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ON HOLOMORPHIC CURVES EXTREMAL FOR THE μ_n-DEFECT RELATION

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To the memory of the late Professor Kiyoshi Noshiro on the occasion of the centennial anniversary of his birth

1. Introduction

Let $f = [f_1, \dots, f_{n+1}]$ be a holomorphic curve from $\boldsymbol{C}$ into the n-dimensional complex projective space $P^n(\boldsymbol{C})$ with a reduced representation

$$(f_1, \dots, f_{n+1}) : \boldsymbol{C} \to \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\},$$

where n is a positive integer. We use the following notations:

$$\|f(z)\| = (|f_1(z)|^2 + \cdots + |f_{n+1}(z)|^2)^{1/2}$$

and for a vector $\boldsymbol{a} = (a_1, \dots, a_{n+1}) \in \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$

$$\|\boldsymbol{a}\| = (|a_1|^2 + \cdots + |a_{n+1}|^2)^{1/2}, \quad (\boldsymbol{a}, f) = a_1 f_1 + \cdots + a_{n+1} f_{n+1},$$
$$(\boldsymbol{a}, f(z)) = a_1 f_1(z) + \cdots + a_{n+1} f_{n+1}(z).$$

The characteristic function of f is defined as follows (see [14]):

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log\|f(re^{i\theta})\| \, d\theta - \log\|f(0)\|.$$

It is known ([1]) that for $U(z) = \max_{1 \le j \le n+1} |f_j(z)|$

$$(1) \qquad T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log U(re^{i\theta}) \, d\theta + O(1).$$

We suppose throughout the paper that f is transcendental; that is to say,

$$\lim_{r \to \infty} \frac{T(r, f)}{\log r} = \infty$$

and that f is linearly non-degenerate over $\boldsymbol{C}$; namely, $f_1, \dots, f_{n+1}$ are linearly independent over $\boldsymbol{C}$. It is well-known that f is linearly non-degenerate over $\boldsymbol{C}$ if

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and only if the Wronskian $W = W(f_1, \dots, f_{n+1})$ of $f_1, \dots, f_{n+1}$ is not identically equal to zero.

We call the quantity:

$$\rho(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}$$

the order of f.

For meromorphic functions in the complex plane we use the standard notation of the Nevanlinna theory of meromorphic functions ([4, 7]).

For $\boldsymbol{a} \in \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$, we write

$$m(r, \boldsymbol{a}, f) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\|\boldsymbol{a}\| \, \|f(re^{i\theta})\|}{|(\boldsymbol{a}, f(re^{i\theta}))|} \, d\theta, \quad N(r, \boldsymbol{a}, f) = N\left(r, \frac{1}{(\boldsymbol{a}, f)}\right).$$

We then have the **First Fundamental Theorem** ([14, p. 76]):

$$T(r, f) = m(r, \boldsymbol{a}, f) + N(r, \boldsymbol{a}, f) + O(1).$$

We call the quantity

$$\delta(\boldsymbol{a}, f) = 1 - \limsup_{r \to \infty} \; N(r, \boldsymbol{a}, f)/T(r, f) = \liminf_{r \to \infty} \; m(r, \boldsymbol{a}, f)/T(r, f)$$

the deficiency (or defect) of $\boldsymbol{a}$ with respect to f. We have $0 \le \delta(\boldsymbol{a}, f) \le 1$.

Let $\nu(c)$ be the order of zero of $(\boldsymbol{a}, f(z))$ at $z = c$ and for a positive integer k

$$n_k(r, \boldsymbol{a}, f) = \sum_{|c| \le r} \min\{\nu(c), k\}.$$

We put for $r > 0$

$$N_k(r, \boldsymbol{a}, f) = \int_0^r \frac{n_k(t, \boldsymbol{a}, f) - n_k(0, \boldsymbol{a}, f)}{t} \, dt + n_k(0, \boldsymbol{a}, f) \log r$$

and put

$$\delta_k(\boldsymbol{a}, f) = 1 - \limsup_{r \to \infty} \; N_k(r, \boldsymbol{a}, f)/T(r, f).$$

Then, it is easy to see that

(1.a) the sequemce $\{\delta_k(\boldsymbol{a}, f)\}_{k=1}^{\infty}$ is decreasing;

(1.b) for any k,

$$0 \le \delta(\boldsymbol{a}, f) \le \delta_k(\boldsymbol{a}, f) \le 1.$$

We denote by $S(r, f)$ the quantity satisfying

$$S(r, f) = \begin{cases} O(\log r) & (r \to \infty) & \text{if } \rho(f) < \infty \\ O(\log T(r, f) + \log r) & (r \to \infty, r \notin E) & \text{if } \rho(f) = \infty, \end{cases}$$

where E is a subset of $(0, \infty)$ of finite linear measure.

Let X be any subset of $\boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$ in N-subgeneral position satisfying $2N - n + 2 \le \# X \le \infty$, where N is an integer satisfying $N \ge n$. We say that X is in general position when X is in n-subgeneral position.

Cartan ([1], $N = n$) and Nochka ([8], $N > n$) gave the following

THEOREM 1.A (see [3, Corollary 3.3.9]). *For any q elements $\boldsymbol{a}_j$ $(j = 1, \dots, q)$ of X $(2N - n + 1 \le q < \infty)$, we have the following inequalities*:
(I) $(q - 2N + n - 1)T(r, f) \le \sum_{j=1}^{q} N_n(r, \boldsymbol{a}_j, f) + S(r, f)$;
(II) (*the truncated defect relation*) $\sum_{j=1}^{q} \delta_n(\boldsymbol{a}_j, f) \le 2N - n + 1$.

Let $\boldsymbol{a}$ be any vector in $\boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$. We say that
"$\boldsymbol{a}$ has multiplicity m if $(\boldsymbol{a}, f)$ has at least one zero and all the zeros of the function $(\boldsymbol{a}, f(z))$ have multiplicity at least m, while at least one zero has multiplicity m."
When $(\boldsymbol{a}, f)$ has no zero, we set $m = \infty$.
Then, as a corollary of Theorem 1.A(II), Cartan ([1], $N = n$) and Nochka ([8], $N > n$) gave the following theorem:

THEOREM 1.B. *For any $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q \in X$, let $\boldsymbol{a}_j$ have multiplicity m_j. Then,*

$$\sum_{j=1}^{q} \left(1 - \frac{n}{m_j}\right) \le 2N - n + 1,$$

where $2N - n + 1 \le q < \infty$ (see [3, Theorem 3.3.15]).

As the numbers "$1 - n/m_j$" are not always non-negative in this theorem, we gave a new defect in [12, Definition 4.1] (see also [6, p. 171]). More generally we give the following

DEFINITION 1.1. For $\boldsymbol{a} \in \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$ with multiplicity m and for a positive integer k we put

$$\mu_k(\boldsymbol{a}, f) = \left(1 - \frac{k}{m}\right)^+ = 1 - \frac{k}{\max(m, k)},$$

where $a^+ = \max(a, 0)$ for any real number a.

