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THE BOREL DIRECTION OF THE LARGEST TYPE OF ALGEBROID FUNCTIONS DEALING WITH MULTIPLE VALUES*

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Abstract

In this paper, we prove that for algebroid functions of order ρ satisfying that $0 < \rho < +\infty$, there exists a Borel direction of the largest type dealing with multiple values and there is a sequence of filling disks in this direction.

1. Introduction and main results

The value distribution theory of meromorphic functions due to R. Nevanlinna (see [1], [4] for standard references) was extended to the corresponding theory of algebroid functions by H. Selberg [8, 9], E. Ullrich [14] and G. Valiron [15] around 1930. The singular direction is one of the main objects studied in the theory of value distribution of algebroid functions. Several types of singular directions have been introduced in [1]. Their existence and some connections between them have also been established. G. Valiron [16] conjectured that there exists at least a Borel direction for any ν-valued algebroid function of finite positive order. A. Rauch [7] proved that there exists a direction such that the corresponding Borel exceptional values form a set of linear measure zeros. N. Toda [11] proved that there exists a direction such that the set of corresponding Borel exceptional values is countable. Later Y. Lü and Y. Gu [6] proved that there exists a direction such that the number of Borel exceptional values is equal to 2ν at most.

For algebroid functions of finite positive order defined in z-plane, D. Sun in [10] proved the existence of the sequence of filling disks, Z. Gao proved in [2] that for such algebroid functions there exists a Borel direction of the largest type and there is a sequence of filling disks in this direction. In this paper, we will consider the existence of the Borel direction of the largest type and the sequence of filling disks dealing with multiple values.

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Let $w = w(z)$ $(z \in \mathbf{C})$ be the ν-valued algebroid function defined by the irreducible equation

$$A_\nu(z)w^\nu + A_{\nu-1}(z)w^{\nu-1} + \cdots + A_0(z) = 0, \tag{1}$$

where $A_\nu(z), \dots, A_0(z)$ are entire functions without any common zeros. The single valued domain of definition of $w(z)$ is a ν-valued covering of the z-plane, a Riemann surface, denoted by $\tilde{R}_z$.

A point in $\tilde{R}_z$ is denoted by $\tilde{z}$ if its projection in the z-plane is z. The open set which lies over $|z| < r$ is denoted by $|\tilde{z}| < r$. Let $n(r, a)$ be the number of zeros, counted according to their multiplicities, of $w(z) - a$ in $|\tilde{z}| \le r$, $\bar{n}^{l)}(r, a)$ be the number of distinct zeros with multiplicity $\le l$ of $w(z) - a$ in $|\tilde{z}| \le r$. Let

$$S(r, w) = \frac{1}{\pi}\iint_{|\tilde{z}| \le r}\left[\frac{|w'(z)|}{1 + |w(z)|^2}\right]^2 d\omega = \frac{1}{\pi}\int_0^{2\pi}\int_0^r \left(\frac{|w'(re^{i\theta})|}{1 + |w(re^{i\theta})|^2}\right)^2 r\,drd\theta,$$

$$T(r, w) = \frac{1}{\nu}\int_o^r \frac{S(t, w)}{t}\,dt,$$

$$N(r, a) = \frac{1}{\nu}\int_0^r \frac{n(t, a) - n(0, a)}{t}\,dt + \frac{n(0, a)}{\nu}\log r,$$

$$\overline{N}^{l)}(r, a) = \frac{1}{\nu}\int_0^r \frac{\bar{n}^{l)}(t, a) - \bar{n}^{l)}(0, a)}{t}\,dt + \frac{\bar{n}^{l)}(0, a)}{\nu}\log r,$$

$$m(r, w) = \frac{1}{2\pi\nu}\int_{|\tilde{z}| = r}\log^+|w(re^{i\theta})|\,d\theta, \quad z = re^{i\theta},$$

where $|\tilde{z}| = r$ is the boundary of $|\tilde{z}| \le r$. Moreover, $S(r, w)$ is called the mean covering number of $|\tilde{z}| \le r$ into w-sphere. We call $T(r, w)$ the characteristic function of $w(z)$. It is known from [5, pp. 84] that $T(r, w) = m(r, w) + N(r, \infty) + O(1)$.

The order of the algebroid function $w(z)$ is defined by

$$\rho = \limsup_{r \to \infty}\frac{\log T(r, w)}{\log r}.$$

Let $n(r, \tilde{R}_z)$ be the number of the branch points of $\tilde{R}_z$ in $|\tilde{z}| \le r$, counted with the order of branch and

$$N(r, \tilde{R}_z) = \frac{1}{\nu}\int_0^r \frac{n(t, \tilde{R}_z) - n(0, \tilde{R}_z)}{t}\,dt + \frac{n(0, \tilde{R}_z)}{\nu}\log r.$$

From [5] or [11], we know that $N(r, \tilde{R}_z) \le 2(\nu - 1)T(r, w) + O(1)$.

