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ON A KOBAYASHI HYPERBOLIC MANIFOLD N MODULO A CLOSED SUBSET Δ_N AND ITS APPLICATIONS

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Abstract

We show that the degeneration locus of the Kobayashi pseudodistance on a complex manifold is always a pseudoconcave set of order 1. We give some results cocerning the degeneration locus of the Kobayashi pseudodistance. Next we prove a generalization of the little Picard theorem relevantly. Finally, we consider the case $N = \Delta_N$.

0. Introduction

We introduced the degeneration locus $S_M(X)$ of the Kobayashi pseudodistance on a complex manifold M in some complex manifold X in [3] and we proved that $S_M(X)$ is a pseudoconcave set of order 1 in X. By using this results, we generalized the big Picard theorem in [1] and Montel's theorem in [2] of a two dimensional case. In this paper, we study the degeneration locus of a complex manifold N and modify some results concerning it in [7, Chaper 3-2]. For example, Theorems 1.12, 1.13, 2.3, 2.5 and Corollary 2.6. Next we study an example of hyperbolic manifold modulo a closed subset Δ_N (Theorem 3.8) and prove Proposition 4.2 and Theorem 4.4 which are types of the little Picard theorem. In the last section, we study examples such that $\Delta_N = N$.

1. Degeneration locus of the Kobayashi pseudodistance on a manifold N

In what follows, we call a manifold if it is a connected complex one. Let N be a manifold of dimension n $(n \geq 2)$ and d_N the Kobayashi pseudodistance on N. For its definition, see [7, p. 50].

Definition 1.1 (cf. [7]). We denote that

$$\Delta_N = \{p \in N; d_N(p,q) = 0 \text{ for some } q \in N \text{ such as } q \neq p\}$$

and for $p \in \Delta_N$

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$$\Delta_N(p) = \{q \in N; d_N(p,q) = 0\}.$$

By the same reason in the proof of Lemma 1.2 in [1], the following two propositions are proved by the Schwarz lemma essentially.

Proposition 1.2. *If $p \in \Delta_N$ and every closed coordinate neighborhood $\overline{U}(p)$ of N which is biholomorphic to the closed unit ball in $\mathbf{C}^n$, then there is a point $q \in \partial\overline{U}$ such that $d_N(p,q) = 0$.*

Proposition 1.3. *If $q \in \Delta_N(p)$ and every closed coordinate neighborhood $\overline{U}(q)$ of N which is biholomorphic to the closed unit ball in $\mathbf{C}^n$, then there is a point $r \in \partial\overline{U}$ such that $d_N(p,r) = 0$.*

Since $d_N : N \times N \to \mathbf{R}$ is a continuous function (see Proposition (3.1.13) in [7]), we have the following propositions:

Proposition 1.4. *The set $\Delta_N(p)$ is a closed set of N.*

Proposition 1.5 (cf. Proposition 1.3 in [1]). *The set Δ_N is a closed set of N.*

Definition 1.6 (cf. [10] and [4]). A closed subset E of N will be called a pseudoconcave set of order 1, if for any coordinate neighborhood

$$U : |z_1| < 1, \dots, |z_n| < 1$$

of N and any positive number r, s with $0 < r, s < 1$ such that $U^* \cap E = \emptyset$, one obtains $U \cap E = \emptyset$, where

$$U^* = \{p \in U; |z_1(p)| \le r\} \cup \{p \in U; s \le \max_{2 \le i \le n} |z_i(p)|\}.$$

Remark 1.7. In the case where dimension of N equals 2, every pseudoconcave set of order 1 is a pseudoconcave set, that is, the complement of a pseudoconvex set.

Proposition 1.8 ([10], pp. 282–286). *The set E of N is a pseudoconcave set order* 1, *if and only if, for every point $p \in E$, for every coordinate neighborhood $|z_1| < 1, \dots, |z_n| < 1$ such that p corresponds to the origin $(0, \dots, 0)$ and $\{z_1 = 0\} \cap E = \{(0, \dots, 0)\}$, and for every ξ_1 with $|\xi_1| < \rho$, there are ξ_i with $|\xi_i| < r$ $(i = 2, \dots, n)$ such that $(\xi_1, \xi_2, \dots, \xi_n) \in E$ for every $1 > r > 0$ and for some sufficientry small $\rho > 0$.*

By Einbettungszats in [9] and Proposition 1.8, it is easy to see the following:

Proposition 1.9 ([10], p. 282). *An analytic curve S of N, that is, an analytic subset of pure dimension* 1 *of N, is a pseudoconcave set of order* 1.