We call $\mu_k(\boldsymbol{a}, f)$ the *μ_k-defect* of $\boldsymbol{a}$ with respect to f. It is easy to see that the sequence $\{\mu_k(\boldsymbol{a}, f)\}_{k=1}^{\infty}$ is decreasing. Later we shall see that for any $\boldsymbol{a} \in X$ and for any positive integer k

$$\mu_k(\boldsymbol{a}, f) \le \delta_k(\boldsymbol{a}, f)$$

(Corollary 2.2).
From Theorem 1.B we have the following defect relation for $\mu_n(\boldsymbol{a}, f)$:

THEOREM 1.C (see [6, Corollary (3.B.46)]). *For any $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q \in X$ $(q < \infty)$, we have the following inequality*:

$$\sum_{j=1}^{q} \mu_n(\boldsymbol{a}_j, f) \le 2N - n + 1.$$

We call this inequality the μ_n-*defect relation* for f over X. In Section 6 we shall give an example of holomorphic curve f from $\boldsymbol{C}$ into $P^n(\boldsymbol{C})$ and a subset X of $\boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$ in N-subgeneral position such that if $k < n$ then

$$\sum_{\boldsymbol{a} \in X} \mu_k(\boldsymbol{a}, f) = +\infty.$$

In [12] we gave some results on $\mu_n(\boldsymbol{a}, f)$ when $\#\{\boldsymbol{a} \in X \mid \mu_n(\boldsymbol{a}, f) = 1\}$ is large. For example,

Theorem 1.D. *Suppose that* $N > n \geq 2$. *If there exist* $n+1$ *linearly independent vectors* $\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n+1} \in X$ *satisfying* $\mu_n(\boldsymbol{a}_j, f) = 1$ $(j = 1, \dots, n+1)$, *then*
(I) ([12, Theorem 4.2]) *For any* $\boldsymbol{a} \in X$ *satisfying* $\mu_n(\boldsymbol{a}, f) > 0$, $\mu_n(\boldsymbol{a}, f) = 1$.
(II) ([12, Theorem 4.3]) $\sum_{\boldsymbol{a} \in X} \mu_n(\boldsymbol{a}, f) \leq N + N/n$.

Remark 1.1. $N + N/n < 2N - n + 1$ when $N > n \geq 2$.

We are interested in a holomorphic curve f for which the μ_n-defect relation is extremal.

The purpose of this paper is to give several results on the μ_n-defect relation for f and on $\mu_n(\boldsymbol{a}, f)$ when the μ_n-defect relation is extremal.

2. Preliminaries and lemma

Let $f = [f_1, \dots, f_{n+1}]$, X, etc. be as in Section 1 and q an integer satisfying $2N - n + 1 \leq q < \infty$. We put $Q = \{1, 2, \dots, q\}$. Let $\{\boldsymbol{a}_j \mid j \in Q\}$ be a subset of X. For a non-empty subset P of Q, we denote by $V(P)$ the vector space spanned by $\{\boldsymbol{a}_j \mid j \in P\}$ and by $d(P)$ the dimension of $V(P)$. We put

$$\mathcal{O} = \{P \subset Q \mid 0 < \#P \leq N + 1\}.$$

Lemma 2.1 (see [3, (2.4.3), p. 68]). *If* $P \in \mathcal{O}$, *then* $\#P - d(P) \leq N - n$.

For $\{\boldsymbol{a}_j \mid j \in Q\}$, let $\omega : Q \to (0, 1]$ be the Nochka weight function and θ the reciprocal number of the Nochka constant given in [3, p. 72]. Then they have the following properties:

Lemma 2.2 (see [3, Theorem 2.4.11], [2]).
(a) $0 < \omega(j)\theta \leq 1$ *for all* $j \in Q$;
(b) $q - 2N + n - 1 = \theta(\sum_{j=1}^{q} \omega(j) - n - 1)$;
(c) $N/n^{(*)} \leq \theta \leq (2N - n + 1)/(n + 1)$;
(d) *If* $P \in \mathcal{O}$, *then* $\sum_{j \in P} \omega(j) \leq d(P)$.
((*): see [11, Note 2.1]).

Definition 2.1 ([9, Definition 1]). We put

$$\lambda = \min_{P \in \mathcal{O}} d(P)/\#P \quad \text{and} \quad \sigma : Q \to (0,1] \text{ such that } \sigma(j) = \lambda \ (j \in Q).$$

Then, λ and σ have the following properties.

LEMMA 2.3 ([9, Proposition 2]). (a) $1/(N-n+1) \le \lambda \le (n+1)/(N+1)$; (b) *For any* $P \in \mathcal{O}$, $\sum_{j \in P} \sigma(j) \le d(P)$.

Remark 2.1. (a) If $\lambda < (n+1)/(2N-n+1)$, then $\lambda = \min_{1 \le j \le q} \omega(j)$, $\omega(j) = \lambda$ and $\theta\omega(j) < 1$ $(j \in P_0)$ for an element $P_0 \in \mathcal{O}$ satisfying $\lambda = d(P_0)/\#P_0$.

(b) If $\lambda \ge (n+1)/(2N-n+1)$, then $\omega(j) = 1/\theta = (n+1)/(2N-n+1)$ $(j = 1, \dots, q)$.

(See [11, Remark 2].)

Let h be an entire function. For $a \in \boldsymbol{C}$, let $\nu(a,h)$ be the order of zero of $h(z)$ at $z = a$:

$$h(z) = c_1(z-a)^{\nu(a,h)} + c_2(z-a)^{\nu(a,h)+1} + \cdots, \quad (c_1 \ne 0).$$

For $\boldsymbol{a}_j$ $(j \in Q)$ we put $F_j = (\boldsymbol{a}_j, f)$ $(j \in Q)$. Then, we have the following

LEMMA 2.4 ([3, (3.2.14), p. 102]).

$$\sum_{j=1}^{q} \omega(j)(\nu(a,F_j) - n)^+ \le \nu(a, W).$$

From this inequality we have the following

LEMMA 2.5. *For* $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q \in X$,

(I) $\sum_{j=1}^{q} \omega(j)\{n(r, \boldsymbol{a}_j, f) - n_n(r, \boldsymbol{a}_j, f)\} \le n(r, 1/W)$.

(II) $\sum_{j=1}^{q} \omega(j)\{(n(r, \boldsymbol{a}_j, f) - n_n(r, \boldsymbol{a}_j, f)) - (n(0, \boldsymbol{a}_j, f) - n_n(0, \boldsymbol{a}_j, f))\}$
$\le n(r, 1/W) - n(0, 1/W)$.

