma),$$

by which we can distinguish two sheets $\hat{\mathbf{C}}\backslash\gamma$ of the covering surface $(\hat{\mathbf{C}}\backslash\gamma) \uplus_\gamma (\hat{\mathbf{C}}\backslash\gamma)$ as $\hat{\mathbf{C}}_1\backslash\gamma$ and $\hat{\mathbf{C}}_2\backslash\gamma$.

Consider two disjoint nonpolar compact subsets A and B in $\mathbf{C}$ such that both of $\hat{\mathbf{C}}\backslash A$ and $\hat{\mathbf{C}}\backslash B$ and hence of course $\hat{\mathbf{C}}\backslash A \cup B$ are connected. Let the pasting arc γ of $\hat{\mathbf{C}}_\gamma$ be taken from $\hat{\mathbf{C}}\backslash A \cup B$. There are two ways to embed $A \cup B$ in $\hat{\mathbf{C}}_\gamma$: either $A \cup B \subset \hat{\mathbf{C}}_j$ or $A \subset \hat{\mathbf{C}}_i$ and $B \subset \hat{\mathbf{C}}_j$ $(i \neq j)$. In the former case we say that A and B are embedded in the same sheet of $\hat{\mathbf{C}}_\gamma$ and in the latter case we say that A and B are embedded in the different sheets of $\hat{\mathbf{C}}_\gamma$. Observe that $(\hat{\mathbf{C}}_1\backslash(A \cup B \cup \gamma)) \uplus_\gamma (\hat{\mathbf{C}}_2\backslash\gamma)$ and $(\hat{\mathbf{C}}_1\backslash\gamma) \uplus_\gamma (\hat{\mathbf{C}}_2\backslash(A \cup B \cup \gamma))$ are conformally the identical surfaces and the same is true of $(\hat{\mathbf{C}}_1\backslash(A \cup \gamma)) \uplus_\gamma (\hat{\mathbf{C}}_2\backslash(B \cup \gamma))$ and $(\hat{\mathbf{C}}_1\backslash(B \cup \gamma)) \uplus_\gamma (\hat{\mathbf{C}}_2\backslash(A \cup \gamma)$ so that $\hat{\mathbf{C}}_\gamma\backslash(A \cup B)$ has only two types: either A and B are in the same sheet or in the different sheets. Unless stated explicitly which is the case we allow for $\hat{\mathbf{C}}_\gamma\backslash(A \cup B)$ to be either one of the above two cases.

Let X be a compact subset of a Riemann surface R and U be an open subset of R such that $X \subset U$. We denote by $\operatorname{cap}(X, U)$ the capacity or more precisely the variational 2-capacity of X relative to U which is given by

$$\operatorname{cap}(X, U) := \inf_{\varphi} D(\varphi; R), \tag{1.1}$$

where φ runs over every function φ in $C^\infty(R)$ such that $\varphi|X \geqq 1$ and the support of φ is contained in U. Here $D(\varphi; R)$ is the Dirichlet integral $\int_R d\varphi \wedge *d\varphi$ of φ over R. The fact that $\operatorname{cap}(X, U) > 0$ is equivalent to that both of X and $R\backslash U$ are nonpolar. In the sequel we will restrict ourselves only to the case of $\operatorname{cap}(X, U) > 0$ so that we always assume that both of X and $R\backslash U$ are nonpolar whenever we consider $\operatorname{cap}(X, U)$. There is a unique bounded locally Sobolev function u with $D(u; R) < +\infty$ such that $u = 1$ ($u = 0$, resp.) on X ($R\backslash U$, resp.) except for a polar subset of X ($R\backslash U$, resp.) and $u\,|\,(U\backslash X)$ is harmonic and, as the most important property,

$$\operatorname{cap}(X, U) = D(u; R). \tag{1.2}$$

The function u is referred to as the *capacitary function* for $\operatorname{cap}(X, U)$ (cf. e.g. [14], [15], etc.).

In addition to the interest in its own sake it is important in connection with the type problem (cf. e.g. [13], [16], [15], [10], [11], [12], [4], [6], [9], etc.) to clarify the connection between two capacities $\operatorname{cap}(A, \hat{\mathbf{C}}_\gamma\backslash B)$ and $\operatorname{cap}(A, \hat{\mathbf{C}}\backslash B)$, which we have been studying from various view points ([5], [6], [7], [8]). The central theme is to determine the range I of the function $\gamma \mapsto \operatorname{cap}(A, \hat{\mathbf{C}}_\gamma\backslash B)$. It is known that I is an open interval $(0, c(A, B))$ with $\operatorname{cap}(A, \hat{\mathbf{C}}\backslash B) \in (0, c(A, B))$. The purpose of this paper is to show that $c(A, B) \leqq 2 \cdot \operatorname{cap}(A, \hat{\mathbf{C}}\backslash B)$ and this is, in a sense, the best possible. Namely, we will prove the following result.

THEOREM. *For any two disjoint nonpolar compact subsets A and B in the complex plane $\hat{\mathbf{C}}$ with connected complements and a simple arc γ in $\hat{\mathbf{C}}\backslash(A \cup B)$, the following capacity inequality holds*:

$$0 < \operatorname{cap}(A, \hat{\mathbf{C}}_\gamma \backslash B) < 2 \cdot \operatorname{cap}(A, \hat{\mathbf{C}} \backslash B). \tag{1.3}$$

Here the bound 2 *in the above inequality is the best possible in the sense that, for any* $0 < \tau < 2$, *there exists a triple* A, B, *and* γ *such that* $\operatorname{cap}(A, \hat{\mathbf{C}}_\gamma \backslash B)$ *is strictly greater than* $\tau \cdot \operatorname{cap}(A, \hat{\mathbf{C}} \backslash B)$, *where* A *and* B *may either in the same sheet or in the different sheets of* $\hat{\mathbf{C}}_\gamma$.