By Theorem IV in [10] and Proposition 1.8, it is easy to see the following

Proposition 1.10. *If the nonempty set E is a pseudoconcave set of order* 1 *of N and is contained in an analytic curve S of N, then E consists of some irreducible components of S.*

Proposition 1.11 (Lenmma of T. Ueda in [5]). *The subset E of N is a pseudoconcave set of order* 1, *if and only if, for every point $p \in E$ and every strictly plurisubharmonic function φ with $\varphi(p) = 0$, $E \cap \{q \in U; \varphi(q) > 0\} \neq \emptyset$, where U is a coordinate neighborhood of p.*

Theorem 1.12. *The set Δ_N is a pseudoconcave set of order* 1 *of N.*

Proof. It is easy to see that for every point $p \in \Delta_N$, Theorem 1_U in [3] holds good. By Proposition 1.11, our theorem is proved by the same method in the proof of Theorem 2 in [3]. □

According to the same method of the proof of the above theorem, the following theorem is proved.

Theorem 1.13. *The set $\Delta_N(p)$ is a pseudoconcave set of order* 1 *of N.*

By the triangle inequality, the following proposition is proved easily.

Proposition 1.14. *For every points $q, r \in \Delta_N(p)$, $d_N(q, r) = 0$.*

Remark 1.15. In general, there is no nonconstant holomorphic map $\varphi : \mathbf{C} \to \Delta_N(p)$. For example, there is a manifold N (which is not Stein) such as an Example (3.6.6) in [7, p. 104] which is not hyperbolic, that is, $\Delta_N(p) \neq \emptyset$ and there is no nonconstant holomorphic map $\varphi : \mathbf{C} \to N$, that is, Brody hyperbolic.

Remark 1.16. The set Δ_N is not always an analytic curve. The following example N is such a manifold. Let $N = \{(x, y) \in \mathbf{C} \times \Delta(1); |x| < e^{-\varphi(y)}\}$ where $\Delta(1) = \{y \in \mathbf{C}; |y| < 1\}$, $\varphi(y)$ is a subharmonic function on $\Delta(1)$ such that $\{\varphi(y) = -\infty\} = \{y = a_i\}$ where $\{a_i\}_{i=1,2,\dots}$ are discrete points converging to $\{y = 0\}$ and $\varphi(y)$ is continuous elsewhere of $\{a_i\}$ (For constructing φ, see Example (3.1.26) in [7]). It is easy to see that $\{y = a_i\} \subset \Delta_N$. For $a \notin \{a_i\}$ there is a small neighborhood $U(a)$ which does not contain the points $\{a_i\}$. Since $(\mathbf{C} \times U(a)) \cap N$ is a bounded domain in $\mathbf{C}^2$, we can prove that $\bigcup_{i=1}^{\infty} \{y = a_i\} = \Delta_N$ by the same method of the proof of Theorem 3.6. It is easy to see that Δ_N is not analytic curve in N. Since $|x| e^{\varphi(y)}$ is plurisubharmonic in $\mathbf{C} \times \Delta(1)$, N is pseudoconvex, and, by Oka's theorem, N is a Stein manifold.

Remark 1.17. There is a case where the set Δ_N contains an open subset. Let $N = \{(x, y); |y| < e^{-|x|} + 1\}$. Then it is easy to see that N is a Stein manifold and $\Delta_N = \{(x, y) \in \mathbf{C}^2; |y| \leq 1\}$ by the same reason of the discussion of Remark 1.16.

Remark 1.18. Let M be a relatively compact subdomain of a manifold of X. We extend d_M onto the closure of $\overline{M}$ of M (extended d_M is not the pseudodistance (cf. [2, p. 386])) and we denote the set of the degeneracy points of d_M on $\overline{M}$ by $S_M(X)$ in [1]. It is trivial by the definition that $S_M(X)|_M = \Delta_M$.