Proof. We note that

$$\begin{aligned} \sum_{|a| \le r} (\nu(a,F_j) - n)^+ &= \sum_{|a| \le r} (\nu(a,F_j) - \min\{\nu(a,F_j), n\}) \\ &= \sum_{|a| \le r} \nu(a,F_j) - \sum_{|a| \le r} \min\{\nu(a,F_j), n\} \\ &= n(r, \boldsymbol{a}_j, f) - n_n(r, \boldsymbol{a}_j, f); \end{aligned}$$

$$\sum_{|a| \le r} \nu(a, W) = n(r, 1/W)$$

and that

$$\sum_{0<|a|\le r}(\nu(a,F_j)-n)^+ = \sum_{0<|a|\le r}(\nu(a,F_j)-\min\{\nu(a,F_j),n\})$$
$$= \sum_{0<|a|\le r}\nu(a,F_j) - \sum_{0<|a|\le r}\min\{\nu(a,F_j),n\}$$
$$= n(r,\boldsymbol{a}_j,f)-n_n(r,\boldsymbol{a}_j,f)-(n(0,\boldsymbol{a}_j,f)-n_n(0,\boldsymbol{a}_j,f));$$
$$\sum_{0<|a|\le r}\nu(a,W) = n(r,1/W)-n(0,1/W).$$

By using these equalities we obtain this lemma from Lemma 2.4. □

LEMMA 2.6 (see [3, p. 105]). *For* $\boldsymbol{a}_1,\dots,\boldsymbol{a}_q \in X$, *we have the inequality*

$$\sum_{j=1}^{q}\omega(j)\{N(r,\boldsymbol{a}_j,f)-N_n(r,\boldsymbol{a}_j,f)\}\le N\left(r,\frac{1}{W}\right)\quad(r\ge 1).$$

Proof. This inequality is essentially the same one as that in [3, p. 105]. But there is a little difference between them for a small term. To make sure of it we give a proof of our lemma. By definition

$$N(r,\boldsymbol{a}_j,f)=\int_0^r\frac{n(t,\boldsymbol{a}_j,f)-n(0,\boldsymbol{a}_j,f)}{t}\,dt+n(0,\boldsymbol{a}_j,f)\log r;$$
$$N_n(r,\boldsymbol{a}_j,f)=\int_0^r\frac{n_n(t,\boldsymbol{a}_j,f)-n_n(0,\boldsymbol{a}_j,f)}{t}\,dt+n_n(0,\boldsymbol{a}_j,f)\log r;$$
$$N(r,1/W)=\int_0^r\frac{n(t,1/W)-n(0,1/W)}{t}+n(0,1/W)\log r.$$

For simplicity we put

$$n(r)=\sum_{j=1}^{q}\omega(j)\{n(r,\boldsymbol{a}_j,f)-n_n(r,\boldsymbol{a}_j,f)\}.$$

Then, we have

$$\sum_{j=1}^{q}\omega(j)\{N(r,\boldsymbol{a}_j,f)-N_n(r,\boldsymbol{a}_j,f)\}=\int_0^r\frac{n(t)-n(0)}{t}\,dt+n(0)\log r=(*).$$

Put

$$n(0,1/W)-n(0)=c_o,$$

which is non-negative by Lemma 2.5(I). Then, by Lemma 2.5(II)

$$(*) \leq \int_0^r \frac{n(t,1/W)-n(0,1/W)}{t}\,dt + n(0,1/W)\log r - c_o \log r$$
$$= N(r,1/W) - c_o \log r \leq N(r,1/W) \quad (r \geq 1).$$

We have this lemma. □

LEMMA 2.7 ([3, Theorem 3.3.8]). *For $\boldsymbol{a}_1,\dots,\boldsymbol{a}_q \in X$, we have the inequality*

$$\sum_{j=1}^{q} \omega(j)\delta_n(\boldsymbol{a}_j, f) \leq n+1.$$

LEMMA 2.8 ([10, Lemma 2.4]). *Suppose that $N > n$. For $\boldsymbol{a}_1,\dots,\boldsymbol{a}_q \in X$ the maximal deficiency sum*

$$\sum_{j=1}^{q} \delta_n(\boldsymbol{a}_j, f) = 2N-n+1$$

holds if and only if the following two relations hold:

1) $(1-\theta\omega(j))(1-\delta_n(\boldsymbol{a}_j,f)) = 0 \ (j=1,\dots,q)$;
2) $\sum_{j=1}^{q} \omega(j)\delta_n(\boldsymbol{a}_j,f) = n+1$.

LEMMA 2.9. *Suppose that there are q elements $\boldsymbol{a}_j$ $(j=1,\dots,q)$ in X $(2N-n+1 \leq q < \infty)$ satisfying*

$$\sum_{j=1}^{q} \delta_n(\boldsymbol{a}_j, f) = 2N-n+1. \tag{2}$$

Then, there is a subset E of $(0,\infty)$ such that for $j=1,\dots,q$

$$\limsup_{r\to\infty} \frac{N_n(r,\boldsymbol{a}_j,f)}{T(r,f)} = \lim_{r\to\infty;\, r\notin E} \frac{N_n(r,\boldsymbol{a}_j,f)}{T(r,f)},$$

where E is empty when $\rho(f) < \infty$ and E is of finite linear measure when $\rho(f) = \infty$.

Proof. (a) When $\rho(f) < \infty$. Let j be any integer satisfying $1 \leq j \leq q$. From Theorem 1.A and (2) we obtain the inequality

$$\begin{aligned} q-(2N-n+1) &\leq \liminf_{r\to\infty} \frac{N_n(r,\boldsymbol{a}_j,f)}{T(r,f)} + \sum_{k=1;\,k\neq j}^{q} \limsup_{r\to\infty} \frac{N_n(r,\boldsymbol{a}_k,f)}{T(r,f)} \\ &\leq \limsup_{r\to\infty} \frac{N_n(r,\boldsymbol{a}_j,f)}{T(r,f)} + \sum_{k=1;\,k\neq j}^{q} \limsup_{r\to\infty} \frac{N_n(r,\boldsymbol{a}_k,f)}{T(r,f)} \\ &= q-(2N-n+1), \end{aligned}$$

which reduces to our lemma.

(b) When $\rho(f) = \infty$. Let j be any integer satisfying $1 \le j \le q$. From Theorem 1.A and (2) there is a subset E of $(0, \infty)$ of finite linear measure for which we obtain the inequality

$$\begin{aligned} q - (2N - n + 1) &\le \liminf_{r\to\infty;\, r\notin E} \frac{N_n(r, \boldsymbol{a}_j, f)}{T(r,f)} + \sum_{k=1;\,k\ne j}^{q} \limsup_{r\to\infty;\, r\notin E} \frac{N_n(r, \boldsymbol{a}_k, f)}{T(r,f)} \\ &\le \limsup_{r\to\infty;\, r\notin E} \frac{N_n(r, \boldsymbol{a}_j, f)}{T(r,f)} + \sum_{k=1;\,k\ne j}^{q} \limsup_{r\to\infty;\, r\notin E} \frac{N_n(r, \boldsymbol{a}_k, f)}{T(r,f)} \\ &\le \limsup_{r\to\infty} \frac{N_n(r, \boldsymbol{a}_j, f)}{T(r,f)} + \sum_{k=1;\,k\ne j}^{q} \limsup_{r\to\infty} \frac{N_n(r, \boldsymbol{a}_k, f)}{T(r,f)} \\ &= q - (2N - n + 1) \end{aligned}$$

and for $j = 1, \dots, q$, we have the inequality

$$\liminf_{r\to\infty;\, r\notin E} \frac{N_n(r, \boldsymbol{a}_j, f)}{T(r,f)} \le \limsup_{r\to\infty;\, r\notin E} \frac{N_n(r, \boldsymbol{a}_j, f)}{T(r,f)} \le \limsup_{r\to\infty} \frac{N_n(r, \boldsymbol{a}_j, f)}{T(r,f)}$$

apriori. From these inequalities we obtain our lemma. □

Corollary 2.1. *Under the same assumption as in Lemma* 2.9, *for any* $j = 1, \dots, q$ *and for any sequence* $\{r_\nu\}_{\nu=1}^{\infty} \subset (0, \infty) - E$ *tending to* $+\infty$

$$\limsup_{r\to\infty} \frac{N_n(r, \boldsymbol{a}_j, f)}{T(r,f)} = \lim_{\nu\to\infty} \frac{N_n(r_\nu, \boldsymbol{a}_j, f)}{T(r_\nu, f)},$$

where E *is as in Lemma* 2.9.