The proof of the first part of the above theorem, i.e. the proof of (1.3) will be given in §§2–4. In §2 (Regular squeezer) approximations of A and B by smooth A_n and B_n are discussed. In the next §3 (Wiener functions) the convergence of capacity functions for capacities $\operatorname{cap}(A_n, \hat{\mathbf{C}} \backslash B_n)$ and $\operatorname{cap}(A_n, \hat{\mathbf{C}}_\gamma \backslash B_n)$ to those for capacities $\operatorname{cap}(A, \hat{\mathbf{C}} \backslash B)$ and $\operatorname{cap}(A, \hat{\mathbf{C}}_\gamma \backslash B)$ are considered both in locally uniform convergence and also the convergence in the Dirichlet integrals. The results in these two sections are used in §4 (Generalized Dirichlet principle) to complete the proof of (1.3). The proof of the second part of our theorem, i.e. the proof of the best possibleness of the bound 2 in the inequality (1.3) in the sense that 2 cannot be replaced by any smaller one will be given in §§5–6. To show the best possibleness of the bound 2, our plan is to make $\operatorname{cap}(A, \hat{\mathbf{C}}_\gamma \backslash B)/\operatorname{cap}(A, \hat{\mathbf{C}} \backslash B)$ as close to 2 as possible by choosing $A := [-b, -a]$ and $B := [a, b]$ with $1 < a < b < 2$ and $\gamma := [-1, 1]$. This particular capacity $\operatorname{cap}(A, \hat{\mathbf{C}} \backslash B)$ can be estimated easily by using the Teichmüller extremal annulus, which is performed in §5 (Teichmüller annulus). To compute $\operatorname{cap}(A, \hat{\mathbf{C}}_\gamma \backslash B)$ we use the Joukowski map $J : \hat{\mathbf{C}} \to \hat{\mathbf{C}}_\gamma$ to transform the annulus $\hat{\mathbf{C}}_\gamma \backslash A \cup B$ to the same kind of annulus $\hat{\mathbf{C}} \backslash J(A) \cup J(B)$ as $\hat{\mathbf{C}} \backslash A \cup B$, which completes our plan and this is done in §6 (Joukowski mapping).

To point out the importance of generalizing our present theorem to the case of n sheeted sphere for arbitrary positive integer $n \geqq 2$, we state a conjecture (partly a theorem already), which is described in the final short §7 (Open question). The difficult part is to show the best possibleness of the bound n whose validity is entirely uncertain at present. The success of the $n = 2$ case heavily depend upon the existence of the Joukowski mapping while no counterpart to it can be expected for the general $n > 2$ case.

2. Regular squeezers

We divide the proof of Theorem into two parts: the part treating the validness of the inequality (1.3) and the part showing the best possibleness of the bound 2 in the inequality (1.3). We start from the first part. If A and B were smooth in the sense that $\hat{\mathbf{C}} \backslash A$ and $\hat{\mathbf{C}} \backslash B$ are regular subregions or we were to prove the weaker version of (1.3) that the strict inequality $<$ in (1.3) is replaced by the nonstrict one $\leqq$, then the proof would be straightforward in view of the standard Dirichlet principle. However in the present setting some kind of labor to an extent as described below may be in order. An extra work is, however, mostly the reduction to the case $\hat{\mathbf{C}} \backslash A$ and $\hat{\mathbf{C}} \backslash B$ being regular.

For a nonpolar compact subset A of $\mathbf{C}$ with the connected complement $\hat{\mathbf{C}} \backslash A$, we now consider, what we call, a regular squeezer or simply squeezer of A. A *regular squeezer*, or often more simply *squeezer*, of A is a sequence $(A_n)_{n\in\mathbf{N}}$, $\mathbf{N}$ being the set of positive integers, of compact subsets A_n of $\mathbf{C}$ satisfying the following 5 conditions: each A_n $(n\in\mathbf{N})$ is a union of a finite number of mutually disjoint closed analytic Jordan regions (where a closed analytic Jordan region is the closure of a Jordan region whose boundary Jordan curve is analytic); the interior of A_n contains A_{n+1} $(n\in\mathbf{N})$; the interior of each A_n contains A; each component of A_n $(n\in\mathbf{N})$ has a nonempty intersection with A; $\bigcap_{n\in\mathbf{N}} A_n = A$.

A sequence $(A_n)_{n\in\mathbf{N}}$ is a regular squeezer of A if and only if $(\hat{\mathbf{C}}\backslash A_n)_{n\in\mathbf{N}}$ is a regular exhaustion of $\hat{\mathbf{C}}\backslash A$, where the fourth condition in the above definition of squeezers corresponds to one of the conditions for $(\hat{\mathbf{C}}\backslash A_n)_{n\in\mathbf{N}}$ to be a regular exhaustion of $\hat{\mathbf{C}}\backslash A$ that each complement of $\hat{\mathbf{C}}\backslash A_n$ $(n\in\mathbf{N})$ (i.e. A_n) has no compact component in $\hat{\mathbf{C}}\backslash A$. Hence the conditions of a regular squeezer of A is the complete dual of the conditions of a regular exhaustion of $\hat{\mathbf{C}}\backslash A$. This last observation assures the existence of a regular squeezer of A since the existence of a regular exhaustion of $\hat{\mathbf{C}}\backslash A$ is a basic knowledge.

Now we choose an arbitrary pair of mutually disjoint nonpolar compact subsets A and B of $\mathbf{C}$ with connected complements $\hat{\mathbf{C}}\backslash A$ and $\hat{\mathbf{C}}\backslash B$ and an arbitrary simple arc γ in $\hat{\mathbf{C}}\backslash(A\cup B)$. We embed A and B in $\hat{\mathbf{C}}_\gamma$ in either in the same sheet or in the different sheets. In order to reduce the study of the relation between $\operatorname{cap}(A,\hat{\mathbf{C}}_\gamma\backslash B)$ and $\operatorname{cap}(A,\hat{\mathbf{C}}\backslash B)$ for general A and B to that for regular A and B, we take squeezers $(A_n)_{n\in\mathbf{N}}$ of A and $(B_n)_{n\in\mathbf{N}}$ of B such that $A_1\cap B_1=\emptyset$. Since $X\mapsto \operatorname{cap}(X,U)$ is increasing for compact subsets X moving in a fixed open subset U of a Riemann surface R and $U\mapsto\operatorname{cap}(X,U)$ is decreasing for open subsets U moving in R containing a fixed compact subset X (cf. e.g. [3]),

$$\begin{aligned}\operatorname{cap}(A,\hat{\mathbf{C}}_\gamma\backslash B) &\leqq \operatorname{cap}(A_n,\hat{\mathbf{C}}_\gamma\backslash B)\leqq \operatorname{cap}(A_n,\hat{\mathbf{C}}_\gamma\backslash B_n)\\ &\leqq \operatorname{cap}(A_n,\hat{\mathbf{C}}_\gamma\backslash B_m)\leqq \operatorname{cap}(A_m,\hat{\mathbf{C}}_\gamma\backslash B_m)\end{aligned}$$