2. Theorems of a hyperbolic manifold modulo a closed set Δ

Definition 2.1 ([7], p. 68). Let N be a manifold and Δ a closed subset of N. We say that N is hyperbolic modulo Δ if for every pair of distinct points p, q of N we have $d_N(p,q) > 0$ unless both are contained in Δ.

Remark 2.2. It is easy to see from Proposition 1.5 that if N is hyperbolic modulo Δ, we can take Δ_N as the smallest Δ.

Theorem 2.3. *Let N be a manifold of dimension n $(n \geq 2)$ such that hyperbolic modulo proper subset Δ_N. Let M be a manifold of dimension n and suppose that is a holomorphic map $\Phi : M \to N$ with the Jacobian of $\Phi \not\equiv 0$. Then, M is hyperbolic modulo $T = \{\Phi^{-1}(\Delta_N)\} \cup \{J\Phi = 0\}$ where $J\Phi$ is the Jacobian of Φ, that is, $\Delta_M \subset T$.*

Proof. Let $p, q \in M$ with $p \neq q$ and suppose that they are not both contained in T. If $\Phi(p) \neq \Phi(q)$, $d_N(\Phi(p), \Phi(q)) > 0$ because $\Phi(p)$ and $\Phi(q)$ are not both contained in Δ_N. Hence we set $\Phi(p) = \Phi(q) = r$. By the assumption, both p, q are not contained in $\Phi^{-1}(\Delta_N)$, $r \notin \Delta_N$. Unless both p, q are contained in $\{J\Phi = 0\}$, we may assume that $p \notin \{J\Phi = 0\}$. Then there are coordinate neighborhood $U(r)$ of $N - \Delta_N$ and $V(p)$ of M which are biholomorphic to each other. If we assume that $d_M(p,q) = 0$, then $p \in \Delta_M$ and for every closed neighborhood $\overline{V}_1(p)$ which is biholomorphic to the closed unit ball in $\mathbf{C}^n$ such as $V(p) \Supset \overline{V}_1(p)$ there is a point $p' \in \partial \overline{V}_1$ with $d_M(p,p') = 0$ by Proposition 1.2. This is a contradiction because $\Phi(p), \Phi(p') \notin \Delta_N$ and then $0 = d_M(p,p') \geq d_N(\Phi(p), \Phi(p')) > 0$. Thus $d_M(p,q) \neq 0$. □

Remark 2.4. In the same situation of above theorem in the case $n = 2$, Δ_M is contained in an analytic curve of M if Δ_N is an analytic curve. Therefore Δ_M is also an analytic curve of M or $\emptyset$ by Proposition 1.10.

Let $\pi : \tilde{N} \to N$ be a covering manifold of a manifold N of dimension n $(n \geq 2)$.

Theorem 2.5 (cf. Theorem (3.2.32) in [7]). $\Delta_{\tilde{N}} = \pi^{-1}(\Delta_N)$.

Proof. (1) If $p \in \Delta_N$, there is a closed coordinate neighborhood $\overline{U}(p)$ which is biholomorphic to the closed unit ball in $\mathbf{C}^n$ and every connected component of $\pi^{-1}(\overline{U}(p))$ is biholomorphic to $\overline{U}(p)$ by π. Then there is a point $q \in \partial \overline{U}(p)$ such that $d_N(p,q) = 0$ by Proposition 1.2. Let $\overline{V}(\tilde{p})$ be a connected component of $\pi^{-1}(\overline{U}(p))$ which contains $\tilde{p}$ where $\tilde{p}$ is an arbitrary point of $\pi^{-1}(p)$. By Theorem (3.2.8) in [7], $0 = d_N(p,q) = \inf_{\tilde{q} \in \pi^{-1}(q)} d_{\tilde{N}}(\tilde{p},\tilde{q})$. If $\tilde{p} \notin \Delta_{\tilde{N}}$, that is, there is not a point $\tilde{r} \in \partial\overline{V}(\tilde{p})$ such that $d_{\tilde{N}}(\tilde{p},\tilde{r}) = 0$, then $d_{\tilde{N}}(\tilde{p},\tilde{q}) \geq \delta = d_{\tilde{N}}(\tilde{p},\partial\overline{V}) > 0$. This contradicts to above equation.