We define an angular domain

$$\Delta(\theta_0, \varepsilon) = \{z \,|\, |\arg z - \theta_0| < \varepsilon\}, \quad 0 < \varepsilon < \frac{\pi}{2}.$$

The part of $\tilde{R}_z$ which lies over $\Delta(\theta_0, \varepsilon)$ is denoted by $\tilde{\Delta}(\theta_0, \varepsilon)$. Let $n(r, \Delta(\theta_0, \varepsilon), a)$ be the number of the zeros of $w(z) - a$ in $\tilde{\Delta}(\theta_0, \varepsilon) \cap \{|\tilde{z}| \leq r\}$ and $n(r, \Delta(\theta_0, \varepsilon), \tilde{R}_z)$ be the number of the branch points in the same region. Similarly, we can define $\bar{n}^{l)}(r, \Delta(\theta_0, \varepsilon), a)$.

Now we give two definitions.

Definition 1 ([3]). Let $w = w(z)$ $(z \in \mathbf{C})$ be the v-valued algebroid function of order ρ $(0 < \rho < +\infty)$ defined by (1) and $l(\geq 2v + 1)$ be a positive integer. For arbitrary $\varepsilon > 0$ $\left(0 < \varepsilon < \frac{\pi}{2}\right)$, if

$$\limsup_{r \to \infty} \frac{\log^+ \bar{n}^{l)}(r, \Delta(\theta_0, \varepsilon), a)}{\log r} = \rho$$

holds for any complex value a except at most $2v$ possible exceptions, then the half line $B : \arg z = \theta_0$ $(0 \leq \theta_0 < 2\pi)$ is called a *Borel direction dealing with multiple values of* $w(z)$.

G. Valiron is the first one to introduce the concept of a proximate order $\rho(r)$ for a meromorphic function w with finite positive order and $U(r) = r^{\rho(r)}$ is called type function of f or $T(r, w)$ such that $\rho(r)$ is nondecreasing, piecewise continuous and differentiable, and

$$\lim_{r \to \infty} \rho(r) = \rho \quad \text{and} \quad \limsup_{r \to \infty} \frac{T(r, w)}{U(r)} = 1$$

and for a fixed positive number d

$$\lim_{r \to \infty} \frac{U(dr)}{U(r)} = d^{\rho}.$$

For an algebroid function w of finite positive order, we can use the same method to get its type function $U(r)$.

Definition 2. As Definition 1, for arbitrary $\varepsilon > 0$ $\left(0 < \varepsilon < \frac{\pi}{2}\right)$, if

$$\limsup_{r \to \infty} \frac{\bar{n}^{l)}(r, \Delta(\theta_0, \varepsilon), a)}{U(r)} > 0$$

holds for any complex value a except at most $2v$ possible exceptions, then the half line $B : \arg z = \theta_0$ $(0 \leq \theta_0 < 2\pi)$ is called a *Borel direction of the largest type dealing with multiple values of* $w(z)$.

In this paper, ρ is a constant and satisfies $0 < \rho < \infty$, C is a positive constant and it may be of different meaning when it appears in different position. Our main results are

Theorem 1. *Let $w = w(z)$ $(z \in \mathbf{C})$ be the v-valued algebroid function of order ρ defined by (1), $l(\geq 2v + 1)$ be a positive integer, then there exists a Borel direction of the largest type dealing with multiple values of $w(z)$.*

THEOREM 2. *Let* $w = w(z)$ $(z \in \mathbf{C})$ *be the* ν*-valued algebroid function of order* ρ *defined by* (1) *and* $l(\geq 2\nu + 1)$ *be a positive integer. If* $B : \arg z = \theta_0$ $(0 \leq \theta_0 < 2\pi)$ *is a Borel direction of the largest type dealing with multiple values of* $w(z)$, *then, in this direction, there exists a sequence of filling disks of* $w(z)$

$$\Gamma_n : \{|z - z_n| < r_n\sigma_n\}, \quad n = 1, 2, \ldots$$

$$z_n = r_n e^{i\theta_n},\ \lim_{n\to\infty} r_n = \infty,\ \lim_{n\to\infty} \sigma_n = 0\ \ (\sigma_n > 0),$$

such that for any complex value α

$$\bar{n}^{l)}(\Gamma_n, w = \alpha) \geq U^{1-\varepsilon_n}(|z_n|),$$

except at most countable possible values enclosed by spherical circles with radius $\delta = U^{-\frac{1}{11}}(|z_n|)$ *on the Remman sphere, where* $\varepsilon_n \to 0$ $(n \to \infty)$.

2. Proof of the theorems

First we will introduce several lemmas.

LEMMA 1 ([3, Theorem 5]). *Suppose that* $w(z)$ *is the* ν*-valued algebroid function in* $|z| < R$ *defined by* (1) *and* $l(\geq 2\nu + 1)$ *is a positive integer. Suppose* $a_1, a_2, a_3, \ldots, a_q$ $(q \geq 3)$ *are distinct points given arbitrarily in* w*-sphere and the spherical distance of any two points is no small than* $\delta \in (0, 1/2)$. *Then for any* $r \in (0, R)$, *we have*

$$\left(q - 2 - \frac{2}{l}\right) S(r, w) \leq \sum_{j=1}^{q} \bar{n}^{l)}(R, a_j) + \frac{l+1}{l} n(R, \tilde{R}_z) + \frac{CR}{(R - r)\delta^{10}}.$$