for every pair of m and n in $\mathbf{N}$ with $m\leqq n$. Thus $(\operatorname{cap}(A_n,\hat{\mathbf{C}}_\gamma\backslash B_n))_{n\in\mathbf{N}}$ is a decreasing sequence with

$$\operatorname{cap}(A,\hat{\mathbf{C}}_\gamma\backslash B)\leqq \operatorname{cap}(A_n,\hat{\mathbf{C}}_\gamma\backslash B_n)\leqq \operatorname{cap}(A_n,\hat{\mathbf{C}}_\gamma\backslash B_m)$$

for every $n\geqq m$ with an arbitrarily fixed $m\in\mathbf{N}$. Hence, on making $n\uparrow\infty$ in the above displayed inequality, we deduce

$$\operatorname{cap}(A,\hat{\mathbf{C}}_\gamma\backslash B)\leqq \lim_{n\to\infty}\operatorname{cap}(A_n,\hat{\mathbf{C}}_\gamma\backslash B_n)\leqq \operatorname{cap}(A,\hat{\mathbf{C}}_\gamma\backslash B_m), \tag{2.1}$$

for any $m\in\mathbf{N}$, since $X\mapsto\operatorname{cap}(X,U)$ is right continuous in the sense that $\operatorname{cap}(X_n,U)\downarrow\operatorname{cap}(X,U)$ if $X_n\supset X_{n+1}$ $(n\in\mathbf{N})$ and $\bigcap_{n\in\mathbf{N}}X_n=X$ for a sequence $(X_n)_{n\in\mathbf{N}}$ of compact subsets X_n in R (cf. e.g. [3]). Observe that

$$\begin{gathered}\operatorname{cap}(A,\hat{\mathbf{C}}_\gamma\backslash B_m)=\operatorname{cap}(B_m,\hat{\mathbf{C}}_\gamma\backslash A)\downarrow\\ \operatorname{cap}(B,\hat{\mathbf{C}}_\gamma\backslash A)=\operatorname{cap}(A,\hat{\mathbf{C}}_\gamma\backslash B)\quad(m\to\infty),\end{gathered}$$

we deduce from (2.1) on making $m\uparrow\infty$ that

$$\lim_{n\to\infty} \operatorname{cap}(A_n, \hat{\mathbf{C}}_\gamma \backslash B_n) = \operatorname{cap}(A, \hat{\mathbf{C}}_\gamma \backslash B). \tag{2.2}$$

Similarly we see that

$$\lim_{n\to\infty} \operatorname{cap}(A_n, \hat{\mathbf{C}} \backslash B_n) = \operatorname{cap}(A, \hat{\mathbf{C}} \backslash B). \tag{2.3}$$

3. Wiener functions

For simplicity we set $V := \hat{\mathbf{C}}_\gamma \backslash (A \cup B)$ and $W := \hat{\mathbf{C}} \backslash (A \cup B)$. Similarly $V_n := \hat{\mathbf{C}}_\gamma \backslash (A_n \cup B_n)$ and $W_n := \hat{\mathbf{C}} \backslash (A_n \cup B_n)$ for every $n \in \mathbf{N}$. Let $A' := \pi^{-1}(A) \backslash A$, $B' := \pi^{-1}(B) \backslash B$, $A'_n := \pi^{-1}(A_n) \backslash A_n$, and $B'_n := \pi^{-1}(B_n) \backslash B_n$ for every $n \in \mathbf{N}$. We also need to consider $V' := V \backslash (A' \cup B') = \pi^{-1}(W)$ and $V'_n := V_n \backslash (A'_n \cup B'_n)$. Choose a function $g \in C^\infty(\hat{\mathbf{C}})$ such that $0 \leqq g \leqq 1$ on $\hat{\mathbf{C}}$, $g = 1$ on a neighborhood of A_1, and $g = 0$ on a neighborhood of $B_1 \cup \gamma \cup \{\infty\}$ and set $f := g \circ \pi$ so that $f \in C^\infty(\hat{\mathbf{C}}_\gamma)$ with $0 \leqq f \leqq 1$ on $\hat{\mathbf{C}}_\gamma$, $f = 1$ on a neighborhood of $A_1 \cup A'_1 = \pi^{-1}(A_1)$, and $f = 0$ on a neighborhood of $B_1 \cup B'_1 \cup \pi^{-1}(\gamma \cup \{\infty\}) = \pi^{-1}(B_1 \cup \gamma \cup \{\infty\})$. Clearly $D(g; \hat{\mathbf{C}}) < +\infty$ and $D(f; \hat{\mathbf{C}}_\gamma) < +\infty$ and thus g is a Dirichlet function on $\hat{\mathbf{C}}$ and f is a Dirichlet function on $\hat{\mathbf{C}}_\gamma$ (cf. e.g. [2], [15]). Therefore g is a Wiener function on W and f is a Wiener function on V and also on V' (cf. e.g. [2], [15]). Hence if we denote by e.g. H_f^V the Dirichlet solution on V with boundary values $f|\partial V$, then

$$H_f^V = \lim_{n\to\infty} H_f^{V_n} \quad \left(H_f^{V'} = \lim_{n\to\infty} H_f^{V'_n}, \text{resp.}\right) \tag{3.1}$$

locally uniformly on V (V', resp.) and similarly

$$H_g^W = \lim_{n\to\infty} H_g^{W_n} \tag{3.2}$$

locally uniformly on W. It is also clear that

$$H_f^{V'} = H_g^W \circ \pi, \quad H_f^{V'_n} = H_g^{W_n} \circ \pi \quad (n \in \mathbf{N}). \tag{3.3}$$