(2) If $\tilde{p} \in \Delta_{\tilde{N}}$, there is a point $\tilde{q} \neq \tilde{p}$ such as $d_{\tilde{N}}(\tilde{p},\tilde{q}) = 0$. Then $0 = d_{\tilde{N}}(\tilde{p},\tilde{q}) \geq d_N(\pi(\tilde{p}),\pi(\tilde{q})) = d_N(p,q)$. If $p \neq q$, then $p \in \Delta_N$. Hence we will say that there is a $\tilde{q}$ such that $p \neq q$. We take a sufficient small closed coordinate neighborhood $\overline{V}(\tilde{p})$ of $\tilde{p}$, where $\overline{V}(\tilde{p}) \cap \{\pi^{-1}(p)\} = \{\tilde{p}\}$ and $\overline{V}(\tilde{p})$ is biholomorphic to the closed unit ball in $\mathbf{C}^n$. By Proposition 1.2, there is a point $\tilde{q} \in \partial\overline{V}(\tilde{p})$ such that $d_{\tilde{N}}(\tilde{p},\tilde{q}) = 0$ and $p \neq q$. □

Corollary 2.6. *Let $\pi : \tilde{N} \to N$ be a covering manifold of N of dimension 2. If $\Delta_{\tilde{N}}$ is an analytic curve, then Δ_N is an analytic curve.*

Proof. If $\Delta_{\tilde{N}}$ is an analytic curve of $\tilde{N}$, $\pi(\Delta_{\tilde{N}})$ is a locally analytic curve in N. Since $\pi(\Delta_{\tilde{N}}) = \Delta_N$ by Theorem 2.5 and Δ_N is a closed set in N by Proposition 1.5, Δ_N is an analytic curve in N. □

3. An example of hyperbolic manifold N modulo Δ_N

Let $P(x,y)$ be a nonconstant polynomial. We say that an irreducible component of a level curve of P is of (g,n) type if its genus is g and its boundaries are n points (counting n by the normalization of such a level curve). It is well-known that every irreducible components of almost all level curves are same type and nonsingular except finite ones. We call that P is a polynomial of type (g,n) if irreducible components of general level curves are of (g,n) type. If an irreducible component of exceptional level curves is of (g',n') type, $g' \leq g$ and $g' + n' \leq g + n$ (cf. Theorem I in [8]).

Definition 3.1. When $P(x,y)$ is a plynomial of (g,n) type, we say that it is a general type if $2g - 2 + n > 0$ and it is exceptional type if $2g - 2 + n \leq 0$.

Definition 3.2. We call that $P(x,y)$ is a primitive polynomial if almost all level curves are irreducible except finite ones.

The following proposition is well-known.

Proposition 3.3. *For every polynomial $P(x,y)$, there is a primitive polynomial $P_0(x,y)$ and a plynomial $\pi(z)$ such that $P = \pi \circ P_0$.*

Theorem 3.4 (Griffiths [6]). *Let $U_\rho = \{z \in \mathbf{C}; |z - \beta| < \rho, \beta \in \mathbf{C}, \rho > 0\}$ and for every $\alpha \in U_\rho$, $\{P(x,y) = \alpha\}$ is irreducible, nonsingular and of (g,n) type where $2g - 2 + n > 0$. We set $N_0 = \{(x,y) \in \mathbf{C}^2; P(x,y) = \alpha, \alpha \in U_\rho\}$. Then universal covering manifold $\tilde{N}_0$ of N_0 is a bounded Bergman domain in $\mathbf{C}^2$.*

The following corollary follows from Theorem 2.5.

