Z.-J. WU
KODAI MATH. J.
30 (2007), 55–60

ON SINGULAR DIRECTION OF MEROMORPHIC FUNCTION AND ITS DERIVATIVES

Zhao-Jun Wu

Abstract

In this paper, by using Ahlfors' theory of covering surface, we study singular direction of meromorphic functions and their derivatives, which is a continuous research of Yang Lo in J. London Math. Soc. 25 (1982) 2: 288–296.

1. Introduction and results

Let $f(z)$ be a transcendental meromorphic function defined on the whole complex plane. In this paper, the standard notations of Nevanlinna are used. The singular direction for f is one of main objects studied in the theory of value distribution for meromorphic function. There is a brief history of this research in [4] and the details can be found in book [3] or [10]. Hayman inequality (see [5]) states that $T(r,f)$ can be bounded by the counting functions of the zero points of $f(z)$ and 1 point of $f^{(k)}$ for any positive integer k. Based the inequality, in 1982 Yang lo [9] proved the following results. Suppose that $f(z)$ is a meromorphic function and satisfies the growth condition

$$\limsup_{r\to\infty}\frac{T(r,f)}{\log^3 r}=\infty, \tag{1}$$

then there exists a ray $\arg z=\theta$ such that for any $\varepsilon>0$ and positive integer k, and any finite complex number a, b $(b\neq 0)$, we have

$$\lim_{r\to\infty}\{n(r,\theta_0,\varepsilon,f=a)+n(r,\theta_0,\varepsilon,f^{(k)}=b)\}=\infty.$$

where $n(r,\theta,\varepsilon,a)$ is the number of the solutions of $f(z)=a$ in $\{z:\theta-\varepsilon<\arg z<\theta+\varepsilon\}\cap\{|z|<r\}$, counting with multiplicities. After that, Chen Huaihui [1] prove a quantitative version of Yang Lo [9]. He proved that

---

2000 *Mathematics Subject Classification*: 30D35, 30D30.
Keywords: Hayman direction; covering surface; derivatives function.
Supported by NNSF of China (Grant No.: 10471048).
Received May 31, 2006.

Theorem A (see [2]). *Let $f(z)$ be meromorphic in* **C** *satisfies*

$$\limsup_{r\to\infty}\frac{T(r,f)}{\log^{\lambda} r}=\infty, \quad (\lambda\geq 3) \tag{2}$$

then there exists a ray $\arg z=\theta$ *such that for any* $\varepsilon>0$ *and positive integer* k, *and any* a, b $(b\neq 0)$, *holds*

$$\limsup_{r\to\infty}\frac{n(r,\theta_0,\varepsilon,f=a)+n(r,\theta_0,\varepsilon,f^{(k)}=b)}{\log^{\lambda-1} r}=\infty. \tag{3}$$

There is a difficult question to be asked: "When $\lambda=2$, does Theorem A hold?" The main purpose of this paper is to answer the question and prove the following theorems

Theorem 1. *Suppose that $f(z)$ is a meromorphic function defined on the whole complex plane, and satisfies the growth condition*

$$\limsup_{r\to\infty}\frac{T(r,f)}{\log^{\lambda} r}=\infty, \quad \lambda\geq 2. \tag{4}$$

Then there exists a ray $\arg z=\theta$ *such that for any* $\varepsilon>0$ *and positive integer* k, *and any finite complex number* a, b $(b\neq 0)$, *we have*

$$\limsup_{r\to\infty}\frac{n(r,\theta_0,\varepsilon,f=\infty)+n(r,\theta_0,\varepsilon,f=a)+n(r,\theta_0,\varepsilon,f^{(k)}=b)}{\log^{\lambda-1} r}=\infty.$$

From Theorem 1, we can derive the following theorem easily.

Theorem 2. *Suppose that $f(z)$ is an entire function, then Theorem A holds for $\lambda\geq 2$.*

2. The proof of theorem

We shall prove the Theorem by using Ahlfors' theory of covering surface. First of all, we recall his definition as following (see [8]). Let $f(z)$ be meromorphic in an angular domain $\Omega(\theta,\delta)=\{z:|\arg z-\theta|\leq\delta\}$, where $\theta\in[0,2\pi)$. Let $\Omega(r)$ be the part of $\Omega(\theta,\delta)$, which is contained in $|z|\leq r$ and put

$$S(r,\Omega,f)=\frac{A(r)}{\pi}=\frac{1}{\pi}\int\int_{\Omega(r)}\left(\frac{|f'(z)|}{(1+|f(z)|^2)}\right)^2 r\,d\theta dr, \quad z=re^{i\theta}.$$

$$T(r,\Omega,f)=\int_0^r\frac{S(t,\Omega(\theta,\delta),f)}{t}\,dt.$$

Generally, suppose that E is a plane domain, put

$$S(E,f)=\frac{A(r)}{\pi}=\frac{1}{\pi}\int_E\left(\frac{|f'(z)|}{(1+|f(z)|^2)}\right)^2 r\,d\theta dr, \quad z=re^{i\theta},$$

and $n(E,a)$ is the number of the solution of $f(z)=a$ in E, counting with multiplicities.

In order to prove the Theorem, we shall need several lemmas.

LEMMA 1 (see [6]). *Let $f(z)$ be meromorphic in $|z|<1$, for any $a,b\in C$ and $b\neq 0$, put $N=n(1,f=\infty)+n(1,f=a)+n(1,f^{(k)}=b)+2$, $D=\frac{1}{2}|a,\infty|^2\min\left\{|b|,\frac{1}{|b|}\right\}$. Then for $0<r<1$, we have*

$$S(r,f)<\frac{c}{(1-r)^2}\left\{N\log\frac{80e}{1-r}+\log\frac{1}{|D|}\right\}.$$

where c is a constant which at mostly depends on the k.