We extend $H_f^{V_n}$ and $H_f^{V'_n}$ to $\hat{\mathbf{C}}_\gamma$ by setting $H_f^{V_n} = 1$ on A_n and $H_f^{V_n} = 0$ on B_n and similarly $H_f^{V'_n} = 1$ on $A_n \cup A'_n$ and $H_f^{V'_n} = 0$ on $B_n \cup B'_n$. Then by the Stokes formula

$$\begin{aligned} D(H_f^{V_n} - H_f^{V_m}, H_f^{V_n}; V) &:= \int_V d(H_f^{V_n} - H_f^{V_m}) \wedge *dH_f^{V_n} \\ &= \int_{\partial V_n} (H_f^{V_n} - H_f^{V_m}) * dH_f^{V_n} = 0 \quad (n \geqq m) \end{aligned}$$

and thus $D(H_f^{V_m}, H_f^{V_n}; V) = D(H_f^{V_n}; V)$ so that

$$\begin{aligned} D(H_f^{V_n} - H_f^{V_m}; V) &= D(H_f^{V_n}; V) + D(H_f^{V_m}; V) - 2D(H_f^{V_n}, H_f^{V_m}; V) \\ &= D(H_f^{V_n}; V) + D(H_f^{V_m}; V) - 2D(H_f^{V_n}; V) \\ &= D(H_f^{V_m}; V) - D(H_f^{V_n}; V). \end{aligned}$$

Hence the sequence $(D(H_f^{V_n}; V))_{n\in\mathbf{N}}$ is a decreasing sequence and the sequence $(H_f^{V_n})_{n\in\mathbf{N}}$ is $D(\cdot; V)$-Cauchy, which assure, by the Fatou lemma and (3.1), that

$$\lim_{n\to\infty} D(H_f^{V_n} - H_f^V; V) = 0. \tag{3.4}$$

By exactly the same fashion we see that

$$\lim_{n\to\infty} D(H_f^{V_n'} - H_f^{V'}; V') = 0 \tag{3.5}$$

and

$$\lim_{n\to\infty} D(H_g^{W_n} - H_g^W; W) = 0. \tag{3.6}$$

Since the usual standard Dirichlet principle assures that $\mathrm{cap}(A_n, \hat{\mathbf{C}}_\gamma\backslash B_n) = D(H_f^{V_n}; V_n)$, we conclude also by (2.2) and (3.4) that

$$\mathrm{cap}(A, \hat{\mathbf{C}}_\gamma\backslash B) = D(H_f^{\hat{\mathbf{C}}_\gamma\backslash(A\cup B)}; \hat{\mathbf{C}}_\gamma\backslash(A\cup B)). \tag{3.7}$$

Similarly we see that

$$\mathrm{cap}(A, \hat{\mathbf{C}}\backslash B) = D(H_g^{\hat{\mathbf{C}}\backslash(A\cup B)}; \hat{\mathbf{C}}\backslash(A\cup B)). \tag{3.8}$$

4. Generalized Dirichlet principle

Our task of comparing $\mathrm{cap}(A, \hat{\mathbf{C}}_\gamma\backslash B)$ and $\mathrm{cap}(A, \hat{\mathbf{C}}\backslash B)$ has been reduced, in view of (3.7) and (3.8), to that of $D(H_f^V; V)$ $(V = \hat{\mathbf{C}}_\gamma\backslash(A\cup B))$ and $D(H_g^W; W)$ $(W = \hat{\mathbf{C}}\backslash(A\cup B))$. By (3.3) we see the following crucial relation that

$$D(H_f^{V'}; V') = D(H_{g\circ\pi}^W; \pi^{-1}(W)) = 2D(H_g^W; W).$$

Therefore (3.8) takes the form

$$2\cdot\mathrm{cap}(A, \hat{\mathbf{C}}\backslash B) = D(H_f^{\hat{\mathbf{C}}_\gamma\backslash(A\cup B\cup A'\cup B')}; \hat{\mathbf{C}}_\gamma\backslash(A\cup B\cup A'\cup B')). \tag{4.1}$$

Hence the proof of (1.3): $\mathrm{cap}(A, \hat{\mathbf{C}}_\gamma\backslash B) < 2\cdot\mathrm{cap}(A, \hat{\mathbf{C}}\backslash B)$ is reduced to

$$\begin{aligned} &D(H_f^{\hat{\mathbf{C}}_\gamma\backslash(A\cup B)}; \hat{\mathbf{C}}_\gamma\backslash(A\cup B)) \\ &\qquad < D(H_f^{\hat{\mathbf{C}}_\gamma\backslash(A\cup B\cup A'\cup B')}; \hat{\mathbf{C}}_\gamma\backslash(A\cup B\cup A'\cup B')). \end{aligned} \tag{4.2}$$

The above relation can be understood to be the assertion that, since $H_f^{\hat{\mathbf{C}}_\gamma\backslash(A\cup B)}$ is obtained from $H_f^{\hat{\mathbf{C}}_\gamma\backslash(A\cup B\cup A'\cup B')} = H_f^{(\hat{\mathbf{C}}_\gamma\backslash(A\cup B))\backslash(A'\cup B')}$ by harmonizing it on $A'\cup B'$, the Dirichlet integral of $H_f^{\hat{\mathbf{C}}_\gamma\backslash(A\cup B)}$ must be less than that of $H_f^{(\hat{\mathbf{C}}_\gamma\backslash(A\cup B))\backslash(A'\cup B')}$ by the "Dirichlet principle". However the applicability of the usual Dirichlet principle requires the continuity of $H_f^{(\hat{\mathbf{C}}_\gamma\backslash(A\cup B))\backslash(A'\cup B')}$ on $A'\cup B'$, which in general fails in the present situation, i.e. $H_f^{(\hat{\mathbf{C}}_\gamma\backslash(A\cup B))\backslash(A'\cup B')}$ can be discontinuous on some nonpolar subset of $\partial((\hat{\mathbf{C}}_\gamma\backslash(A\cup B))\backslash(A'\cup B'))\cap(A'\cup B')$. Thus a proof for (4.2) is here in order.