Corollary 3.5. *The manifold N_0 is hyperbolic, that is, $\Delta_{N_0} = \emptyset$.*

Theorem 3.6. *Let $P(x, y)$ be a primitive general type polynomial and set $N_1 = \{(x, y) \in \mathbf{C}^2; P(x, y) \neq a, b\}$ where a and b are arbitrary different complex number. Then $\Delta_{N_1} \subset S$, where S is the exceptional level curves of P in N_1.*

Proof. We assume that $p, q \in N_1$ with $p \neq q$ and both p, q are not contained in S. We will prove that $d_{N_1}(p, q) > 0$. In case p, q are not both contained in a same level curve, it is easy to see that $d_{N_1}(p, q) \geq d_{\mathbf{C}^2 - \{a,b\}}(P(p), P(q)) > 0$. We assume that p, q are both contained in a same level curve $\{P(x, y) = \beta\}$. Let $U_{2s} = \{z \in \mathbf{C}; d_{\mathbf{C}-\{a,b\}}(\beta, z) < 2s, s > 0\}$. We take a number s sufficientry small such that $2s = \rho$ where ρ satisfies the condition in Theorem 3.4. Then $N_0 = \{(x, y) \in \mathbf{C}^2; P(x, y) = \alpha, \alpha \in U_{2s}\}$ is hyperbolic by Corollary 3.5. We take positive number r $(r < 1)$ sufficiently small such that $d_{\Delta(1)}(0, z) < s$ for every $z \in \Delta(r)$ where $\Delta(r) = \{z \in \mathbf{C}; |z| < r\}$. Thus if $f : \Delta(1) \to N_1$ is holomorphic and $P(f(0)) \in U_s$, then $f(\Delta(r)) \subset N_0$.

Let $f_i : \Delta(1) \to N_1$ be holomorphic mappings and a_i, b_i be points of $\Delta(1)$ such that $p = f_1(a_1), f_1(b_1) = f_2(a_2), \dots, f_k(b_k) = q$. By homogenity of $\Delta(1)$ we may assume that $a_i = 0$ for all i. By inserting extra terms in this chain if necessary, we may assume also that $b_i \in \Delta(r/2)$ for all $i = 1, \dots, k$. Choose $c > 0$ such that $d_{\Delta(1)}(0, a) \geq c \cdot d_{\Delta(r)}(0, a)$ for every $a \in \Delta(r/2)$. We set $p_0 = p, p_1 = f(b_1), \dots, p_k = f_k(b_k) = q$.

We have two cases to consider. Consider the first case where at least one of the $P(p_i)$'s is not contained in U_s. Then it is easy to see

$$\sum_{i=1}^{k} d_{\Delta(1)}(0, b_i) \geq \sum_{i=1}^{k} d_{N_1}(f_i(0), f_i(b_i)) \geq \sum_{i=1}^{k} d_{\mathbf{C}-\{a,b\}}(P(f_i(0)), P(f_i(b_i))) \geq s.$$

Consider the next case where all $P(p_i)$'s are in U_s. Then

$$\sum_{i=1}^{k} d_{\Delta(1)}(0, b_i) \geq c \sum_{i=1}^{k} d_{\Delta(r)}(0, b_i) \geq c \sum_{i=1}^{k} d_{N_0}(p_{i-1}, p_i) \geq c \cdot d_{N_0}(p, q) > 0.$$

This shows that $d_{N_1}(p, q) \geq \min\{s, c \cdot d_{N_0}(p, q)\} > 0$. Thus N_1 is hyperbolic modulo S, that is, $\Delta_{N_1} \subset S$. □

Example 3.7. Set $N_1 = \{(x, y) \in \mathbf{C}^2; y^2 - x^3 \neq 0, 1\}$. Then N_1 is hyperbolic by Theorem 3.6.

Theorem 3.8. *Let $P(x, y)$ be a general type polynomial. Then, for $N = \{(x, y) \in \mathbf{C}^2; P(x, y) \neq a, b\}$ $\Delta_N \subset S$, where S is a curve consists of the exceptional level curves of $P(x, y)$ in N.*

Proof. From Proposition 3.3, there is a primitive polynomial $P_0(x, y)$ and a polynomial $\pi(z)$ such that $P = \pi \circ P_0$. Hence there is an injection $i : N \to N_1$

where N_1 is the same in Theorem 3.6 and we take P_0 instead of P, a point of $\pi^{-1}(a)$ instead of a and a point of $\pi^{-1}(b)$ instead of b. From Theorem 2.3, $\Delta_N \subset S$. □

Remark 3.9. In the same notation of Theorem 3.8, Δ_N is an algebraic curve or $\emptyset$ by Proposition 1.10.