LEMMA 2. *Let* $w = w(z)$ $(z \in \mathbf{C})$ *be the* ν*-valued algebroid function of finite positive order* ρ *defined by* (1) *and* $l(\geq 2\nu + 1)$ *be a positive integer.* $\rho(r)$ *is a precise order of* $w(z)$, $U(r) = r^{\rho(r)}$. *For a complex number* a, *if*

$$\limsup_{r\to\infty} \frac{\bar{n}^{l)}(r, a)}{U(r)} = 0,$$

then

$$\limsup_{r\to\infty} \frac{\overline{N}^{l)}(r, a)}{U(r)} = 0.$$

Proof. Otherwise, there exists a positive number h such that

$$\limsup_{r\to\infty} \frac{\overline{N}^{l)}(r, a)}{U(r)} = h.$$

So there is $\{r_q\}$ satisfying $r_q \to \infty$ $(q \to \infty)$ such that

$$\lim_{q\to\infty} \frac{\overline{N}^{l)}(r_q, a)}{U(r_q)} = h.$$

Thus we have q_0 such that

$$\frac{\overline{N}^{l)}(r_q, a)}{U(r_q)} > \frac{h}{2}, \quad \text{when } q > q_0. \tag{2}$$

Furthermore,

$$\limsup_{r\to\infty} \frac{\bar{N}^{l)}(r,a)}{U(r)} \le 1, \quad \lim_{r\to\infty} \frac{U(2r)}{U(r)} = 2^{\rho},$$

and

$$\bar{n}^{l)}(r,a) \le \frac{\bar{n}^{l)}(r,a)}{\log 2}\int_r^{2r}\frac{dt}{t} \le \frac{v}{\log 2}\bar{N}^{l)}(2r,a) \quad (r \ge 1).$$

Then, for any positive number ε $(0 < \varepsilon < \rho)$, there exists a natural number N_1 such that

$$\bar{n}^{l)}(r,a) \le \frac{v}{\log 2}(1+\varepsilon)^2 2^{\rho} U(r), \tag{3}$$

when $r \ge N_1$. By $\rho(r) \to \rho$ $(r \to \infty)$, there exists N_2 such that

$$\rho - \varepsilon < \rho(r) < \rho + \frac{\varepsilon}{2}, \tag{4}$$

when $r \ge N_2$. Furthermore, there exists N_3 such that

$$\frac{\log r}{\log 2}(1+\varepsilon)^2 2^{\rho} v \frac{1}{r^{\rho-\varepsilon}-1} < \frac{h}{3}, \tag{5}$$

when $r \ge N_3$. Put $N = \max\{N_1, N_2, N_3\}$. Suppose $K > N$. Then for $q > q_0$, there exists a natural number p such that $r_q \in (K^p, K^{p+1})$. Hence we have

$$\begin{aligned} 0 < \frac{h}{2} &< \frac{\bar{N}^{l)}(r_q,a)}{U(r_q)} \\ &< \frac{1}{U(r_q)}\left(\sum_{j=1}^{p}\int_{K^j}^{K^{j+1}} \frac{\bar{n}^{l)}(t,a)}{t}dt + \bar{N}^{l)}(K,a)\right) \\ &< \frac{1}{U(K^p)}\left(\sum_{j=1}^{p} \bar{n}^{l)}(K^{j+1},a)\log K + \bar{N}^{l)}(K,a)\right) \\ &\le \frac{v}{\log 2}(1+\varepsilon)^2 2^{\rho}\log K \cdot \frac{\sum_{j=1}^{p-1} U(K^j) + 2\bar{n}^{l)}(K^{p+1},a)}{U(K^p)} \\ &\quad + \frac{\bar{N}^{l)}(K,a)}{U(K^p)} \\ &\le \frac{\log K}{\log 2}(1+\varepsilon)^2 2^{\rho} v \sum_{j=1}^{\infty}\frac{1}{K^{j(\rho-\varepsilon)}} + \frac{2\bar{n}^{l)}(K^{p+1},a)}{U(K^p)}\frac{\log K}{\log 2}\cdot(1+\varepsilon)^2 2^{\rho} v \\ &\quad + \frac{\bar{N}^{l)}(K,a)}{U(K^p)} \end{aligned} \tag{6}$$

$$= \frac{\log K}{\log 2}(1+\varepsilon)^2 2^\rho v \frac{1}{K^{\rho-\varepsilon}-1} + \frac{2\bar{n}^{l)}(K^{p+1},a)}{U(K^p)}\frac{\log K}{\log 2}\cdot(1+\varepsilon)^2 2^\rho v$$

$$+ \frac{\overline{N}^{l)}(K,a)}{U(K^p)}$$

$$< \frac{h}{3} + \frac{2\bar{n}^{l)}(K^{p+1},a)}{U(K^p)}\frac{\log K}{\log 2}\cdot(1+\varepsilon)^2 2^\rho v + \frac{\overline{N}^{l)}(K,a)}{U(K^p)}$$

Letting $\varepsilon \to 0$ in (6) we get

$$\frac{h}{6} \le \frac{\overline{N}^{l)}(K,a)}{U(K^p)} + \frac{\log K}{\log 2}\cdot 2^{\rho+1} v \frac{\bar{n}^{l)}(K^{p+1},a)}{U(K^p)}.$$

