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SOME CONVERGENCE THEOREMS FOR ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPPINGS

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Abstract

Let K be a nonempty closed convex subset of a real Banach space $E, T : K \to K$ a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n\ge0} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ such that $p \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}_{n\ge0} \subset [0,1]$ be such that $\sum_{n\ge0} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$. For arbitrary $x_0 \in K$ and $\{v_n\}_{n\ge0}$ in K let $\{x_n\}_{n\ge0}$ be iteratively defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n v_n, \quad n \ge 0,$$

satisfying $\lim_{n\to\infty}\|v_n - x_n\| = 0$. Suppose there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$, $\phi(0) = 0$ such that

$$\langle T^n x - p, j(x-p)\rangle \le k_n\|x-p\|^2 - \phi(\|x-p\|), \quad \forall x \in K.$$

Then $\{x_n\}_{n\ge0}$ converges strongly to $p \in F(T)$.

The remark at the end is important.

1. Introduction

Let E be a real Banach space and K be a nonempty convex subset of E. Let J denote the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2 \text{ and } \|f^*\| = \|x\|\},$$

where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. We shall denote the single-valued duality map by j.

Let $T : D(T) \subset E \to E$ be a mapping with domain $D(T)$ in E.

Definition 1. The mapping T is said to be uniformly L-Lipschitzian if there exists $L > 0$ such that for all $x, y \in D(T)$

$$\|T^n x - T^n y\| \le L\|x-y\|.$$

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Definition 2. T is said to be nonexpansive if for all $x, y \in D(T)$, the following inequality holds:

$$\|Tx - Ty\| \le \|x - y\| \quad \text{for all } x, y \in D(T).$$

Definition 3. T is said to be asymptotically nonexpansive [6], if there exists a sequence $\{k_n\}_{n\ge 0} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\| \quad \text{for all } x, y \in D(T),\ n \ge 1.$$

Definition 4. T is said to be asymptotically pseudocontractive if there exists a sequence $\{k_n\}_{n\ge 0} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ and there exists $j(x-y) \in J(x-y)$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \le k_n \|x - y\|^2 \quad \text{for all } x, y \in D(T),\ n \ge 1.$$

Remark 1. It is easy to see that every asymptotically nonexpansive mapping is uniformly L-Lipschitzian.

Remark 2. If T is asymptotically nonexpansive mapping then for all $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$\begin{aligned}\langle T^n x - T^n y, j(x-y)\rangle &\le \|T^n x - T^n y\|\,\|x - y\| \\ &\le k_n \|x - y\|^2, \quad n \ge 1.\end{aligned}$$

Hence every asymptotically nonexpansive mapping is asymptotically pseudocontractive.

Remark 3. Rhoades in [11] showed that the class of asymptotically pseudocontractive mappings properly contains the class of asymptotically nonexpansive mappings.

The asymptotically pseudocontractive mappings were introduced by Schu [12] who proved the following theorem:

Theorem 1. *Let K be a nonempty bounded closed convex subset of a Hilbert space $H, T : K \to K$ a completely continuous, uniformly L-Lipschitzian and asymptotically pseudocontractive with sequence $\{k_n\} \subset [1,\infty)$; $q_n = 2k_n - 1$, $\forall n \in N$; $\sum (q_n^2 - 1) < \infty$; $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$; $\in < \alpha_n < \beta_n \le b$, $\forall n \in N$, and some $\in > 0$ and some $b \in (0, L^{-2}[(1+L^2)^{1/2} - 1])$; $x_1 \in K$ for all $n \in N$, define*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n.$$

Then $\{x_n\}$ converges to some fixed point of T.

The recursion formula of theorem 1 is a modification of the well-known Mann iteration process (see [9]).

Recently, Chang [1] extended Theorem 1 to real uniformly smooth Banach space; in fact, he proved the following theorem:

Theorem 2. *Let K be a nonempty bounded closed convex subset of a real uniformly smooth Banach space $E, T : K \to K$ an asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$, and $x^* \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\} \subset [0, 1]$ satisfying the following conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$. For arbitrary $x_0 \in K$ let $\{x_n\}$ be iteratively defined by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 0.$$

If there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$, $\phi(0) = 0$ such that

$$\langle T^n x - x^*, j(x - x^*)\rangle \leq k_n\|x - x^*\|^2 - \phi(\|x - x^*\|), \quad \forall n \in N,$$

then $x_n \to x^* \in F(T)$.