4. A generalization of the little Picard theorem

It is easy to see the following:

Proposition 4.1. *Let N_1 and N_2 be manifolds of dimension n $(n \geq 2)$. If a holomorphic map $F : N_1 \to N_2$ is nondegenerate, that is, $F(N_1)$ contains an open set in N_2, if and only if $JF \not\equiv 0$.*

Proposition 4.2. *Let N be a manifold of dimension 2 and $d_N \equiv 0$. Let $F : N \to \mathbf{C}^2$ be a holomorphic map such that $P \circ F \neq a, b$, where $P(x, y)$ is a polynomial and a, b are different complex numbers. Then F is a degenerate map.*

Proof. For every points $p, q \in N$ such as $p \neq q$, $0 = d_N(p, q) \geq d_{\mathbf{C}^2 - \{a,b\}}(P \circ F(p), P \circ F(q))$. Hence $P \circ F(p) = P \circ F(q)$. Therefore $F(N)$ is contained in a same level curve. □

Proposition 4.3. *Let N be a manifold of dimension 2 such that $\Delta_N \neq \emptyset$ and let $F : N \to \mathbf{C}^2$ is a holomorphic map such that $P \circ F \neq a, b$, where $P(x, y)$ is a polynomial. Then $P \circ F(\Delta_N(p)) = \alpha$ (constant) and $\Delta_N(p)$ is an analytic curve in N.*

Proof. Since for every $q, r \in \Delta_N(p)$, $d_N(q, r) = 0$ by Proposition 1.14, $F(\Delta_N(p))$ is contained in a same level curve by the same reason of Proposition 4.2. Since $\Delta_N(p)$ is contained an analytic cuve of N, $\Delta_N(p)$ is an analytic curve of N by Proposition 1.10 and Theorem 1.13. □

Theorem 4.4. *Let N be a manifold of dimension 2 and let the nonempty set Δ_N be not an analytic curve of N. Let $F : N \to \mathbf{C}^2$ be a holomorphic map such that $P \circ F \neq a, b$, where $P(x, y)$ be a general type polynomial. Then F is a degenerate map.*

Proof. Since $M = \{(x, y) \in \mathbf{C}^2; P(x, y) \neq a, b\}$ is hyperbolic modulo algebraic curve or $\emptyset$ by Theorem 3.8, Remark 3.9, Remark 2.4, Propositions 4.1 and 4.3, F is a degenerate map. □

Remark 4.5. The condition that $P(x, y)$ is a general type polynomial is indispensable for Theorem 4.4. For example, if $N = \mathbf{C} \times (\mathbf{C} - \{a, b\})$, F is an identity map and $p(x, y) \equiv y$, then $N = \{P \circ F \neq a, b\}$.

5. Examples of a manifold N such that $\Delta_N = N$

Problem 5.1. Enumurate the Stein manifold N of dimension 2 such that $\Delta_N = N$ or $d_N \equiv 0$ specially.

We study the case where a manifold N is a quasi-projective Stein manifold.

Proposition 5.2. *Let C be a curve of degree ≤ 2. Then $N = \mathbf{P}^2 - C$ satisfies $\Delta_N = N$ and hence $d_N \equiv 0$.*

Proof. In the case where degree of C equals 1, N is biholomorphic to $\mathbf{C}^2$ and the conclusion is trivial.

In the case where the degree of C equals 2, C consists of two lines or a conic. The former case, N is biholomorphic to $\mathbf{C} \times \mathbf{C}^*$ and the conclusion is trivial. The latter case, for every distinct points p and $q \in N$ the line L through p and q meets with C at most two points. Then $d_N(p, q) = 0$ because $L - C$ is biholomorphic to $\mathbf{C}$ or $\mathbf{C}^*$. □

Proposition 5.3. *Let C be a curve of degree equals to 3. Then $N = \mathbf{P}^2 - C$ satisfies $\Delta_N = N$.*

Proof. In case C consists of three lines in general position, N is biholomorphic to $\mathbf{C}^* \times \mathbf{C}^*$ and it is easy to see that $\Delta_N = N$ and $d_N \equiv 0$.