Now we prove (4.2): $D(H_f^V; V) < D(H_f^{V'}; V')$ $(V' = V \backslash (A' \cup B') = (\hat{\mathbf{C}}_\gamma \backslash (A \cup B)) \backslash (A' \cup B'))$. Recall that $V_n = \hat{\mathbf{C}}_\gamma \backslash (A_n \cup B_n)$ exhausts V and $V_n' = V_n \backslash (A_n' \cup B_n')$ exhausts V' as $n \to \infty$. Observe that, by the Stokes formula,

$$\begin{aligned} D(H_f^{V_n'}, H_f^{V_n}; V_n') &= \int_{\partial V_n} H_f^{V_n'} * dH_f^{V_n} + \int_{\partial V_n' \backslash \partial V_n} H_f^{V_n'} * dH_f^{V_n} \\ &= \int_{\partial V_n} H_f^{V_n} * dH_f^{V_n} + \int_{(\partial V_n' \backslash \partial V_n) \cap A_n'} * dH_f^{V_n} \\ &= D(H_f^{V_n}; V_n) + \int_{A_n'} d(*dH_f^{V_n}) = D(H_f^{V_n}; V_n) \end{aligned}$$

since $H_f^{V_n'} \,|\, (\partial V_n) \cap A_n' = 1 = H_f^{V_n} \,|\, (\partial V_n) \cap A_n'$ and $H_f^{V_n'} \,|\, (\partial V_n) \cap B_n' = 0 = H_f^{V_n} \,|\, (\partial V_n) \cap B_n'$ and $H_f^{V_n}$ is harmonic on A_n'. Thus

$$\begin{aligned} D(H_f^{V_n'} - H_f^{V_n}; V_n') &= D(H_f^{V_n'}; V_n') + D(H_f^{V_n}; V_n') - 2D(H_f^{V_n'}, H_f^{V_n}; V_n') \\ &= D(H_f^{V_n'}; V_n') + D(H_f^{V_n}; V_n') - 2D(H_f^{V_n}; V_n) \\ &\leqq D(H_f^{V_n'}; V_n') + D(H_f^{V_n}; V_n) - 2D(H_f^{V_n}; V_n) \end{aligned}$$

or

$$D(H_f^{V_n}; V_n) + D(H_f^{V_n'} - H_f^{V_n}; V_n') \leqq D(H_f^{V_n'}; V')$$

for every $n \in \mathbf{N}$. Hence on making $n \to \infty$ in these inequalities we see that

$$D(H_f^V; V) + D(H_f^{V'} - H_f^V; V') \leqq D(H_f^{V'}; V'), \tag{4.3}$$

where $V = \hat{\mathbf{C}}_\gamma \backslash (A \cup B)$ and $V' = \hat{\mathbf{C}}_\gamma \backslash (A \cup B \cup A' \cup B')$. Then (4.3) implies (4.2) if

$$D(H_f^{V'} - H_f^V; V') > 0 \tag{4.4}$$

is valid. Contrary to the assertion assume that $D(H_f^{V'} - H_f^V; V') = 0$. First of all this assures that $H_f^{V'} - H_f^V$ is a constant on V'. Since the boundary values of $H_f^{V'} - H_f^V$ at ∂V are $f - f = 0$ except for the subset of irregular points of ∂V which is a polar set. Hence we must conclude that $H_f^{V'} = H_f^V$ identically on V'. However H_f^V is harmonic in a neighborhood of $A' \cup B'$ and hence $0 < H_f^V < 1$ there and in particular on $\partial V' \backslash \partial V$. On the other hand the boundary values of $H_f^{V'}$ which is of course identical with H_f^V is either 0 or 1 at $\partial V' \backslash \partial V$ except for its polar subset. This is clearly a contradiction and the proof of (1.3) is herewith complete.

5. Teichmüller annulus

An *annulus* Y in $\hat{\mathbf{C}}$ is a doubly connected subregion $Y := \hat{\mathbf{C}} \backslash (F_1 \cup F_2)$ of $\hat{\mathbf{C}}$, where F_1 and F_2 are disjoint closed subsets of $\hat{\mathbf{C}}$ such that each of $\hat{\mathbf{C}} \backslash F_j$ $(j = 1, 2)$

is connected and hence a subregion of $\hat{\mathbf{C}}$. In particular, if $F_1 = \{|z-a| \leqq m\}$ and $F_2 = \{|z-a| \geqq M\} \cup \{\infty\}$ with $a \in \mathbf{C}$ fixed and $0 \leqq m < M \leqq +\infty$, then $Y = \hat{\mathbf{C}} \backslash (F_1 \cup F_2) = \{m < |z-a| < M\}$ is said to be a circular annulus. In this paper we only consider nondegenerate annulus in the sense that F_1 and F_2 are nondegenerate, i.e. not single point sets, so that we simply say annuli meaning nondegenerate annuli. Hence in the case of a circular annulus $\{m < |z-a| < M\}$ the inner radius m and the outer radius M are assumed to satisfy $0 < m < M < +\infty$. Any annulus Y is conformally a circular annulus $\{1 < |z| < M\}$ and the conformal invariant $\log M$ associated with Y is referred to as the *modulus* of Y and denoted by mod Y. Probably the present definition is adopted more frequently than to use $(1/2\pi)$ mod Y as the modulus of Y which is but not too rarely used. Anyway, if $Y = \hat{\mathbf{C}} \backslash (F_1 \cup F_2)$ is any annulus, then Y is conformally equivalent to the circular annulus $\{1 < |z| < \exp(\text{mod } Y)\}$ and thus we see that

$$\operatorname{cap}(F_1, \hat{\mathbf{C}} \backslash F_2) = 2\pi / \operatorname{mod}(\hat{\mathbf{C}} \backslash (F_1 \cup F_2)). \tag{5.1}$$

Based on the Teichmüller theorem that among annuli separating the pair $\{-1, 0\}$ from the pair $\{w, \infty\}$ with $\infty > |w| = R > 0$ the annulus

$$T(R) := \hat{\mathbf{C}} \backslash ([-1, 0] \cup [R, +\infty]) \tag{5.2}$$

has the greatest modulus, where $[-1, 0]$ is the interval (the line segment) $\{x \in \mathbf{R} : -1 \leqq x \leqq 0\}$ and $[R, +\infty]$ is the half straight line segment $\{x \in \mathbf{R} : x \geqq R\} \cup \{\infty\}$ with $\mathbf{R}$ the real line in $\mathbf{C}$. The above extremal annulus $T(R)$ is referred to as the *Teichmüller annulus* with index R. We need to know the concrete value of mod $T(R)$ for our purpose. The following estimate (cf. e.g. [1]) is handy to use:

$$\log 16 + \log R \leqq \operatorname{mod} T(R) \leqq \log 16 + \log(R+1). \tag{5.3}$$

The estimate is getting better and better as R is becoming larger and larger. To obtain a good estimate for small R, noting $1/R$ is large for small R, we use the relation $(\operatorname{mod} T(R)) \cdot (\operatorname{mod} T(1/R)) = \pi^2$ reducing the estimate of mod $T(R)$ to that of mod $T(1/R)$ with large $1/R$.