Letting $p \to \infty$ we obtain

$$\frac{h}{6} \le \limsup_{p\to\infty} \frac{\overline{N}^{l)}(K,a)}{U(K^p)} + \frac{\log K}{\log 2}\cdot 2^{\rho+1} v \limsup_{p\to\infty} \frac{\bar{n}^{l)}(K^{p+1},a)}{U(K^{p+1})}\frac{U(K^{p+1})}{U(K^p)}$$

$$\le 2^{\rho+1} v K^\rho \frac{\log K}{\log 2} \limsup_{p\to\infty} \frac{\bar{n}^{l)}(K^{p+1},a)}{U(K^{p+1})}.$$

We have

$$\limsup_{p\to\infty} \frac{\bar{n}^{l)}(K^{p+1},a)}{U(K^{p+1})} \ge \frac{\log 2}{\log K}\frac{1}{K^\rho}\frac{h}{3v\cdot 2^{\rho+2}} > 0.$$

This contradicts $\limsup_{r\to\infty} \dfrac{\bar{n}^{l)}(r,a)}{U(r)} = 0$ and we prove Lemma 2. □

LEMMA 3. *Let $w = w(z)$ $(z \in \mathbf{C})$ be the v-valued algebroid function defined by (1) and $l(\ge 2v+1)$ be a positive integer. For $0 < \varepsilon < \varepsilon_0$, $0 \le \theta_0 < 2\pi$, let*

$$\triangle_0 = \{z \mid |\arg z - \theta_0| < \varepsilon_0\},$$

$$\overline{\triangle} = \{z \mid |\arg z - \theta_0| \le \varepsilon\}.$$

The part of $\tilde{R}_z$ which lies over $\overline{\triangle} \cap \{|z| \le r\}$ is denoted by $\tilde{\overline{\triangle}}$,

$$S(r,\overline{\triangle},w) = \frac{1}{\pi}\iint_{\tilde{\overline{\triangle}}} \left[\frac{|w'(z)|}{1+|w(z)|^2}\right]^2 d\omega.$$

Then for any positive number $\lambda > 1$, any positive integer α and any q $(q \ge 3)$ distinct points $a_1, a_2, \ldots, a_q, a_i \in V$ $(i = 1, 2, \ldots, q)$, we have

$$\left(q-2-\frac{2}{l}\right)S(r,\overline{\triangle},w) \le 2\sum_{j=1}^{q} \bar{n}^{l)}(\lambda^{2\alpha}r,\triangle_0,a_j) + \frac{l+1}{l}\left(1+\frac{1}{\alpha}\right)n(\lambda^{2\alpha}r,\triangle_0,\tilde{R}_z)$$

$$+\left(q-2-\frac{2}{l}\right)S(\lambda^\alpha,\overline{\triangle},w) + \frac{2A}{\alpha\log\lambda}\frac{\log^+ r}{1-\kappa},$$

where κ is a constant satisfying $0 < \kappa < 1$ and A is a constant depending only on a_i $(i = 1, 2, \ldots, q)$.

Proof. Put $r_\mu = \lambda^{\alpha\mu}$, $r_{\mu,k} = \lambda^{\alpha\mu+k}$, $\mu = 0, 1, 2, \dots,$ $0 \le k \le \alpha - 1$. Note that $r_{\mu,0} = r_\mu$. Let $\Omega_{\mu,k} = \{r_{\mu,k} \le |z| < r_{\mu,k+1}\} \cap \triangle_0$. For a fixed positive integer n, we have $\{r_0 \le |z| < r_{n+1}\} \cap \triangle_0 = \sum_{k=0}^{\alpha-1} \sum_{\mu=0}^{n} \Omega_{\mu,k}$. The number of the branch points in $\Omega_{\mu,k}$ is denoted by $n(\Omega_{\mu,k}, \tilde{R}_z)$. Then there exists a k $(0 \le k \le \alpha - 1)$. Without loss of generality, we assume that $k = 0$ such that

$$\sum n(\Omega_{\mu,0}, \tilde{R}_z) \le \frac{1}{\alpha} n(r_{n+1}, \triangle_0, \tilde{R}_z).$$

Let

$$\triangle_\mu^0 = \{z \,|\, |\arg z - \theta_0| \le \varepsilon_0, r_{\mu-1,0} \le |z| < r_{\mu,1}\},$$

$$\overline{\triangle}_\mu = \left\{z \,|\, |\arg z - \theta_0| \le \varepsilon, \frac{r_{\mu-1,0} + r_{\mu-1,1}}{2} \le |z| \le \frac{r_{\mu,0} + r_{\mu,1}}{2}\right\}.$$

Then $\triangle_\mu^0$ can be mapped conformally to the unit disk $|\zeta| < 1$ such that $\left(\frac{r_{\mu-1,0} + r_{\mu-1,1} + r_{\mu,0} + r_{\mu,1}}{4}, \theta_0\right)$, the center of $\overline{\triangle}_\mu$, corresponds to $\zeta = 0$. The image of $\overline{\triangle}_\mu$ is contained in the disk $|\zeta| < \kappa$, where κ is a constant depending only on ε_0, ε and λ^α, and is independent of μ. Since $S(\overline{\triangle}_\mu)$ is a conformal invariant, by Lemma 1

$$\left(q - 2 - \frac{2}{l}\right) S(\overline{\triangle}_\mu) \le \sum_{j=1}^{q} \bar{n}^{l)}(\triangle_\mu^0, a_j) + \frac{l+1}{l} n(\tilde{\triangle}_\mu^0) + \frac{A}{1-\kappa}.$$