In case C consists of three lines in particular position, N is biholomorphic to $\mathbf{C} \times (\mathbf{C} - \{a, b\})$ where $a \neq b$. It is easy to see that $\Delta_N = N$ and $d_N \not\equiv 0$.

In case C consists of a conic and a line L, $\Delta_N = N$ and $d_N \equiv 0$. Because for almost all distinct points p and $q \in N$, tangent line L_p of the conic through p meets with L at a point, $L_p - C$ is biholomorphic to $\mathbf{C}$ or $\mathbf{C}^*$. The similar line L_q meets with L_p with a point r or $L_p = L_q$. Then in the former case $d_N(p, q) \leq d_N(p, r) + d_N(q, r) = 0$ and in the latter case it is easy to see that $d_N(p, q) = 0$. Since Δ_N is a closed set and d_N is continuous, $\Delta_N = N$ and $d_N \equiv 0$.

In case C is a cubic curve, $\Delta_N = N$ and $d_N \equiv 0$. Because for almost all distinct points p and $q \in N$, tangent line L_p of C through p meets with C at most two points, and then $L_p - C$ is biholomorphic to $\mathbf{C}^*$ or $\mathbf{C}$. The similar line L_q meets with L_p with a point r or $L_p = L_q$. Then conclusion is easy to see similary to the above discussion. □

In the case where the degree of C equals 4, we only raise examples.

Example 5.4. If C consists of four lines in general position, it is well-known that Δ_N is a diagonal line (cf. Theorem (3.10.27) in [7]).

If C consists of four lines in particular position, N is biholomorphic to $\mathbf{C} \times (\mathbf{C} - \{a, b, c\})$ or $\mathbf{C}^* \times (\mathbf{C} - \{a, b\})$. Then $\Delta_n = N$ and $d_N \not\equiv 0$.

Example 5.5 (J. Carson, F. Sakai and B. Shiffman). If $C = L_\infty \cup \{y = x^3\}$ where L_∞ is the line at infinity, then $\Delta_N = N$ and $d_N \equiv 0$ where $N = \mathbf{C}(x, y) - \{y = x^3\}$. Because $F : x = z, y = z^3 + e^w$ is a holomorphic map of $\mathbf{C}^2$ onto N.

PROBLEM 5.6. Let N be a Stein manifold with $d_N \equiv 0$. Then, is there a nondegenerate holomorphic map of $\mathbf{C}^2$ to N?

REFERENCES

[1] Y. ADACHI, A generalization of the big Picard theorem, Kodai Math. J. **18** (1995), 408–424.

[2] Y. ADACHI, On the relation between tautly imbedded space modulo an analytic subset S and hyperbolically imbedded space modulo S, Publ. RIMS, Kyoto Univ. **33** (1997), 385–392.

[3] Y. ADACHI AND M. SUZUKI, Degeneracy points of the Kobayashi pseudodistances on complex manifolds, Proc. sympos. pure math. **52**, Amer. Math. Soc., Providence, RI, 1991, 41–51.

[4] O. FUJITA, Sur les familles d'ensemble analytiques, J. Math. Soc. Japan **16** (1964), 379–405.

[5] O. FUJITA, Domaines pseudoconvexes d'ordre général et fonctions pseudoconvexes d'ordre général, J. Math. Kyoto Univ. **30** (1990), 637–649.

[6] P. A. GRIFFITHS, Complex-analytic properties of certain Zariski open sets on algebraic varieties, Ann. Math. **94** (1971), 21–51.

[7] S. KOBAYASHI, Hyperbolic complex spaces, Die Grundlehren der mathematishen Wissenschaften **318**, Springer-Verlag, Berlin-Heidelberg, 1998.

[8] T. NISHINO, Nouvelles recherches sur les fonctions entières de plusieurs variables complexes IV, J. Math. Kyoto Univ. **13** (1973), 217–272.

[9] R. REMMERT UND K. STEIN, Über die wesentlichen Singularitäten analytischer Mengen, Math. Annalen **126** (1953), 263–306.

[10] M. TADOKORO, Sur les ensembles pseudononcaves généraux, J. Math. Soc. Japan **17** (1965), 281–290.

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