Lemma 2.10. *Let* k *be a positive integer. For a vector* $\boldsymbol{a}$ *in* $\boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$ *of multiplicity* m *with respect to* f *and for* $r \ge 1$, *we have the inequality*

$$N_k(r, \boldsymbol{a}, f) \le \frac{k}{\max(m,k)} N(r, \boldsymbol{a}, f) \le \begin{cases} N(r, \boldsymbol{a}, f) \\ \dfrac{k}{\max(m,k)} T(r,f) + O(1). \end{cases}$$

Proof. As this lemma is trivial when $m = \infty$ or $N(r, \boldsymbol{a}, f) = 0$, we have only to prove the following inequality when $m < \infty$.

$$N_k(r, \boldsymbol{a}, f) \le \frac{k}{\max(m,k)} N(r, \boldsymbol{a}, f) \quad (r \ge 1). \tag{3}$$

(a) When $k \ge m$. (3) is trivial.
(b) When $k < m$. For $r \ge 1$, we have the inequality

$$N_k(r,\boldsymbol{a},f) \le k\overline{N}(r,\boldsymbol{a},f) = \frac{k\overline{N}(r,\boldsymbol{a},f)}{N(r,\boldsymbol{a},f)} N(r,\boldsymbol{a},f)$$
$$\le \frac{k\overline{N}(r,\boldsymbol{a},f)}{m\overline{N}(r,\boldsymbol{a},f)} N(r,\boldsymbol{a},f) = \frac{k}{m} N(r,\boldsymbol{a},f) = \frac{k}{\max(m,k)} N(r,\boldsymbol{a},f). \quad \square$$

COROLLARY 2.2. *Let k be a positive integer and let $\boldsymbol{a}$ be a vector in $\boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$ whose multiplicity with respect to f is m. Then,*

(a) *We have the inequality*

$$1 \ge \delta_k(\boldsymbol{a},f) \ge 1 - \frac{k}{\max(m,k)}(1 - \delta(\boldsymbol{a},f)) \ge \begin{cases} \delta(\boldsymbol{a},f) \\ \mu_k(\boldsymbol{a},f). \end{cases}$$

(b) *If $\mu_k(\boldsymbol{a},f) = \delta_k(\boldsymbol{a},f)$, then $m = \infty$ or $\delta(\boldsymbol{a},f) = 0$.*

Proof. (a) From Lemma 2.10 we obtain this proposition immediately.

(b) Suppose that $m < \infty$. By the assumption, we obtain from (a) of this corollary that

$$1 - \frac{k}{\max(m,k)}(1 - \delta(\boldsymbol{a},f)) = \mu_k(\boldsymbol{a},f),$$

from which we obtain the equality

$$\frac{k}{\max(m,k)}\delta(\boldsymbol{a},f) = 0.$$

As $m < \infty$, we obtain that $\delta(\boldsymbol{a},f) = 0$. $\square$

LEMMA 2.11. *Let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{n+1}$ be linearly independent $n+1$ vectors in $\boldsymbol{C}^{n+1}$. Then, there is a subset E of $(0,\infty)$ for which it holds that*

$$\limsup_{r\to\infty, r\notin E} \frac{N(r,1/W)}{T(r,f)} \le n + 1 - \sum_{j=1}^{n+1} \delta(\boldsymbol{u}_j, f),$$

where E is empty when $\rho(f) < \infty$ and E is of finite linear measure when $\rho(f) = \infty$.

Proof. We put $G_j = (\boldsymbol{u}_j, f)$ $(j = 1, \dots, n+1)$. Then, $G_1, \dots, G_{n+1}$ are linearly independent over $\boldsymbol{C}$ and we have the relation

$$W = W(f_1, \dots, f_{n+1}) = cW(G_1, \dots, G_{n+1}) \quad (c \ne 0, \text{ constant}),$$

where $W(f_1, \dots, f_{n+1})$ is the Wronskian of $f_1, \dots, f_{n+1}$.

$$\begin{aligned} N(r,1/W) &= \frac{1}{2\pi}\int_0^{2\pi} \log|W(re^{i\theta})|\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log|W(G_1,\dots,G_{n+1})(re^{i\theta})|\,d\theta + \log|c| \\ &\le \sum_{j=1}^{n+1} N(r,\boldsymbol{u}_j,f) + S(r,f), \end{aligned}$$

since

$$\begin{aligned} \log|W(G_1,\dots,G_{n+1})| &= \sum_{j=1}^{n+1}\log|G_j| + \log\left|\frac{W(G_1,\dots,G_{n+1})}{G_1\cdots G_{n+1}}\right| \\ &\le \sum_{j=1}^{n+1}\log|G_j| + \log^+\left|\frac{W(G_1,\dots,G_{n+1})}{G_1\cdots G_{n+1}}\right| \end{aligned}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{W(G_1,\dots,G_{n+1})}{G_1\cdots G_{n+1}}(re^{i\theta})\right|d\theta = S(r,f)$$

as in [1, pp. 14–15]. Therefore, there is a subset E of $(0,\infty)$ for which

$$\limsup_{r\to\infty,\, r\notin E}\frac{N(r,1/W)}{T(r,f)} \le n+1-\sum_{j=1}^{n+1}\delta(\boldsymbol{u}_j,f),$$

where E is empty when $\rho(f)<\infty$ and E is of finite linear measure when $\rho(f)=\infty$.

3. On the μ_n-defect relation

Let f, X, etc. be as in Section 1 or 2. In this section we shall consider the μ_n-defect relation for f over X.

Proposition 3.1. *For any vectors* $\boldsymbol{a}_1,\dots,\boldsymbol{a}_q \in X$ $(2N-n+1 \le q < \infty)$ *we have the inequality*

$$\sum_{j=1}^{q}\omega(j)\mu_n(\boldsymbol{a}_j,f) \le n+1.$$

Proof. As $0 \le \mu_n(\boldsymbol{a},f) \le \delta_n(\boldsymbol{a},f) \le 1$ for any vector $\boldsymbol{a}$ in $\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$ by Corollary 2.2(a) and $\omega(j)>0$, this proposition is a direct consequence of Lemma 2.7. □

We put for any positive integer k

$$M_k^+(X,f) = \{\boldsymbol{a} \in X \mid \mu_k(\boldsymbol{a},f) > 0\} \quad \text{and} \quad M_k^1(X,f) = \{\boldsymbol{a} \in X \mid \mu_k(\boldsymbol{a},f) = 1\}.$$

It is known that
(3.a) ([12, Theorem 4.1]) $\# M_n^+(X,f) \le (n+1)(2N-n+1)$;
(3.b) ([12, Proposition 4.2]) $\# M_n^1(X,f) \le N + N/n$.