Remark 4. Theorem 2, as stated is a modification of Theorem 2.4 of Chang [1] who actually included error terms in his algorithm.

In [10], E. U. Ofoedu proved the following results.

Theorem 3. *Let K be a nonempty closed convex subset of a real Banach space $E, T : K \to K$ a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ such that $x^* \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}_{n\geq 0} \subset [0, 1]$ be such that $\sum_{n\geq 0} \alpha_n = \infty$, $\sum_{n\geq 0} \alpha_n^2 < \infty$ and $\sum_{n\geq 0} \alpha_n(k_n - 1) < \infty$. For arbitrary $x_0 \in K$ let $\{x_n\}_{n\geq 0}$ be iteratively defined by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 0.$$

Suppose there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$, $\phi(0) = 0$ such that

$$\langle T^n x - x^*, j(x - x^*)\rangle \leq k_n\|x - x^*\|^2 - \phi(\|x - x^*\|), \quad \forall x \in K.$$

Then $\{x_n\}_{n\geq 0}$ is bounded.

Theorem 4. *Let K be a nonempty closed convex subset of a real Banach space $E, T : K \to K$ a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ such that $x^* \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}_{n\geq 0} \subset [0, 1]$ be such that $\sum_{n\geq 0} \alpha_n = \infty$, $\sum_{n\geq 0} \alpha_n^2 < \infty$ and $\sum_{n\geq 0} \alpha_n(k_n - 1) < \infty$. For arbitrary $x_0 \in K$ let $\{x_n\}_{n\geq 0}$ be iteratively defined by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 0.$$

Suppose there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$, $\phi(0) = 0$ such that

$$\langle T^n x - x^*, j(x - x^*)\rangle \leq k_n\|x - x^*\|^2 - \phi(\|x - x^*\|), \quad \forall x \in K.$$

Then $\{x_n\}_{n\geq 0}$ converges strongly to $x^* \in F(T)$.

THEOREM 5. *Let K be a nonempty closed convex subset of a real Banach space $E, T : K \to K$ a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n \geq 0} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ such that $x^* \in F(T) = \{x \in K : Tx = x\}$. Let $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$, $\{c_n\}_{n\geq 0}$ be real sequences in $[0, 1]$ satisfying the following conditions*:

i): $a_n + b_n + c_n = 1$;
ii): $\sum_{n\geq 0}(b_n + c_n) = \infty$;
iii): $\sum_{n\geq 0}(b_n + c_n)^2 < \infty$;
iv): $\sum_{n\geq 0}(b_n + c_n)(k_n - 1) < \infty$; *and*
v): $\sum_{n\geq 0} c_n < \infty$.

For arbitrary $x_0 \in K$ let $\{x_n\}_{n\geq 0}$ be iteratively defined by

$$x_{n+1} = a_n x_n + b_n T^n x_n + c_n u_n, \quad n \geq 0,$$

where $\{u_n\}_{n\geq 0}$ is a bounded sequence of error terms in K. Suppose there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$, $\phi(0) = 0$ such that

$$\langle T^n x - x^*, j(x - x^*)\rangle \leq k_n\|x - x^*\|^2 - \phi(\|x - x^*\|), \quad \forall x \in K.$$

Then $\{x_n\}_{n\geq 0}$ converges strongly to $x^* \in F(T)$.

The purpose of this paper is to introduce the following Mann iteration process associated with uniformly L-Lipschitzian asymptotically pseudocontractive mappings to have a strong convergence in the setting of real Banach spaces.

Let K be a closed convex subset of a real normed space E and $T : K \to K$ be a mapping. For a sequence $\{v_n\}_{n\geq 0}$ in K, define $\{x_n\}_{n\geq 0}$ in the following way:

(AU-M) $$x_0 \in K,$$
$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n v_n, \quad n \geq 0,$$

where $\{\alpha_n\}_{n\geq 0}$ be a real sequence in $[0, 1]$ satisfying some appropriate conditions.