Hence

$$\left(q - 2 - \frac{2}{l}\right) \sum_{\mu=2}^{n} S(\overline{\triangle}_\mu) \le \sum_{j=1}^{q} \sum_{\mu=2}^{n} \bar{n}^{l)}(\triangle_\mu^0, a_j) + \frac{l+1}{l} \sum_{\mu=2}^{n} n(\tilde{\triangle}_\mu^0) + \frac{A}{1-\kappa}(n-1). \tag{7}$$

We have

$$\sum_{\mu=2}^{n} S(\overline{\triangle}_\mu) = S(r_n, \overline{\triangle}, w) - S(r_1, \overline{\triangle}, w).$$

Since $\triangle_\mu^0$ $(\mu = 1, 2, \dots, n)$ overlap $\Omega_{\mu,0}$ twice at most, we have

$$\sum_{\mu=2}^{n} \bar{n}^{l)}(\triangle_\mu^0, a_j) \le 2\bar{n}^{l)}(r_{n+1}, \triangle_0, a_j),$$

$$\sum_{\mu=2}^{n} n(\tilde{\triangle}_\mu^0) \le \left(1 + \frac{1}{\alpha}\right) n(r_{n+1}, \triangle_0, \tilde{R}_z),$$

where $n(\tilde{\triangle}_\mu^0)$ is the number of the branch points in $\tilde{\triangle}_\mu^0$.

From $r_n = \lambda^{\alpha n}$, we have $n = \dfrac{\log r_n}{\alpha \log \lambda}$.
By (7)

$$\left(q-2-\frac{2}{l}\right)S(r_n, \overline{\triangle}, w) \leq 2\sum_{j=1}^{q} \bar{n}^{l)}(r_{n+1}, \triangle_0, a_j) + \frac{l+1}{l}\left(1+\frac{1}{\alpha}\right)n(r_{n+1}, \triangle_0, \tilde{R}_z) + \left(q-2-\frac{2}{l}\right)S(r_1, \overline{\triangle}, w) + \frac{A}{\alpha(1-\kappa)\log\lambda}\log^+ r_n. \tag{8}$$

For $r \geq r_1$, there exists an $n > 1$ such that $r_{n-1} \leq r < r_n$. From (8) and noticing that $r_1 = \lambda^{\alpha}$, $r_{n+1} = \lambda^{2\alpha} r_{n-1} \leq \lambda^{2\alpha} r$, $r_n = \lambda^{\alpha} r_{n-1} \leq r^2$ and $\log r_n \leq 2\log r$, it follows that

$$\left(q-2-\frac{2}{l}\right)S(r, \overline{\triangle}, w) \leq 2\sum_{j=1}^{q} \bar{n}^{l)}(\lambda^{2\alpha} r, \triangle_0, a_j) + \frac{l+1}{l}\left(1+\frac{1}{\alpha}\right)n(\lambda^{2\alpha} r, \triangle_0, \tilde{R}_z) + \left(q-2-\frac{2}{l}\right)S(\lambda^{\alpha}, \overline{\triangle}, w) + \frac{2A}{\alpha(1-\kappa)\log\lambda}\log^+ r.$$

This inequality is certainly valid for $r < \lambda^{\alpha}$. We have proved this lemma. □

Lemma 4. *Let $w = w(z)$ $(z \in \mathbf{C})$ be the v-valued algebroid function of order ρ defined by* (1), *$l (\geq 2v+1)$ and m $(m > 1)$ be two positive integers. Put $\varphi_0 = 0, \varphi_1 = \dfrac{2\pi}{m}, \dots, \varphi_{m-1} = (m-1)\dfrac{2\pi}{m}$. Let*

$$\triangle(\varphi_i) = \left\{z \,|\, |\arg z - \varphi_i| < \frac{2\pi}{m}\right\} \quad (0 \leq i \leq m-1).$$

Then there exists a $\triangle(\varphi_i)$ among $\triangle(\varphi_i)$ $(i = 0, 1, \dots, m-1)$ such that

$$\limsup_{r\to\infty} \frac{\bar{n}^{l)}(r, \triangle(\varphi_i), a)}{U(r)} > 0$$

for any value a with $2v$ possible exceptions.

Proof. Suppose that the conclusion is false. Then for every $\triangle(\varphi_i)$ $(i = 0, 1, \dots, m-1)$ and any $\varepsilon > 0$, there exist $R_0 > 0$ and $q = 2v+1$ exceptional values $\{a_i^j\}_{j=1}^{q}$ such that

$$\limsup_{r\to\infty} \frac{\bar{n}^{l)}(r, \triangle(\varphi_i), a_i^j)}{U(r)} = 0. \tag{9}$$

Let α, β be any two positive integers. Put $\varphi_{i,k} = \dfrac{2\pi}{m} i + \dfrac{2k\pi}{\beta m}$, $0 \leq i \leq m-1$, $0 \leq k \leq \beta - 1$. For any given numbers $r > 1$ and $\lambda > 1$, writing

$$\triangle_{i,k} = \{z \mid |z| < \lambda^{2\alpha} r, \varphi_{i,k} \le \arg z < \varphi_{i,k+1}\}.$$

Then

$$\{|z| < \lambda^{2\alpha} r\} = \sum_{k=0}^{\beta-1} \sum_{i=0}^{m-1} \triangle_{i,k}.$$