Note 3.1. As an improvement of (3.a), we have the inequality

$$\# M_n^+(X,f) + n\# M_n^1(X,f) \le (n+1)(2N-n+1).$$

Proof. From the μ_n-defect relation (Theorem 1.C) we have the inequality

$$\begin{aligned}\sum_{\boldsymbol{a} \in M_n^+(X,f)} \mu_n(\boldsymbol{a},f) &= \sum_{\boldsymbol{a} \in M_n^+(X,f)-M_n^1(X,f)} \mu_n(\boldsymbol{a},f) + \sum_{\boldsymbol{a} \in M_n^1(X,f)} \mu_n(\boldsymbol{a},f)\\ &= \sum_{\boldsymbol{a} \in M_n^+(X,f)-M_n^1(X,f)} \mu_n(\boldsymbol{a},f) + \# M_n^1(X,f) \le 2N-n+1,\end{aligned}$$

from which we obtain the inequality

$$\#(M_n^+(X,f) - M_n^1(X,f))/(n+1) + \# M_n^1(X,f) \le 2N-n+1 \tag{4}$$

since $\mu_n(\boldsymbol{a},f) \ge 1/(n+1)$ for $\boldsymbol{a} \in M_n^+(X,f) - M_n^1(X,f)$ ([14]). The inequality (4) reduces to our result. □

We put $M_n^+(X,f) = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_q\}$.

PROPOSITION 3.2. *Suppose that*

$$\sum_{j=1}^{q} \mu_n(\boldsymbol{a}_j,f) = 2N-n+1. \tag{5}$$

Then,
(a) $\sum_{j=1}^q \delta_n(\boldsymbol{a}_j,f) = 2N-n+1$.
(b) $\mu_n(\boldsymbol{a}_j,f) = \delta_n(\boldsymbol{a}_j,f)$ $(1 \le j \le q)$.

Proof. (a) By Corollary 2.2(a), Theorem 1.A(II) and the assumption (5) we obtain (a) immediately.

(b) This is a direct consequence of Corollary 2.2(a) for $k = n$, the assumption (5) and (a) of this proposition. □

PROPOSITION 3.3. *Suppose that*
(i) $N > n \ge 1$;
(ii) $\sum_{j=1}^q \mu_n(\boldsymbol{a}_j,f) = 2N-n+1$.
Then, there is a subset E of $(0,\infty)$ such that for any $\boldsymbol{a}_j \in M_n^+(X,f) - M_n^1(X,f)$ and for any sequence $\{r_\nu\}_{\nu=1}^\infty \subset (0,\infty) - E$ tending to $+\infty$, it holds that

$$\liminf_{\nu\to\infty} \frac{N(r_\nu, \boldsymbol{a}_j, f)}{T(r_\nu, f)} = 1,$$

where E is empty when $\rho(f) < \infty$ and E is of finite linear measure when $\rho(f) = \infty$.

Proof. By the assumption (ii) we have from Proposition 3.2(b) that

$$\delta_n(\boldsymbol{a}_j, f) = \mu_n(\boldsymbol{a}_j, f) \quad (j = 1, \dots, q).$$

Let $\boldsymbol{a}_j \in M_n^+(X, f) - M_n^1(X, f)$. Then, by Corollary 2.1 and Lemma 2.10 for $k = n$ we have that

$$\frac{n}{\max(m_j, n)} = \limsup_{r\to\infty} \frac{N_n(r, \boldsymbol{a}_j, f)}{T(r, f)} = \lim_{\nu\to\infty} \frac{N_n(r_\nu, \boldsymbol{a}_j, f)}{T(r_\nu, f)}$$
$$\leq \liminf_{\nu\to\infty} \frac{n}{\max(m_j, n)} \frac{N(r_\nu, \boldsymbol{a}_j, f)}{T(r_\nu, f)} \leq \frac{n}{\max(m_j, n)},$$

where m_j is the multilicitiy of $\boldsymbol{a}_j$ with respect to f. This inequality implies that this proposition holds. □

PROPOSITION 3.4. *Suppose that*
(i) $N > n \geq 1$;
(ii) $\sum_{j=1}^q \mu_n(\boldsymbol{a}_j, f) = 2N - n + 1$.
Then, we have the inequality

$$n + 1 = \sum_{j=1}^q \omega(j)\delta_n(\boldsymbol{a}_j, f) \leq \limsup_{r\to\infty; r\notin E} \frac{N(r, 1/W)}{T(r, f)} + \sum_{\{j \mid \boldsymbol{a}_j \in M_n^1(X, f)\}} \omega(j),$$

where E is as in Proposition 3.3.

Proof. First we note that

$$\limsup_{r\to\infty; r\notin E} \frac{N(r, 1/W)}{T(r, f)} \leq n + 1$$

by Lemma 2.11.

By the assumption (ii) we have from Proposition 3.2(a) that

$$\sum_{j=1}^q \delta_n(\boldsymbol{a}_j, f) = 2N - n + 1,$$

and so from Lemma 2.8 we have that

$$n + 1 = \sum_{j=1}^q \omega(j)\delta_n(\boldsymbol{a}_j, f). \tag{6}$$

Next, from Lemma 2.6, we have the inequality

$$\sum_{j=1}^{q} \omega(j) N(r, \boldsymbol{a}_j, f) \leq N\left(r, \frac{1}{W}\right) + \sum_{j=1}^{q} \omega(j) N_n(r, \boldsymbol{a}_j, f) \quad (r \geq 1). \tag{7}$$

Let $\{r_\nu\}_{\nu=1}^{\infty} \subset (0, \infty) - E$ be any sequence tending to $+\infty$. Then from (7) we obtain the inequality

$$\begin{aligned}\sum_{j=1}^{q} \omega(j) \liminf_{\nu\to\infty} \frac{N(r_\nu, \boldsymbol{a}_j, f)}{T(r_\nu, f)} &\leq \liminf_{\nu\to\infty} \frac{N(r_\nu, 1/W)}{T(r_\nu, f)} + \sum_{j=1}^{q} \omega(j) \limsup_{\nu\to\infty} \frac{N_n(r_\nu, \boldsymbol{a}_j, f)}{T(r_\nu, f)} \\ &\leq \limsup_{r\to\infty;\, r\notin E} \frac{N(r, 1/W)}{T(r, f)} + \sum_{j=1}^{q} \omega(j) \limsup_{r\to\infty} \frac{N_n(r, \boldsymbol{a}_j, f)}{T(r, f)},\end{aligned}$$

from which we obtain our proposition by Proposition 3.3 and (6) since $N(r, \boldsymbol{a}_j, f) = 0$ for $\boldsymbol{a}_j \in M_n^1(X, f)$. □

From now on throughout this section we suppose that $N > n \geq 2$. In [12] we proved that
"*If* $\# M_n^1(X, f) \geq N + 1$, *the* μ_n*-defect relation is not extremal*:

$$\sum_{\boldsymbol{a} \in M_n^+(X, f)} \mu_n(\boldsymbol{a}, f) < 2N - n + 1."$$