We improve the results of Ofoedu [10] in a significantly more general context by removing the conditions $\sum_{n\geq 0} \alpha_n^2 < \infty$ and $\sum_{n\geq 0} \alpha_n(k_n - 1) < \infty$ from the theorems 3–4. We also significantly extend theorem 2 from uniformly smooth Banach space to arbitrary real Banach space. The boundedness assumption imposed on K in the theorem is also dispensed with. A related result involving bounded sequence of error terms is also obtained.

2. Main results

The following lemmas are now well known.

LEMMA 1. *Let $J : E \to 2^E$ be the normalized duality mapping. Then for any $x, y \in E$, we have*

(2.1) $$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y).$$

Suppose there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$.

LEMMA 2. *Let* $\{\theta_n\}$ *be a sequence of nonnegative real numbers,* $\{\lambda_n\}$ *be a real sequence satisfying*

$$0 \le \lambda_n \le 1, \quad \sum_{n=0}^{\infty} \lambda_n = \infty.$$

If there exists a positive integer n_0 *such that*

$$\theta_{n+1}^2 \le \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n,$$

for all $n \ge n_0$, *with* $\sigma_n \ge 0$, $\forall n \in \mathbf{N}$, *and* $\sigma_n = 0(\lambda_n)$, *then* $\lim_{n\to\infty} \theta_n = 0$.

THEOREM 6. *Let* K *be a nonempty closed convex subset of a real Banach space* $E, T : K \to K$ *a uniformly L-Lipschitzian asymptotically pseudo-contractive mapping with sequence* $\{k_n\}_{n\ge 0} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ *such that* $p \in F(T) = \{x \in K : Tx = x\}$. *Let* $\{\alpha_n\}_{n\ge 0} \subset [0,1]$ *be such that* $\sum_{n\ge 0} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$. *For arbitrary* $x_0 \in K$ *and* $\{v_n\}_{n\ge 0}$ *in* K, *define the sequence* $\{x_n\}_{n\ge 0}$ *by (AU-M) satisfying* $\lim_{n\to\infty} \|v_n - x_n\| = 0$. *Suppose there exists a strictly increasing function* $\phi : [0,\infty) \to [0,\infty)$, $\phi(0) = 0$ *such that*

$$\langle T^n x - p, j(x-p)\rangle \le k_n \|x-p\|^2 - \phi(\|x-p\|), \quad \forall x \in K. \tag{2.2}$$

Then $\{x_n\}_{n\ge 0}$ *converges strongly to* $p \in F(T)$.

Proof. By $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \|v_n - x_n\| = 0$ and $\lim_{n\to\infty} k_n = 1$, imply there exists $n_0 \in \mathbf{N}$ such that $\forall n \ge n_0$, $\alpha_n \le \delta$, $\|v_n - x_n\| \le \delta'$;

$$0 < \delta = \min\left\{\frac{1}{1+2(1+L)}, \frac{\phi(2\phi^{-1}(a_0))}{48(1+L)^2[\phi^{-1}(a_0)]^2}\right\},$$

$$0 < \delta' = \min \frac{1}{L}\left\{\phi^{-1}(a_0), \frac{\phi(2\phi^{-1}(a_0))}{48\phi^{-1}(a_0)}\right\},$$

and

$$k_n - 1 \le \frac{\phi(2\phi^{-1}(a_0))}{36[\phi^{-1}(a_0)]^2}.$$

Define $a_0 := \|x_{n_0} - T^{n_0} x_{n_0}\| \, \|x_{n_0} - p\| + (k_{n_0} - 1)\|x_{n_0} - p\|^2$. Then from (2.2), we obtain that $\|x_{n_0} - p\| \le \phi^{-1}(a_0)$.

CLAIM. $\|x_n - p\| \le 2\phi^{-1}(a_0)$ $\forall n \ge n_0$.