There exists a k_0, without loss of generality, we may assume that $k_0 = 0$, such that

$$\sum_{i=0}^{m-1} n(\triangle_{i,0}, \tilde{R}_z) \le \frac{1}{\beta} n(\lambda^{2\alpha} r, \tilde{R}_z).$$

Put

$$\overline{\triangle}_i = \left\{ z \left| \frac{\varphi_{i,0} + \varphi_{i,1}}{2} \le \arg z \le \frac{\varphi_{i+1,0} + \varphi_{i+1,1}}{2} \right. \right\},$$

$$\triangle_i^0 = \{z \mid \varphi_{i,0} < \arg z < \varphi_{i+1,1}\}, \quad 0 \le i \le m-1.$$

Since $\triangle_i^0$ overlap $\triangle_{i,0}$ twice at most, then

$$\sum_{i=0}^{m-1} n(\lambda^{2\alpha} r, \triangle_i^0, \tilde{R}_z) \le \left(1 + \frac{1}{\beta}\right) n(\lambda^{2\alpha} r, \tilde{R}_z).$$

By Lemma 3 we have

$$\left(q - 2 - \frac{2}{l}\right) S(r, \overline{\triangle}_i, w) \le 2 \sum_{j=1}^{q} \bar{n}^{l)}(\lambda^{2\alpha} r, \triangle_i^0, a_j) + \frac{l+1}{l}\left(1 + \frac{1}{\alpha}\right) n(\lambda^{2\alpha} r, \triangle_i^0, \tilde{R}_z)$$
$$+ \left(q - 2 - \frac{2}{l}\right) S(\lambda^{\alpha}, \overline{\triangle}_i, w) + \frac{2A}{\alpha \log \lambda} \frac{\log^+ r}{1 - \kappa}.$$

Adding from $i = 0$ to $m-1$, dividing both sides of this inequality by r, and then integrating both sides from 1 to r, thus we obtain

$$(10) \quad \left(q - 2 - \frac{2}{l}\right) T(r, w)$$
$$\le 2 \sum_{j=1}^{q} \sum_{i=0}^{m-1} \overline{N}^{l)}(\lambda^{2\alpha} r, \triangle_i^0, a_j) + \frac{l+1}{l}\left(1 + \frac{1}{\alpha}\right)\left(1 + \frac{1}{\beta}\right) N(\lambda^{2\alpha} r, \tilde{R}_z)$$
$$+ \left(q - 2 - \frac{2}{l}\right) S(\lambda^{\alpha}, w) \log r + \left(q - 2 - \frac{2}{l}\right) T(1, w)$$
$$+ \frac{2A}{\alpha(1 - \kappa) \log \lambda} \log^2 r,$$

where $A = \sum_{i=0}^{m-1} A_i$. Dividing both sides of (10) by $U(r)$ and letting $r \to \infty$, by Lemma 1 and (3), we have

$$q - 2 - \frac{2}{l} \le 2\frac{l+1}{l}(v-1)\lambda^{2\alpha\rho}\left(1+\frac{1}{\alpha}\right)\left(1+\frac{1}{\beta}\right).$$

Letting $\lambda \to 1$, $\alpha \to \infty$, $\beta \to \infty$ respectively we get $l \le 2v$. This contradicts $l \ge 2v+1$ and Lemma 4 follows. □

Now we begin to prove Theorem 1.

Proof of Theorem 1. By Lemma 4, for any given positive integer m, there exists

$$\triangle_m = \left\{ z \,|\, |\arg z - \theta_m| < \frac{2\pi}{m} \right\}$$

such that

$$\limsup_{r\to\infty} \frac{\bar{n}^{l)}(r, \triangle_m, a)}{U(r)} > 0,$$

for any value of a with $2v$ possible exceptions at most. By choosing a subsequence, we can assume that $\theta_m \to \theta_0$, when $m \to \infty$. Then $B : \arg z = \theta_0$ is of the properties of Theorem 1. □

Corollary. *A Borel direction of the largest type of* $w(z)$ *dealing with multiple values in Theorem* 1 *must be a Borel direction of* $w(z)$ *dealing with multiple values.*

Lemma 5. *Let* $w = w(z)$ $(z \in \mathbf{C})$ *be the* v*-valued algebroid function of order* ρ *defined by* (1) *and* $l(\ge 2v+1)$ *be a positive integer. If* $B : \arg z = \theta_0$ $(0 \le \theta_0 < 2\pi)$ *is a Borel direction of the largest type dealing with multiple values of* $w(z)$, *then*

$$\limsup_{r\to\infty} \frac{S(r, \triangle(\theta_0, \varepsilon), w)}{U(r)} > 0,$$

where $\Delta(\theta_0, \varepsilon) = \{z \,|\, |\arg z - \theta_0| < \varepsilon\}$, $0 < \varepsilon < \frac{\pi}{2}$ *and*

$$S(r, \triangle(\theta_0, \varepsilon), w) = \frac{1}{\pi} \iint_{\tilde{\Delta}(\theta_0,\varepsilon)\cap\{|\tilde{z}|\le r\}} \left[\frac{|w'(z)|}{1+|w(z)|^2}\right]^2 d\omega.$$