One of the main purposes of this section is to give a generalization of this result. We put

$$D_n^+(X, f) = \{\boldsymbol{a} \in X \mid \delta_n(\boldsymbol{a}, f) > 0\} \quad \text{and} \quad D_n^1(X, f) = \{\boldsymbol{a} \in X \mid \delta_n(\boldsymbol{a}, f) = 1\}.$$

THEOREM 3.1. *If there exists one* $\boldsymbol{a}_o \in X - M_n^1(X, f)$ *satisfying* $\delta(\boldsymbol{a}_o, f) > 0$, *then the* μ_n*-defect relation for* f *over* X *is not extremal.*

Proof. By Corollary 2.2(a) for $k = n$ we have that

$$M_n^+(X, f) \subset D_n^+(X, f), \quad \boldsymbol{a}_o \in D_n^+(X, f) \quad \text{and} \quad \mu_n(\boldsymbol{a}_o, f) < \delta_n(\boldsymbol{a}_o, f),$$

from which and by Theorem 1.A(II) we obtain the inequality

$$\sum_{\boldsymbol{a} \in M_n^+(X, f)} \mu_n(\boldsymbol{a}, f) < \sum_{\boldsymbol{a} \in D_n^+(X, f)} \delta_n(\boldsymbol{a}, f) \leq 2N - n + 1.$$ □

For any subset A of X, we denote by $\dim(A)$ the dimension of the vector space spanned by elements of A.

THEOREM 3.2. *Suppose that*
(i) $N > n \geq 2$;
(ii) $\sum_{j=1}^{q} \mu_n(\boldsymbol{a}_j, f) = 2N - n + 1$.

Then, we have that

$$\# M_n^1(X,f) \le (2N-n+1)/2 \quad \textit{and} \quad \dim(M_n^1(X,f)) \le (n+1)/2.$$

Proof. By Theorem 1.D, it holds that $\dim(M_n^1(X,f)) \le n$ and $\# M_n^1(X,f) \le N$, which implies that $q > 2N-n+1$ by the assumption (ii). As there is nothing to prove when $\dim(M_n^1(X,f)) = 0$, we suppose that $\dim(M_n^1(X,f)) \ge 1$. By Proposition 3.4 we have the inequality

$$n+1 = \sum_{j=1}^{q} \omega(j)\delta_n(\boldsymbol{a}_j, f) \le \limsup_{r\to\infty;\, r\notin E} \frac{N(r,1/W)}{T(r,f)} + \sum_{\{j \mid \mu_n(\boldsymbol{a}_j,f)=1\}} \omega(j), \tag{8}$$

where E is as in Proposition 3-3.

In Lemma 2.11, let $\boldsymbol{u}_1,\dots,\boldsymbol{u}_{n+1}$ be linearly independent $n+1$ vectors in $\boldsymbol{C}^{n+1}$ such that $\boldsymbol{u}_1,\dots,\boldsymbol{u}_p$ are in $M_n^1(X,f)$, where $p = \dim(M_n^1(X,f))$. Then, we have

$$\limsup_{r\to\infty;\, r\notin E} \frac{N(r,1/W)}{T(r,f)} \le n+1-\dim(M_n^1(X,f)), \tag{9}$$

since $\delta(\boldsymbol{u}_j, f) = 1$ for $\boldsymbol{u}_j \in M_n^1(X,f)$.

From (8), (9) and from Lemma 2.2(d) we obtain

$$\dim(M_n^1(X,f)) = \sum_{\boldsymbol{a}_j \in M_n^1(X,f)} \omega(j) \tag{10}$$

as $\# M_n^1(X,f) \le N$. Further, by Proposition 3.2 we have the equality

$$\sum_{j=1}^{q} \delta_n(\boldsymbol{a}_j, f) = 2N-n+1 \tag{11}$$

and the relation $\delta_n(\boldsymbol{a}_j,f) = \mu_n(\boldsymbol{a}_j,f)$ $(j=1,\dots,q)$, so that we have

$$D_n^+(X,f) = M_n^+(X,f) \quad \text{and} \quad D_n^1(X,f) = M_n^1(X,f). \tag{12}$$

From (6), (10) and (12) we obtain the equality

$$n+1-\dim(D_n^1(X,f)) = \sum_{\boldsymbol{a}_j \in D_n^+(X,f)-D_n^1(X,f)} \omega(j)\delta_n(\boldsymbol{a}_j,f),$$

which is equal to

$$\frac{1}{\theta}\{2N-n+1-\# D_n^1(X,f)\}$$

by (11) since $\theta\omega(j) = 1$ for j satisfying $\boldsymbol{a}_j \notin D_n^1(X,f)$ by Lemma 2.8. From this relation we obtain that

$$\theta = \frac{2N-n+1-\# D_n^1(X,f)}{n+1-\dim(D_n^1(X,f))}.$$

Note that $\dim(D_n^1(X,f)) = \dim(M_n^1(X,f)) \le n$. As $\theta \le (2N-n+1)/(n+1)$ by Lemma 2.2(c), we obtain the inequality

$$\frac{\#D_n^1(X,f)}{\dim(D_n^1(X,f))} \ge \frac{2N-n+1}{n+1}. \tag{13}$$

As $\#D_n^1(X,f) = \#M_n^1(X,f) \le N$, we have the inequality

$$\#D_n^1(X,f) - \dim(D_n^1(X,f)) \le N-n \tag{14}$$

by Lemma 2.1. From (13) and (14) we obtain that

$$\#M_n^1(X,f) = \#D_n^1(X,f) \le (2N-n+1)/2$$

and

$$\dim(M_n^1(X,f)) = \dim(D_n^1(X,f)) \le (n+1)/2,$$

which are to be proved. □

Corollary 3.1. *Suppose that* $N > n \ge 2$. *If either* $\#M_n^1(X,f) > (2N-n+1)/2$ *or* $\dim(M_n^1(X,f)) > (n+1)/2$ *holds, then the* μ_n*-defect relation for* f *over* X *is not extremal.*

Remark 3.1. When $n=1$, Theorem 3.2 or Corollary 3.1 does not hold as Example 6.2 shows.

4. Extremal case of the μ_n-defect relation I: $n = 2m$

Let f, X, $\delta_n(\boldsymbol{a}, f)$, $\mu_n(\boldsymbol{a}, f)$, etc. be as in Section 1, 2 or 3.

Lemma 4.1 (see [10, Theorem 5.1 and its proof]). *Suppose that*
(i) $N > n = 2m$ $(m \in \boldsymbol{N})$;
(ii) *there exist vectors* $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q \in X$ $(q < \infty)$ *satisfying*

$$\sum_{j=1}^{q} \delta_n(\boldsymbol{a}_j, f) = 2N - n + 1.$$

Then, there exists a non-empty subset P_0 *of* $Q = \{1, 2, \dots, q\}$ *satisfying*
(a) $d(P_0)/\#P_0 < (n+1)/(2N-n+1)$;
(b) $\delta_n(\boldsymbol{a}_j, f) = 1$ $(j \in P_0)$.
In particular,

$$\#\{j \in Q \,|\, \delta_n(\boldsymbol{a}_j, f) = 1\} > (2N-n+1)/(n+1).$$

Let $q = \#M_n^+(X,f)$ *and* $M_n^+(X,f) = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_q\}$. *Then by* (3.a) $q \le (n+1)(2N-n+1)$.