The proof is by induction. Clearly, the claim holds for $n = n_0$. Suppose it holds for some $n \ge n_0$, i.e., $\|x_n - p\| \le 2\phi^{-1}(a_0)$. We prove that $\|x_{n+1} - p\| \le 2\phi^{-1}(a_0)$. Suppose that this is not true. Then $\|x_{n+1} - p\| > 2\phi^{-1}(a_0)$, so that $\phi(\|x_{n+1} - p\|) > \phi(2\phi^{-1}(a_0))$. Using the recursion formula (AU-M), we have the following estimates

$$\begin{aligned}\|x_n - T^n v_n\| &\le \|x_n - p\| + \|p - T^n v_n\| \\ &\le \|x_n - p\| + L\|v_n - p\| \\ &\le (1+L)\|x_n - p\| + L\|v_n - x_n\| \\ &\le 2(1+L)\phi^{-1}(a_0) + L\|v_n - x_n\|,\end{aligned}$$

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-\alpha_n)x_n + \alpha_n T^n v_n - p\| \\ &= \|x_n - p - \alpha_n(x_n - T^n v_n)\| \\ &\le \|x_n - p\| + \alpha_n\|x_n - T^n v_n\| \\ &\le 2\phi^{-1}(a_0) + \alpha_n[2(1+L)\phi^{-1}(a_0) + L\|v_n - x_n\|] \\ &\le [1 + 2(1+L)]\phi^{-1}(a_0)\alpha_n \\ &\le 3\phi^{-1}(a_0).\end{aligned}$$

With these estimates and again using the recursion formula (AU-M), we obtain by lemma 1 that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)x_n + \alpha_n T^n v_n - p\|^2 \\ &= \|x_n - p - \alpha_n(x_n - T^n v_n)\|^2 \\ &\le \|x_n - p\|^2 - 2\alpha_n\langle x_n - T^n x_n, j(x_{n+1} - p)\rangle \\ &= \|x_n - p\|^2 + 2\alpha_n\langle T^n x_{n+1} - p, j(x_{n+1} - p)\rangle \\ &\quad - 2\alpha_n\langle x_{n+1} - p, j(x_{n+1} - p)\rangle \\ &\quad + 2\alpha_n\langle T^n v_n - T^n x_{n+1}, j(x_{n+1} - p)\rangle \\ &\quad + 2\alpha_n\langle x_{n+1} - x_n, j(x_{n+1} - p)\rangle \\ &\le \|x_n - p\|^2 + 2\alpha_n(k_n\|x_{n+1} - p\|^2 - \phi(\|x_{n+1} - p\|)) \\ &\quad - 2\alpha_n\|x_{n+1} - p\|^2 + 2\alpha_n\|T^n v_n - T^n x_{n+1}\|\,\|x_{n+1} - p\| \\ &\quad + 2\alpha_n\|x_{n+1} - x_n\|\,\|x_{n+1} - p\| \\ &\le \|x_n - p\|^2 + 2\alpha_n(k_n - 1)\|x_{n+1} - p\|^2 - 2\alpha_n\phi(\|x_{n+1} - p\|) \\ &\quad + 2\alpha_n(1+L)\|x_{n+1} - x_n\|\,\|x_{n+1} - p\| \\ &\quad + 2L\alpha_n\|v_n - x_n\|\,\|x_{n+1} - p\|,\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|(1-\alpha_n)x_n + \alpha_n T^n v_n - x_n\| \\ &= \alpha_n\|x_n - T^n v_n\| \\ &\le (1+L)\|x_n - p\|\alpha_n + L\|v_n - x_n\|\alpha_n.\end{aligned} \tag{2.4}$$

Substituting (2.4) in (2.3), we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|x_n-p\|^2+2\alpha_n(k_n-1)\|x_{n+1}-p\|^2-2\alpha_n\phi(\|x_{n+1}-p\|)\\ &\quad +2\alpha_n^2(1+L)^2\|x_n-p\|\,\|x_{n+1}-p\|\\ &\quad +2L(1+L)\alpha_n^2\|v_n-x_n\|\,\|x_{n+1}-p\|\\ &\quad +2L\alpha_n\|v_n-x_n\|\,\|x_{n+1}-p\|\\ &\le \|x_n-p\|^2-2\alpha_n\phi(2\phi^{-1}(a_0))+\alpha_n\phi(2\phi^{-1}(a_0))\\ &= \|x_n-p\|^2-\alpha_n\phi(2\phi^{-1}(a_0)).\end{aligned} \tag{2.5}$$