Taking four real numbers $a_j \in \mathbf{R}$ $(1 \leqq j \leqq 4)$ satisfying $-\infty < a_1 < a_2 < a_3 < a_4 < +\infty$, we consider the annulus $\hat{\mathbf{C}} \backslash [a_1, a_2] \cup [a_3, a_4]$ and we wish to evaluate its modulus $\operatorname{mod}(\hat{\mathbf{C}} \backslash [a_1, a_2] \cup [a_3, a_4])$. We introduce the number $R = R(a_1, a_2, a_3, a_4)$ defined by

$$R = R(a_1, a_2, a_3, a_4) := \frac{a_4 - a_1}{a_4 - a_3} \cdot \frac{a_3 - a_2}{a_2 - a_1}. \tag{5.4}$$

Consider the Möbius transformation $S : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ given by

$$S(z) := -\frac{a_1 - a_4}{a_1 - a_2} \cdot \frac{z - a_2}{z - a_4}.$$

Observe that $S(a_1) = -1$, $S(a_2) = 0$, $S(a_3) = R = R(a_1, a_2, a_3, a_4)$, and $S(a_4) = \infty$. Hence S maps the annulus $\hat{\mathbf{C}} \backslash [a_1, a_2] \cup [a_3, a_4]$ conformally onto the Teichmüller annulus $T(R)$ with index $R = R(a_1, a_2, a_3, a_4)$. Therefore we deduce

$$(5.5)\qquad \log 16+\log R \leqq \operatorname{mod}(\hat{\mathbf{C}}\backslash[a_1,a_2]\cup[a_3,a_4]) \leqq \log 16+\log(R+1),$$

where $R=R(a_1,a_2,a_3,a_4)$ is given by (5.4). Using (5.1) we have

$$(5.6)\qquad \frac{2\pi}{\log 16+\log(R+1)} \leqq \operatorname{cap}([a_1,a_2],\hat{\mathbf{C}}\backslash[a_3,a_4]) \leqq \frac{2\pi}{\log 16+\log R}.$$

Next we specialize the above situation. Choose two arbitrary real numbers a and b with $1<a<b<2$ and let $a_1=-b$, $a_2=-a$, $a_3=a$, and $a_4=b$. Then we have

$$(5.7)\qquad R(a,b):=R(-b,-a,a,b)=\frac{4ab}{(b-a)^2}$$

and (5.6) takes the form

$$\frac{2\pi}{\log 16+\log(R(a,b)+1)} \leqq \operatorname{cap}([-b,-a],\hat{\mathbf{C}}\backslash[a,b]) \leqq \frac{2\pi}{\log 16+\log R(a,b)}.$$

Since $R(a,b)>(2/(b-a))^2$ in view of $b>a>1$ and $R(a,b)+1=(b+a)^2/(b-a)^2<(4/(b-a))^2$ by virtue of $a<b<2$, the above displayed inequalities yield

$$(5.8)\qquad \frac{\pi}{\log 16-\log(b-a)} \leqq \operatorname{cap}([-b,-a],\hat{\mathbf{C}}\backslash[a,b]) \leqq \frac{\pi}{\log 8-\log(b-a)}.$$

Now observe that two compact subsets A and B in $\hat{\mathbf{C}}$ given by

$$(5.9)\qquad A=A(a,b):=[-b,-a]\quad B=B(a,b):=[a,b]$$

are disjoint nonpolar compact subsets of $\hat{\mathbf{C}}$ and both of $\hat{\mathbf{C}}\backslash A$ and $\hat{\mathbf{C}}\backslash B$ are connected. Then (5.8) says that

$$(5.10)\qquad \frac{\pi}{\log 16-\log(b-a)} \leqq \operatorname{cap}(A(a,b),\hat{\mathbf{C}}\backslash B(a,b)) \leqq \frac{\pi}{\log 8-\log(b-a)}.$$

with $1<a<b<2$.

6. Joukowski mapping

Consider the quantity

$$(6.1)\qquad \sigma_j := \sup_{(A,B)}\left(\sup_{\gamma\subset\hat{\mathbf{C}}\backslash A\cup B}\frac{\operatorname{cap}(A,\hat{\mathbf{C}}_\gamma\backslash B)}{\operatorname{cap}(A,\hat{\mathbf{C}}\backslash B)}\right)\quad (j=1,2),$$

where (A,B) runs over every pair of two disjoint nonpolar compact subsets A and B in $\mathbf{C}$ with connected $\hat{\mathbf{C}}\backslash A$ and $\mathbf{C}\backslash B$ and γ runs over every simple arc $\gamma\subset\hat{\mathbf{C}}\backslash A\cup B$ and moreover A and B are embedded in the same sheet of $\hat{\mathbf{C}}_\gamma$ $(j=1)$ or in the different sheets of $\hat{\mathbf{C}}_\gamma$ $(j=2)$. We have seen in §§2–4 that $\sigma_j\leqq 2$ $(j=1,2)$. The best possibleness of the bound 2 in (1.3) thus means that $\sigma_j=2$ $(j=1,2)$ and it suffices to show that $\sigma_j\geqq 2$ $(j=1,2)$.

We now watch the particular pair $(A(a,b), B(a,b))$ given in (5.9) of two disjoint nonpolar compact subsets $A(a,b)$ and $B(a,b)$ in $\mathbf{C}$ with connected complements $\hat{\mathbf{C}}\backslash A(a,b)$ and $\hat{\mathbf{C}}\backslash B(a,b)$ and the particular pasting arc $\gamma = [-1,1] \subset \hat{\mathbf{C}}\backslash A(a,b) \cup B(a,b)$ $(1 < a < b < 2)$. Then

$$\frac{\operatorname{cap}(A(a,b), \hat{\mathbf{C}}_{[-1,1]}\backslash B(a,b))}{\operatorname{cap}(A(a,b), \hat{\mathbf{C}}\backslash B(a,b))} \leqq \sigma_j \quad (j = 1,2) \tag{6.2}$$

for every $1 < a < b < 2$ and we will show that the term on the left hand side of the above inequality tends to 2 as $a \downarrow 1$ first and then $b \downarrow 1$. The estimation of the term we are presently observing is already evaluated by (5.8). The estimation of the numerator of the same term is not as straightforward as that of the denominator above. This will be done below.