Theorem 4.1. *Suppose that*

(i) $N > n = 2m$ $(m \in \boldsymbol{N})$;
(ii) $\sum_{j=1}^{q} \mu_n(\boldsymbol{a}_j, f) = 2N - n + 1$.
Then, there exists a non-empty subset P_0 *of* $Q = \{1, 2, \dots, q\}$ *satisfying*
(a) $d(P_0)/\#P_0 < (n+1)/(2N - n + 1)$;
(b) $\mu_n(\boldsymbol{a}_j, f) = 1$ $(j \in P_0)$.
In particular,

$$\#\{j \in Q \mid \mu_n(\boldsymbol{a}_j, f) = 1\} > (2N - n + 1)/(n + 1).$$

Proof. We note that from (ii) and Theorem 1.D, the number q must satisfy the inequality $2N - n + 1 < q \le (n+1)(2N - n + 1)$.

From Proposition 3.2(a), Lemma 4.1 and Proposition 3.2(b) we obtain this theorem. □

Note 4.1. Let P_0 be the subset of Q given in Theorem 4.1. Then, there are at least two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\{\boldsymbol{a}_j \mid j \in P_0\}$ satisfying $\boldsymbol{a} = c\boldsymbol{b}$ $(c \ne 0)$.

Proof. From the inequality

$$\#P_0 - d(P_0) > \left(\frac{2N - n + 1}{n + 1} - 1\right) d(P_0) = \frac{2(N - n)}{n + 1} d(P_0) > 0,$$

we have that $\#P_0 \ge d(P_0) + 1$.

(a) When $d(P_0) = 1$, our conclusion is trivial.

(b) When $d(P_0) \ge 2$. We suppose that $\boldsymbol{b}_1, \dots, \boldsymbol{b}_{d(P_0)}$ are linearly independent vectors belonging to $\{\boldsymbol{a}_j \mid j \in P_0\}$. Then, any vector $\boldsymbol{a} \in \{\boldsymbol{a}_j \mid j \in P_0\} - \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_{d(P_0)}\}$ can be represented by $\boldsymbol{b}_1, \dots, \boldsymbol{b}_{d(P_0)}$ as a linear combination over $\boldsymbol{C}$:

$$\boldsymbol{a} = c_1 \boldsymbol{b}_1 + \cdots + c_{d(P_0)} \boldsymbol{b}_{d(P_0)}.$$

From this relation, we obtain

$$(\boldsymbol{a}, f) = \sum_{v=1}^{d(P_0)} c_v (\boldsymbol{b}_v, f). \tag{15}$$

As $\boldsymbol{a}, \boldsymbol{b}_1, \dots, \boldsymbol{b}_{d(P_0)} \in M_n^1(X, f)$, from (15) we obtain that there is v_0 $(1 \le v_0 \le d(P_0))$ such that

$$(\boldsymbol{a}, f) = c_{v_0} (\boldsymbol{b}_{v_0}, f) \quad (c_{v_0} \ne 0)$$

by a Borel's theorem (see [1, 1^o, p. 19]). This relation reduces to $\boldsymbol{a} = c_{v_0} \boldsymbol{b}_{v_0}$ since f is linearly non-degenerate over $\boldsymbol{C}$.

Corollary 4.1. *Suppose that* $N > n = 2m$. *If any two vectors in* X *are linearly independent, then for any linearly non-degenerate and transcendental holomorphic curve* f *from* $\boldsymbol{C}$ *into* $P^n(\boldsymbol{C})$, *the* μ_n*-defect relation for* f *over* X *is not extremal.* □

5. Extremal case of the μ_n-defect relation II: $n = 2m - 1$

Let n be odd and f, X, $\delta_n(\boldsymbol{a}, f)$, $\mu_n(\boldsymbol{a}, f)$, etc. be as in Section 1, 2 or 3. The purpose of this section is to give a result when the μ_n-defect relation is extremal.

Lemma 5.1 ([13, Theorem 3.1]). *Suppose that*
(i) $N > n = 2m - 1$ $(m \in \boldsymbol{N})$;
(ii) $\delta_n(\boldsymbol{a}_j, f) > 0$ $(j = 1, \dots, q; q < \infty)$ *and*

$$\sum_{j=1}^{q} \delta_n(\boldsymbol{a}_j, f) = 2N - n + 1.$$

Then, for the set $Q = \{1, \dots, q\}$, *either* (I) *or* (II) *given below holds*:
(I) $\#\{j \in Q \mid \delta_n(\boldsymbol{a}_j, f) = 1\} > (2N - n + 1)/(n + 1)$.
(II) q *is divisible by* $N - m + 1$ *and for* $p = q/(N - m + 1)$, *there are mutually disjoint subsets* $M_1, \dots, M_p$ *of* Q *satisfying*
(a) $Q = \bigcup_{k=1}^{p} M_k$; (b) $d(M_k) = m$, $\#M_k = N - m + 1$ $(1 \le k \le p)$ *and*
(c) *any* m *elements of* $\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_q\}$ *are linearly independent.*

By using this lemma, we obtain the following result when the μ_n-defect relation is extremal. Let $q = \#M_n^+(X, f)$, then $q \le (n+1)(2N - n + 1)$ by (3.a) and we put $M_n^+(X, f) = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_q\}$.

Theorem 5.1. *Suppose that*
(i) $N > n = 2m - 1$ $(m \in \boldsymbol{N})$;
(ii) $\sum_{j=1}^{q} \mu_n(\boldsymbol{a}_j, f) = 2N - n + 1$.
Then, for the set $Q = \{1, \dots, q\}$, *either* (I) *or* (II) *given below holds*:
(I) $\#\{j \in Q \mid \mu_n(\boldsymbol{a}_j, f) = 1\} > (2N - n + 1)/(n + 1)$.
(II) q *is divisible by* $N - m + 1$ *and for* $p = q/(N - m + 1)$, *there are mutually disjoint subsets* $M_1, \dots, M_p$ *of* Q *satisfying*
(a) $Q = \bigcup_{k=1}^{p} M_k$; (b) $d(M_k) = m$, $\#M_k = N - m + 1$ $(1 \le k \le p)$ *and*
(c) *any* m *elements of* $\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_q\}$ *are linearly independent.*

Proof. By the assumptions (i), (ii) and Proposition 3.2 we can apply Lemma 5.1 to this case to obtain the result immediately. □

Remark 5.1. The case (II) occurs. We shall give an example for (II) of this theorem when $m = 1$ in Section 6.

6. Example

In this section we shall give some examples of holomorphic curves.

Example 6.1. There exists a transcendental holomorphic curve f from $\boldsymbol{C}$ into $P^n(\boldsymbol{C})$ and a subset X of $\boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$ in N-subgeneral position whose μ_k-defect relation over X is divergent when $1 \le k \le n - 1$, where $N > n \ge 2$.