Thus

$$\alpha_n\phi(2\phi^{-1}(a_0)) \le \|x_n-p\|^2-\|x_{n+1}-p\|^2,$$

implies

$$\begin{aligned}\phi(2\phi^{-1}(a_0))\sum_{n=n_0}^{j}\alpha_n &\le \sum_{n=n_0}^{j}(\|x_n-p\|^2-\|x_{n+1}-p\|^2)\\ &= \|x_{n_0}-p\|^2,\end{aligned}$$

so that as $j\to\infty$ we have

$$\phi(2\phi^{-1}(a_0))\sum_{n=n_0}^{\infty}\alpha_n \le \|x_{n_0}-p\|^2<\infty,$$

which implies that $\sum\alpha_n<\infty$, a contradiction. Hence, $\|x_{n+1}-x^*\|\le 2\phi^{-1}(a_0)$; thus $\{x_n\}_{n\ge 0}$ is bounded.

Now from (2.5), we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|x_n-p\|^2-2\alpha_n\phi(\|x_{n+1}-p\|)\\ &\quad +8[\phi^{-1}(a_0)]^2\alpha_n(k_n-1)\\ &\quad +8[\phi^{-1}(a_0)]^2(1+L)^2\alpha_n^2\\ &\quad +4L(1+L)\phi^{-1}(a_0)\alpha_n^2\|v_n-x_n\|\\ &\quad +4L\phi^{-1}(a_0)\alpha_n\|v_n-x_n\|\\ &= \|x_n-p\|^2-2\alpha_n\phi(\|x_{n+1}-p\|)\\ &\quad +4\phi^{-1}(a_0)\delta_n\alpha_n;\end{aligned} \tag{2.6}$$

$\delta_n=2\phi^{-1}(a_0)(k_n-1)+2(1+L)^2\phi^{-1}(a_0)\alpha_n+(L(1+L)\alpha_n+L)\|v_n-x_n\|$.

Denote

$$\theta_n = \|x_n - p\|,$$
$$\lambda_n = 2\alpha_n,$$
$$\sigma_n = 4\phi^{-1}(a_0)\delta_n\alpha_n.$$

Condition $\lim_{n\to\infty} \alpha_n = 0$ ensures the existence of a rank $n_0 \in \mathbf{N}$ such that $\lambda_n = 2\alpha_n \le 1$, for all $n \ge n_0$. Now with the help of $\sum_{n\ge 0} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} \|v_n - x_n\| = 0$ and lemma 2, we obtain from (2.6) that

$$\lim_{n\to\infty} \|x_n - p\| = 0,$$

completing the proof. □

THEOREM 7. *Let K be a nonempty closed convex subset of a real Banach space $E, T : K \to K$ a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n\ge 0} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ such that $p \in F(T) = \{x \in K : Tx = x\}$. Let $\{a_n\}_{n\ge 0}$, $\{b_n\}_{n\ge 0}$, $\{c_n\}_{n\ge 0}$ be real sequences in $[0, 1]$ satisfying the following conditions*:

i): $a_n + b_n + c_n = 1$;
ii): $\sum_{n\ge 0} b_n = \infty$;
iii): $c_n = o(b_n)$;
iv): $\lim_{n\to\infty} b_n = 0$.

For arbitrary $x_0 \in K$ and $\{v_n\}_{n\ge 0}$ in K let $\{x_n\}_{n\ge 0}$ be iteratively defined by

$$x_n = a_n x_{n-1} + b_n T^n v_n + c_n u_n, \quad n \ge 0,$$

satisfying $\lim_{n\to\infty} \|v_n - x_n\| = 0$, where $\{u_n\}_{n\ge 0}$ is a bounded sequence of error terms in K. Suppose there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$, $\phi(0) = 0$ such that

$$\langle T^n x - p, j(x - p)\rangle \le k_n\|x - p\|^2 - \phi(\|x - p\|), \quad \forall x \in K.$$

Then $\{x_n\}_{n\ge 0}$ converges strongly to $p \in F(T)$.

3. Multi-step fixed point iterations

Let K be a nonempty closed convex subset of a real normed space E and $T_1, T_2, \dots, T_p : K \to K$ $(p \ge 2)$ be a family of selfmappings.