Proof. Note that $n(r, \triangle(\theta_0, \varepsilon), w) \ge \bar{n}^{l)}(r, \triangle(\theta_0, \varepsilon), w)$, applying a similar argument as Lemma 3.2 ([2]), we can prove this lemma. □

Lemma 6. *Let* $w = w(z)$ $(z \in \mathbf{C})$ *be the* v*-valued algebroid function of order* ρ *defined by* (1) *and* $l(\ge 2v+1)$ *be a positive integer. Put*

$$B_p = \{a^{p-1} \le |z| < a^{p+2}\} \cap \left\{|\arg z - \theta_0| < \frac{a-1}{a}\right\}, \quad p = 1, 2, \dots.$$

For arbitrarily constant $\varepsilon(\in (0, \rho))$ *and* $R > 1$, *there exists an* $a_0(\in (1,2))$ *such that for any* $a(\in (1, a_0))$, *the following assertion is true*:

If $B : \arg z = \theta_0$ $(0 \le \theta_0 < 2\pi)$ *is a Borel direction of the largest type dealing with multiple values of* $w(z)$ *and*

$$\limsup_{r\to\infty} \frac{S\left(r, \triangle\left(\theta_0, \frac{a-1}{2a}\right), w\right)}{U(r)} = b > 0,$$

then there exists at least a p_0 *with* $a^{p_0} > R$, *such that* $\bar{n}^{l)}(B_{p_0}, \alpha) \ge U^{1-\varepsilon}(a^{p_0})$ *holds for any complex value* α *except at most* $\left[\frac{(l+1)a^{3\rho}(\nu-1)2^{\rho+1}}{lb \log 2} + \frac{2}{l}\right] + 2$ *possible exceptions enclosed by spherical circles of radius* $\delta = U^{-1/11}(a^{p_0})$ *on the Riemann sphere*, *where* $[x]$ *denotes the integral part of* x.

Proof. Suppose that the conclusion is false. Then, for any $a > 1$ and any $p > \frac{\log R}{\log a}$, there exists $q = \left[\frac{(l+1)a^{3\rho}(\nu-1)2^{\rho+1}}{lb \log 2} + \frac{2}{l}\right] + 3$ distinct complex numbers $\{\alpha_j = \alpha_j(p)\}_{j=1}^q$ with spherical distance of any two of them is equal to or larger than $\delta = U^{-1/11}(a^p)$. Put

$$\Delta_0 = \left\{z \mid |\arg z - \theta_0| < \frac{a-1}{a}\right\}, \quad \Delta = \left\{z \mid |\arg z - \theta_0| < \frac{a-1}{2a}\right\}.$$

Taking $r > R$ arbitrarily and setting $T = [\log r/\log a]$, then $a^T \le r < a^{T+1}$. For any positive integer M, put

$$b = a^{1/M}, \quad r_{p,t} = b^{Mp+t}, \quad t = 0, 1, \dots, M-1,$$
$$\Omega_{p,t} = \{r_{p,t} \le |z| < r_{p,t+1}\} \cap \Delta_0.$$

Since

$$\{a^{-1} \le |z| < a^T\} \cap \Delta_0 = \bigcup_{t=0}^{M-1} \bigcup_{p=-1}^{T-1} \Omega_{p,t},$$

then there exists a t_0 $(0 \le t_0 \le M-1)$ depending on T, without loss of generality, we may assume that $t_0 = 0$ such that

$$\sum_{p=-1}^{T-1} n(\Omega_{p,0}, \tilde{R}_z) \le \frac{1}{M} n(a^T, \tilde{R}_z).$$

Put

$$B_p^0 = \left\{\frac{b^{Mp} + b^{Mp+1}}{2} \le |z| < \frac{b^{Mp+M} + b^{Mp+M+1}}{2}\right\} \cap \Delta,$$

$$\bar{B}_p = \{b^{Mp} \le |z| < b^{Mp+M+1}\} \cap \Delta_0.$$

Then

$$B_p^0 \subset \bar{B}_p \subset B_p.$$

Since $\{\bar{B}_p\}$ overlap $\bigcup_{p=-1}^{T-1} \Omega_{p,0}$ twice at most, thus

$$\sum_{p=1}^{T-2} n(\bar{B}_p, \tilde{R}_z) \le \left(1 + \frac{1}{M}\right) n(a^T, \tilde{R}_z).$$

Obviously, $\bar{B}_p$ can be mapped conformally to the unit disk $|\zeta| < 1$ such that the center of B_p^0 corresponds to $\zeta = 0$ and the image of B_p^0 is contained in the disk $|\zeta| < \kappa < 1$, where $\kappa > 0$ is a constant and is independent of p. Hence, for α_j $(1 \le j \le q)$, by Lemma 1 we have

$$\left(q - 2 - \frac{2}{l}\right) S(B_p^0) \le \sum_{j=1}^{q} \bar{n}^{l)}(\bar{B}_p, \alpha_j) + \frac{l+1}{l} n(\bar{B}_p, \tilde{R}_z) + \frac{C}{\delta^{10}(1-\kappa)}.$$