Observe that the *Joukowski mapping*

$$w = J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) \tag{6.3}$$

maps the sphere $\hat{\mathbf{C}}$ conformally onto the two sheeted sphere

$$\hat{\mathbf{C}}_{[-1,1]} = (\hat{\mathbf{C}}_1\backslash[-1,1]) \uplus_{[-1,1]} (\hat{\mathbf{C}}_2\backslash[-1,1]).$$

We assume that J^{-1} maps $\hat{\mathbf{C}}_1\backslash[-1,1]$ conformally onto $\{|z| < 1\} \subset \hat{\mathbf{C}}$ and $\hat{\mathbf{C}}_2\backslash[-1,1]$ to $\{1 < |z| \leqq +\infty\} \subset \hat{\mathbf{C}}$. We wish to express the annulus $\hat{\mathbf{C}}\backslash J^{-1}([-b,-a] \cup [a,b])$ as concretely as possible, where we note that $[-b,-a] = A(a,b)$ and $[a,b] = B(a,b)$. For the purpose we must treat two cases separately: the case $[-b,-a]$ and $[a,b]$ are embedded in the same sheet of $\hat{\mathbf{C}}_{[-1,1]}$; the case $[-b,-a]$ and $[a,b]$ are embedded in the different sheets of $\hat{\mathbf{C}}_{[-1,1]}$.

6.1. *The case $[-b,-a]$ and $[a,b]$ are in the same sheet of $\hat{\mathbf{C}}_{[-1,1]}$.* We may assume that $[-b,-a]$ and $[a,b]$ are on $\hat{\mathbf{C}}_2\backslash[-1,1]$. By (6.3) we see that

$$J^{-1}[-b,-a] = [-b - \sqrt{b^2-1}, -a - \sqrt{a^2-1}] =: [-b', -a']$$

and

$$J^{-1}[a,b] = [a + \sqrt{a^2-1}, b + \sqrt{b^2-1}] =: [a', b'].$$

Here we have $1 < a' < b' < 2$ by taking $1 < a < b < 2$ close enough to 1. Then the annulus $\hat{\mathbf{C}}_{[-1,1]}\backslash[-b,-a] \cup [a,b]$ is mapped by J^{-1} conformally onto the annulus $\hat{\mathbf{C}}\backslash[-b',-a'] \cup [a',b']$ and hence by (5.10) and the conformal invariance of capacities we have

$$\begin{aligned} \frac{\pi}{\log 16 - \log(b'-a')} &\leqq \operatorname{cap}(A(a,b), \hat{\mathbf{C}}_{[-1,1]}\backslash B(a,b)) \\ &\leqq \frac{\pi}{\log 8 - \log(b'-a')}, \end{aligned} \tag{6.1.1}$$

where $1 < a < b < 2$ are chosen close enough to 1 so as to have $1 < a' < b' < 2$ and $A(a,b) = [-b,-a]$ and $B(a,b) = [a,b]$. By using the inequality on the most right hand side of (5.10) and the inequality on the most left hand side of (6.1.1) we see that

$$\sigma_1 \geqq \frac{\operatorname{cap}([-b,-a],\hat{\mathbf{C}}_{[-1,1]}\backslash[a,b])}{\operatorname{cap}([-b,-a],\hat{\mathbf{C}}\backslash[a,b])} \geqq \frac{\pi/(\log 16-\log(b'-a'))}{\pi/(\log 8-\log(b-a))},$$

which is true for every $1<a<b<2$ sufficiently close to 1. On letting $a \downarrow 1$ (so that $a' \downarrow 1$) in the above displayed inequality we obtain by noting $b'-a' \to b-1+\sqrt{b^2-1}$ that

$$\frac{\log 8-\log(b-1)}{\log 16-\log(b-1+\sqrt{b^2-1})} \leqq \sigma_1.$$

Since $b-1+\sqrt{b^2-1}=\sqrt{b-1}(\sqrt{b-1}+\sqrt{b+1})$ and hence $\log(b-1+\sqrt{b^2-1})=(1/2)\log(b-1)+\log(\sqrt{b-1}+\sqrt{b+1})$, the above displayed inequality implies that

$$\frac{\log 8/\log(b-1)-1}{\log 16/\log(b-1)-1/2-\log(\sqrt{b-1}+\sqrt{b+1})/\log(b-1)} \leqq \sigma_1$$

for every $1<b<2$ sufficiently close to 1. On making $b \downarrow 1$ in the above inequality we obtain $2 \leqq \sigma_1$, as desired.

6.2. *The case* $[-b,-a]$ *and* $[a,b]$ *are in the different sheets of* $\hat{\mathbf{C}}_\gamma$. We may assume that $[-b,-a]$ is on $\hat{\mathbf{C}}_1\backslash[-1,1]$ and $[a,b]$ is on $\hat{\mathbf{C}}_2\backslash[-1,1]$. By (6.3) we see that

$$J^{-1}[-b,-a]=[-a+\sqrt{a^2-1},-b+\sqrt{b^2-1}]$$

and

$$J^{-1}[a,b]=[a+\sqrt{a^2-1},b+\sqrt{b^2-1}].$$

Thus the annulus $\hat{\mathbf{C}}_{[-1,1]}\backslash[-b,-a]\cup[a,b]$ is mapped by J^{-1} conformally onto the annulus $\hat{\mathbf{C}}\backslash[-a+\sqrt{a^2-1},-b+\sqrt{b^2-1}]\cup[a+\sqrt{a^2-1},b+\sqrt{b^2-1}]$. Back to the original observation we set $a_1=-a+\sqrt{a^2-1}$, $a_2=-b+\sqrt{b^2-1}$, $a_3=a+\sqrt{a^2-1}$, $a_4=b+\sqrt{b^2-1}$ and therefore we have

$$\begin{aligned}R&=R(a_1,a_2,a_3,a_4)\\&=\frac{(b+\sqrt{b^2-1})-(-a+\sqrt{a^2-1})}{(b+\sqrt{b^2-1})-(a+\sqrt{a^2-1})}\cdot\frac{(a+\sqrt{a^2-1})-(-b+\sqrt{b^2-1})}{(-b+\sqrt{b^2-1})-(-a+\sqrt{a^2-1})}\\&=\frac{ab+1+\sqrt{a^2-1}\sqrt{b^2-1}}{ab-1-\sqrt{a^2-1}\sqrt{b^2-1}}\end{aligned}$$