ALGORITHM 1. *For a given $x_0 \in K$, compute the sequence $\{x_n\}_{n\ge 0}$ by the iteration process of arbitrary fixed order $p \ge 2$,*

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T_1^n y_n^1, \\ y_n^i &= (1 - \beta_n^i)x_n + \beta_n^i T_{i+1}^n y_n^{i+1}; \quad i = 1, 2, \dots, p-2, \\ y_n^{p-1} &= (1 - \beta_n^{p-1})x_n + \beta_n^{p-1} T_p^n x_n, \quad n \ge 0, \end{aligned} \tag{4.1}$$

which is called the modified multi-step iteration process, where $\{\alpha_n\}, \{\beta_n^i\} \subset [0,1]$, $i = 1, 2, \ldots, p-1$.

For $p = 3$, we obtain the following modified three-step iteration process:

ALGORITHM 2. *For a given* $x_0 \in K$, *compute the sequence* $\{x_n\}$ *by the iteration process*:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1^n y_n^1,$$
$$y_n^1 = (1-\beta_n^1)x_n + \beta_n^1 T_2^n y_n^2,$$
$$y_n^2 = (1-\beta_n^2)x_n + \beta_n^2 T_3^n x_n, \quad n \geq 0, \tag{4.2}$$

where $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n^1\}_{n\geq 0}$ *and* $\{\beta_n^2\}_{n\geq 0}$ *are three real sequences in* $[0,1]$.

For $p = 2$, we obtain the following modified two-step iteration process:

ALGORITHM 3. *For a given* $x_0 \in K$, *compute the sequence* $\{x_n\}_{n\geq 0}$ *by the iteration process*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1^n y_n^1,$$
$$y_n^1 = (1-\beta_n^1)x_n + \beta_n^1 T_2^n x_n, \quad n \geq 0, \tag{4.3}$$

where $\{\alpha_n\}_{n\geq 0}$ *and* $\{\beta_n^1\}_{n\geq 0}$ *are two real sequences in* $[0,1]$.

If $T_1 = T$, $T_2 = I$, $\beta_n^1 = 0$ in (4.3), we obtain the modified Mann iteration process [12]:

ALGORITHM 4. *For any given* $x_0 \in K$, *compute the sequence* $\{x_n\}_{n\geq 0}$ *by the iteration process*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 0, \tag{4.4}$$

where $\{\alpha_n\}_{n\geq 0}$ *is a real sequence in* $[0,1]$.

By applying Theorem 6 under assumption that T_1 is uniformly L-Lipschitzian asymptotically pseudocontractive mapping, we obtain Theorem 8 which proves strong convergence of the iteration process defined by (4.1). Consider by taking $T_1 = T$ and $v_n = y_n^1$, then

$$\begin{aligned}\|v_n - x_n\| &= \|y_n^1 - x_n\| \\ &= \|(1-\beta_n^1)x_n + \beta_n^1 T_2^n y_n^2 - x_n\| \\ &= \beta_n^1 \|T_2^n y_n^2 - x_n\| \\ &\leq M\beta_n^1;\end{aligned}$$

M depends on L, β_n^i and $\phi^{-1}(a_0)$, $i = 2, \ldots, p$. Now from the condition $\lim_{n\to\infty} \beta_n^1 = 0$, it can be easily seen that $\lim_{n\to\infty} \|v_n - x_n\| = 0$.

THEOREM 8. *Let K be a nonempty closed convex subset of a real Banach space E and $T_1, T_2, \ldots, T_p$ $(p \geq 2)$ be selfmappings of K. Let T_1 be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ and T_i; $i = 2, \ldots, p$ are uniformly L-Lipschitzian mappings. Let $\{\alpha_n\}_{n\geq 0}, \{\beta_n^i\}_{n\geq 0} \subset [0, 1]$, $i = 1, 2, \ldots, p-1$ be real sequences in $[0, 1]$ satisfying $\sum_{n\geq 0} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \beta_n^1 = 0$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}_{n\geq 0}$ by* (4.1). *Then $\{x_n\}_{n\geq 0}$ converges strongly to the common fixed point of $\bigcap_{i=1}^{p} F(T) \neq \phi$.*

Remark 5. Similar results can be found for the iteration processes involved error terms, we omit the details.

Remark 6. If we take $\alpha_n = \dfrac{1}{n^\sigma}$; $0 < \sigma < 1$, then $\sum \alpha_n = \infty$, but also $\sum \alpha_n^2 = \infty$. Hence the conclusions of theorems 3–5 are wrong.

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