Proof. We apply the method used in [5]. Let h_1 and h_2 be entire functions without common zeros such that the meromorphic function $h = h_1/h_2$ is transcendental. We put

$$f_{j+1} = {}_nC_j h_1^{n-j} h_2^j \quad (j = 0, \dots, n).$$

Then, (a) $f_1, \dots, f_{n+1}$ are entire functions without common zeros.

(b) $f_1, \dots, f_{n+1}$ are linearly independent over $\boldsymbol{C}$.

(c) For the curve $f = [f_1, \dots, f_{n+1}]$, we have the relation $T(r, f) = nT(r, h) + O(1)$, so that f is transcendental.

For $\boldsymbol{e}_1 = (1, 0, \dots, 0) \in \boldsymbol{C}^{n+1}$, let

$$X_0 = \{(a^n, a^{n-1}, \dots, a, 1) \mid a \in \boldsymbol{C}\} \cup \{\boldsymbol{e}_1\}$$

and

$$X = X_0 \cup \{v\boldsymbol{e}_1 \mid v = 2, \dots, N - n + 1\}.$$

Then, (d) X_0 is in general position and X is in N-subgeneral position.

Here, we prove (a), (b), (c) and (d) briefly.

(a) As h_1 and h_2 have no common zeros, h_1^n and h_2^n have no common zeros, so that $f_1, \dots, f_{n+1}$ are entire functions without common zeros.

(b) Let $\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_{n+1} f_{n+1} = 0$. Then, we have

$$h_2^n(\alpha_1 h^n + \alpha_{2n}C_1 h^{n-1} + \cdots + \alpha_{nn}C_{n-1} h + \alpha_{n+1}) = 0. \tag{16}$$

As h is transcendental and meromorphic, $h^n, \dots, h, 1$ are linearly independent over $\boldsymbol{C}$, so that from (16)

$$\alpha_1 = \alpha_{2n}C_1 = \cdots = \alpha_{nn}C_{n-1} = \alpha_{n+1} = 0;$$

namely $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha_{n+1} = 0$. This means that $f_1, \dots, f_{n+1}$ are linearly independent over $\boldsymbol{C}$.

(c) By using the following inequality for a positive constant K

$$\begin{aligned} \max\{|h_1(z)|^n, |h_2(z)|^n\} \le U(z) &= \max_{1 \le j \le n+1} |f_j(z)| \\ &= \max_{0 \le j \le n} {}_nC_j |h_1(z)|^{n-j} |h_2(z)|^j \\ &\le K \max\{|h_1(z)|^n, |h_2(z)|^n\} \end{aligned}$$

we have the relation

$$\begin{aligned} T(r, f) &= \frac{1}{2\pi} \int_0^{2\pi} \log U(re^{i\theta})\, d\theta + O(1) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log \max\{|h_1(re^{i\theta})|^n, |h_2(re^{i\theta})|^n\}\, d\theta + O(1) \\ &= \frac{n}{2\pi} \int_0^{2\pi} \log \max\{|h_1(re^{i\theta})|, |h_2(re^{i\theta})|\}\, d\theta + O(1) \\ &= nT(r, h) + O(1). \end{aligned}$$

(d) As any $n+1$ vectors in X_0 are linearly independent, X_0 is in general position.

Let A be any subset of X containing $N+1$ elements. As $\#\{A \cap X_0\} \geq n+1$, the set A contains $n+1$ linearly independent vectors. The holomorphic curve f, X_0 and X satisfy Example 6-1. In fact, we put $\boldsymbol{a}(a) = (a^n, a^{n-1}, \dots, a, 1)$ $(a \in \boldsymbol{C})$. Then, for any $a \in \boldsymbol{C}$ the multiplicity of $\boldsymbol{a}(a)$ with respect to f is at least n since

$$(\boldsymbol{a}(a), f) = a^n f_1 + a^{n-1} f_2 + \cdots + a f_n + f_{n+1}$$
$$= (ah_1)^n + {}_nC_1 (ah_1)^{n-1} h_2 + \cdots + {}_nC_{n-1}(ah_1) h_2^{n-1} + h_2^n = (ah_1 + h_2)^n.$$

Let $m(a)$ be the multiplicity of $\boldsymbol{a}(a)$ with respect to f, then, $m(a) \geq n$ and we have that for $1 \leq k \leq n-1$

$$\mu_k(\boldsymbol{a}(a), f) = \left(1 - \frac{k}{m(a)}\right)^+ \geq 1 - \frac{k}{n} \geq \frac{1}{n} \tag{17}$$

for any $a \in \boldsymbol{C}$. By (17) we have that for $1 \leq k \leq n-1$

$$\infty = \sum_{a \in \boldsymbol{C}} \mu_k(\boldsymbol{a}(a), f) \leq \sum_{\boldsymbol{a} \in X_0} \mu_k(\boldsymbol{a}, f) \leq \sum_{\boldsymbol{a} \in X} \mu_k(\boldsymbol{a}, f).$$

Example 6.2. Let $f = [e^z, 1]$ and $X = \{v(a,1) \mid a \in \boldsymbol{C}; v = 1, \dots, N\} \cup \{v(1,0) \mid v = 1, \dots, N\}$. Then, f is transcendental from $\boldsymbol{C}$ into $P^1(\boldsymbol{C})$ and X is in N-subgeneral position, where $N > 1$. In this case, $n = 1$. We put $\boldsymbol{a}_v = v(1,0)$ and $\boldsymbol{b}_v = v(0,1)$ for $v = 1, \dots, N$. Then, we obtain that for $v = 1, \dots, N$

$$\mu_1(\boldsymbol{a}_v, f) = \mu_1(\boldsymbol{b}_v, f) = 1,$$

and so

$$\sum_{\boldsymbol{a} \in X} \mu_1(\boldsymbol{a}, f) = \sum_{v=1}^{N} \{\mu_1(\boldsymbol{a}_v, f) + \mu_1(\boldsymbol{b}_v, f)\} = 2N. \qquad \square$$

Example 6.3. Let $f = [\cos z, 1]$ and $X = \{v(a,1) \mid a \in \boldsymbol{C}; v = 1, \dots, N\} \cup \{v(1,0) \mid v = 1, \dots, N\}$. Then, f is transcendental from $\boldsymbol{C}$ into $P^1(\boldsymbol{C})$ and X is in N-subgeneral position, where $N > 1$. In this case, $n = 1$. We put for $v = 1, \dots, N$

$$\boldsymbol{a}_v = v(1,1), \quad \boldsymbol{b}_v = v(-1,1) \quad \text{and} \quad \boldsymbol{c}_v = v(0,1)$$

Then, we obtain that for $v = 1, \dots, N$

$$\mu_1(\boldsymbol{a}_v, f) = \mu_1(\boldsymbol{b}_v, f) = 1/2 \quad \text{and} \quad \mu_1(\boldsymbol{c}_v, f) = 1,$$

and so

$$\sum_{\boldsymbol{a} \in X} \mu_1(\boldsymbol{a}, f) = \sum_{v=1}^{N} \{\mu_1(\boldsymbol{a}_v, f) + \mu_1(\boldsymbol{b}_v, f) + \mu_1(\boldsymbol{c}_v, f)\} = 2N. \qquad \square$$

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