and

$$R+1 = \frac{(ab+1+\sqrt{a^2-1}\sqrt{b^2-1}) + (ab-1-\sqrt{a^2-1}\sqrt{b^2-1})}{ab-1-\sqrt{a^2-1}\sqrt{b^2-1}}$$
$$= \frac{2ab}{ab-1-\sqrt{a^2-1}\sqrt{b^2-1}}$$

and a fortiori we see, by (5.6), that

$$\text{(6.2.1)} \qquad \frac{2\pi}{\log 16 + \log(R+1)} \leqq \operatorname{cap}(A(a,b), \hat{\mathbf{C}}_{[-1,1]} \backslash B(a,b)) \leqq \frac{2\pi}{\log 16 + \log R},$$

where $A(a,b) = [-b,-a]$, $B(a,b) = [a,b]$, and

$$R + s = R(-a+\sqrt{a^2-1}, -b+\sqrt{b^2-1}, a+\sqrt{a^2-1}, b+\sqrt{b^2-1}) + s,$$

with $s = 0$ and 1, are computed above. From the inequality on the right most hand side of (5.10) and that on the left most hand side of (6.2.1) above, it follows that

$$\sigma_2 \geqq \frac{\operatorname{cap}([-b,-a], \hat{\mathbf{C}}_{[-1,1]} \backslash [a,b])}{\operatorname{cap}([-b,-a], \hat{\mathbf{C}} \backslash [a,b])}$$
$$\geqq \frac{2\pi \Big/ \left(\log 16 + \log \dfrac{2ab}{ab-1-\sqrt{a^2-1}\sqrt{b^2-1}} \right)}{\pi/(\log 8 - \log(b-a))}.$$

This is true for every $1 < a < b < 2$ sufficiently close to 1. Firstly on making $a \downarrow 1$ in the above displayed inequality we derive

$$\sigma_2 \geqq 2 \cdot \frac{\log 8 - \log(b-1)}{\log 16 + \log \dfrac{2b}{b-1}} = 2 \cdot \frac{\log 8 - \log(b-1)}{\log 16 + \log 2b - \log(b-1)}$$

valid for every $1 < b < 2$ enough close to 1. On making $b \downarrow 1$ in the above displayed inequality we deduce $\sigma_2 \geqq 2$, as desired.

7. Open question

Choose an arbitrary $n \in \mathbf{N}$ with $n \geqq 2$. Let $(\hat{\mathbf{C}}_k)_{1 \leqq k \leqq n}$ be the sequence of replicas $\hat{\mathbf{C}}_k = \hat{\mathbf{C}}$ of $\hat{\mathbf{C}}$ $(1 \leqq k \leqq n)$, A and B be two disjoint nonpolar compact subsets of $\mathbf{C}$ with connected complements $\hat{\mathbf{C}} \backslash A$ and $\hat{\mathbf{C}} \backslash B$, and $\Gamma = (\gamma_k)_{1 \leqq k \leqq n-1}$ be a sequence of simple arcs $\gamma_k \subset \hat{\mathbf{C}} \backslash (A \cup B)$ $(1 \leqq k \leqq n-1)$ such that $\gamma_k \cap \gamma_{k+1} = \emptyset$ $(1 \leqq k \leqq n-1)$ with $\gamma_0 = \emptyset$. Paste $\hat{\mathbf{C}}_1 \backslash \gamma_1$ to $\hat{\mathbf{C}}_2 \backslash \gamma_1 \cup \gamma_2$ crosswise along γ_1 to produce $(\hat{\mathbf{C}}_1 \backslash \gamma_1) \uplus_{\gamma_1} (\hat{\mathbf{C}}_2 \backslash \gamma_1 \cup \gamma_2)$, which is pasted to $\hat{\mathbf{C}}_3 \backslash \gamma_2 \cup \gamma_3$ crosswise along γ_2 to produce

$$((\hat{\mathbf{C}}_1 \backslash \gamma_1) \uplus_{\gamma_1} (\hat{\mathbf{C}}_2 \backslash \gamma_1 \cup \gamma_2)) \uplus_{\gamma_2} (\hat{\mathbf{C}}_3 \backslash \gamma_2 \cup \gamma_3).$$

Repeating this process we obtain $\hat{\mathbf{C}}_\Gamma$ as

$$(\cdots(((\hat{\mathbf{C}}_1\backslash\gamma_1) \uplus_{\gamma_1} (\hat{\mathbf{C}}_2\backslash\gamma_1 \cup \gamma_2)) \uplus_{\gamma_2} (\hat{\mathbf{C}}_3\backslash\gamma_2 \cup \gamma_3))\cdots) \uplus_{\gamma_{n-1}} (\hat{\mathbf{C}}_n\backslash\gamma_{n-1}),$$

which is an n sheeted covering surface of $\hat{\mathbf{C}}$. Embed A and B to $\hat{\mathbf{C}}_\Gamma$ either in the same sheet $\hat{\mathbf{C}}_i$ or in the different sheets $\hat{\mathbf{C}}_i$ and $\hat{\mathbf{C}}_j$ $(i \neq j)$ of $\hat{\mathbf{C}}_\Gamma$. We have proved the following conjecture in the case $n = 2$ as our main theorem of this paper stated in §1.

Conjecture. *For the n sheeted covering surface* $\hat{\mathbf{C}}_\Gamma$ *of* $\hat{\mathbf{C}}$ $(n \in \mathbf{N}, n \geq 2)$ *as constructed above the following inequality*

$$\operatorname{cap}(A, \hat{\mathbf{C}}_\Gamma\backslash B) < n \cdot \operatorname{cap}(A, \hat{\mathbf{C}}\backslash B) \tag{7.1}$$

is valid. The bound n in the above inequality is the best possible in the sense that for any $0 < \tau < n$ *there is a triple A, B, and* $\Gamma = (\gamma_k)_{1 \leqq k \leqq n-1}$ *such that* $\operatorname{cap}(A, \hat{\mathbf{C}}_\Gamma\backslash B)$ *is strictly greater than* $\tau \cdot \operatorname{cap}(A, \hat{\mathbf{C}}\backslash B)$, *where A and B may either in the arbitrarily chosen same sheet or in the arbitrarily chosen different sheets of* $\hat{\mathbf{C}}_\Gamma$.

Actually, by mimicking the proof of (1.3) we can prove (7.1) for every $n \geqq 2$ without any further elaboration beyond the notational complexity. Hence the question is whether the second part of the above conjecture about the best possibleness of the bound n is true or not.

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