Thus

$$\sum_{p=1}^{T-2} \left(q - 2 - \frac{2}{l}\right) S(B_p^0) \le \sum_{p=1}^{T-2} \sum_{j=1}^{q} \bar{n}^{l)}(\bar{B}_p, \alpha_j) + \sum_{p=1}^{T-2} \frac{l+1}{l} n(\bar{B}_p, \tilde{R}_z) + \sum_{p=1}^{T-2} \frac{C}{\delta^{10}(1-\kappa)},$$

$$\begin{aligned}\left(q - 2 - \frac{2}{l}\right) S(a^{T-2}, \Delta, w) &\le \sum_{p=1}^{T-2} \sum_{j=1}^{q} \bar{n}^{l)}(\bar{B}_p, \alpha_j) + \frac{l+1}{l}\left(1 + \frac{1}{M}\right) n(a^T, \tilde{R}_z) \\ &\quad + C' T \delta^{-10} + \left(q - 2 - \frac{2}{l}\right) S(a^2, \Delta, w) \\ &\le T q U^{1-\varepsilon}(a^{T-2}) + \frac{l+1}{l}\left(1 + \frac{1}{M}\right) n(r, \tilde{R}_z) \\ &\quad + C' T U^{\frac{10}{11}}(a^{T-1}) + \left(q - 2 - \frac{2}{l}\right) S(a^2, \Delta, w).\end{aligned}$$

Taking $T(= [\log r / \log a])$ sufficiently large, then r is sufficiently large too and $r \in [a^T, a^{T+1})$. Thus we have

$$\begin{aligned}\text{(11)} \qquad \left(q - 2 - \frac{2}{l}\right) S(a^{-3} r, \Delta, w) &\le U^{1-\varepsilon/2}(r) + \sum_{p=1}^{T-2} \frac{l+1}{l}\left(1 + \frac{1}{M}\right) n(r, \tilde{R}_z) \\ &\quad + C' U^{11/12}(r).\end{aligned}$$

Dividing both sides of (11) by r and then integrating both sides from r to $2r$, we obtain

$$(12)\quad \left(q-2-\frac{2}{l}\right)S(a^{-3}r,\Delta,w)\log 2$$
$$\leq U^{1-\varepsilon/2}(2r)\log 2+\frac{l+1}{l}\left(1+\frac{1}{M}\right)N(2r,\tilde{R}_z)+C'U^{11/12}(2r)\log 2$$
$$\leq U^{1-\varepsilon/2}(2r)\log 2+2\frac{l+1}{l}\left(1+\frac{1}{M}\right)(v-1)T(2r,w)$$
$$+O(1)+C'U^{11/12}(2r)\log 2.$$

Dividing both sides of (12) by $U(r)$ and letting $r\to\infty$, we have

$$\left(q-2-\frac{2}{l}\right)\cdot b\cdot\frac{1}{a^{3\rho}}\log 2\leq 2\frac{l+1}{l}\left(1+\frac{1}{M}\right)(v-1)\cdot 2^{\rho}.$$

Letting $M\to\infty$,

$$q\leq\frac{(l+1)a^{3\rho}(v-1)2^{\rho+1}}{lb\log 2}+\frac{2}{l}+2.$$

This contradicts $q=\left[\frac{(l+1)a^{3\rho}(v-1)2^{\rho+1}}{lb\log 2}+\frac{2}{l}\right]+3$ and we have completed our proof. □

Proof of Theorem 2. Let $\varepsilon_n=\frac{\rho}{2^n}$, $R_n=2^n$. By Lemma 6, we have $a_n\in\left(1,1+\frac{1}{n}\right)$, p_n and

$$B_{p_n}=\{a_n^{p_n-1}\leq|z|<a_n^{p_n+2}\}\cap\left\{|\arg z-\theta_0|<\frac{a_n-1}{a_n}\right\},\quad n=1,2,\ldots.$$

Let $z_n=a_n^{p_n}e^{i\theta_0}$, $|z_n|=a_n^{p_n}>R_n=2^n\to\infty$ $(n\to\infty)$. Put

$$r_n=\frac{a_n^{p_n+2}-a_n^{p_n}}{2}+\frac{a_n-1}{a_n}\cdot a_n^{p_n+2}$$
$$=a_n^{p_n}\left(\frac{a_n^2-1}{2}+a_n(a_n-1)\right)$$
$$\leq 4(a_n-1)a_n^{p_n}=4(a_n-1)|z_n|.$$

Take $\sigma_n=4(a_n-1)\in\left(0,\frac{4}{n}\right)$, then $\sigma_n\to 0$ $(n\to\infty)$. Put $\Gamma_n=\{|z-z_n|<\sigma_n r_n\}$, then $B_{p_n}\subset\Gamma_n$. Let

$$\limsup_{r\to\infty}\frac{S\left(r,\triangle\left(\theta_0,\frac{a_n-1}{2a_n}\right),w\right)}{U(r)}=b(a_n),$$

then $b_n=b(a_n)>0$. By Lemma 6, $\bar{n}^{l)}(\Gamma_n,\alpha)\geq U^{1-\varepsilon_n}(|z_n|)$ holds for any complex value α except at most $\left[\frac{(l+1)a_n^{3\rho}(v-1)2^{\rho+1}}{lb_n\log 2}+\frac{2}{l}\right]+2$ possible exceptions enclosed by spherical circles of radius $\delta = U^{-1/11}(|z_n|)$ on the Riemann sphere. Noticing that $0 < \rho < +\infty$ and $b_n > 0$, the spherical circles in this theorem is countable. □

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