LEMMA 2. *Let $f(z)$ be meromorphic in $\Omega_0:\{z:|\arg z-\theta|<\delta_0\}$, where $0\leq\theta<2\pi$. For any $0<\delta<\delta_0$, put $\Omega:\{z:|\arg z-\theta|<\delta\}$ $(\Omega\subset\Omega_0)$. Then for any two finite complex number a, b $(b\neq 0)$, σ $(\sigma>1)$, positive integer m and sufficiently large r, we have*

$$S(r,\Omega,f)\leq A\{n(r\sigma^{2m},\theta,\delta_0,f=\infty)+n(r\sigma^{2m},\theta,\delta_0,f=a)$$
$$+n(r\sigma^{2m},\theta,\delta_0,f^{(k)}=b)\}+B\log r,$$

where $A=\dfrac{m+1}{m}\dfrac{c}{(1-\kappa)^2}\log\dfrac{80e}{1-\kappa}$, $B=\left[2\dfrac{c}{(1-\kappa)^2}\log\dfrac{80e}{1-\kappa}+\log\dfrac{1}{|D|}\right]\Big/ m\log\sigma+1.$

Proof. Put $r_i=\sigma^{mi}$, $i=0,1,2,\dots$, $r_{ij}=\sigma^{mi+j}$, $j=0,1,\dots,m-1$. Then $r_{i0}=r_i$, $r_{im}=r_{i+1}$. Suppose that E is a plane domain, put $N(E)=n(E,f=\infty)+n(E,f=a)+n(E,f^{(k)}=b)$. For any t, $t\geq r_1$, there exists a positive integer l, such that $r_l\leq t\leq r_{l+1}$. It's easy to verify that there exists a positive integer j_0, $0\leq j_0\leq m-1$, such that $\sum_{i=0}^{k+1}N(\{z:r_{i,j_0}<|z|<r_{i,j_0+1}\}\cap\Omega_0)\leq\frac{1}{m}N(\{z:|z|<r_{k+2}\}\cap\Omega_0)$.

By using the same method of [7] or [11], let $\varsigma=\dfrac{z}{r_{i+1,j_0+1}}$ $(0\leq i\leq k)$, then the domain becomes $\Omega_0\cap\{z:r_{i,j_0}<|z|<r_{i+1,j_0+1}\}$ becomes the domain $\Omega_0\cap\left\{\varsigma:\dfrac{1}{\sigma^{m+1}}<|\varsigma|<1\right\}$ on the ς-plane and $\Omega\cap\{z:r_i'<|z|<r_{i+1}'\}$ becomes $\Omega\cap\left\{\varsigma:\dfrac{1}{\sigma^{m+1/2}}<|\varsigma|<\dfrac{1}{\sigma^{1/2}}\right\}$, and the point $\sqrt{r_{i,j_0}r_{i+1,j_0+1}}$ becomes $\sigma^{-(m+1)/2}$ at the same time, where

$$r_i'=\sqrt{r_{i,j_0}r_{i,j_0+1}},\quad r_{i+1}'=\sqrt{r_{i+1,j_0}r_{i+1,j_0+1}}.$$

Note that the image domain is independent of i. If we map $\Omega_0\cap\left\{\varsigma:\dfrac{1}{\sigma^{m+1}}<|\varsigma|<1\right\}$ conformally on $|\xi|<1$, such that $\sigma^{-(m+1)/2}$ becomes $\xi=0$, then the image of $\Omega \cap \left\{ \varsigma : \frac{1}{\sigma^{m+1/2}} < |\varsigma| < \frac{1}{\sigma^{1/2}} \right\}$ is contained in $|\xi| < \kappa < 1$, where κ is a constant, which is only dependent of m, σ, δ, δ_0 and independent of t. By the Lemma 1, the following inequality

$$S(\Omega \cap \{z : r_i' < |z| < r_{i+1}'\}, f) \le \frac{c}{(1-\kappa)^2} \left\{ [N(\Omega_0 \cap \{z : r_{i,j_0} < |z| < r_{i+1,j_0+1}\} + 2] \log \frac{80e}{1-\kappa} + \log \frac{1}{|D|} \right\},$$

holds for $i = 0, 1, \ldots, k$, then

$$\sum_{i=0}^{k} S(\Omega \cap \{z : r_i' < |z| < r_{i+1}'\}, f) \le \sum_{i=0}^{k} \left\{ \frac{c}{(1-\kappa)^2} \left[(N(\Omega_0 \cap \{z : r_{i,j_0} < |z| < r_{i+1,j_0+1}\}) + 2) \log \frac{80e}{1-\kappa} + \log \frac{1}{|D|} \right] \right\}$$

Since $r_k \le r \le r_{k+1}$ and $r_{k+2} \le r\sigma^{2m}$,

$$S(r, \Omega, f) \le \frac{m+1}{m} A_1 N(\{z : |z| < r\sigma^{2m}\} \cap \Omega_0) + B_1 \log r + S(\sigma^{2m}, \Omega, f)$$

where $A_1 = \frac{c}{(1-\kappa)^2} \log \frac{80e}{1-\kappa}$, $B_1 = \left[2 \frac{c}{(1-\kappa)^2} \log \frac{80e}{1-\kappa} + \log \frac{1}{|D|} \right] \Big/ m \log \sigma$.

Therefore when r is large enough, we have

$$S(r, \Omega, f) \le A\{n(r\sigma^{2m}, \theta, \delta_0, f = \infty) + n(r\sigma^{2m}, \theta, \delta_0, f = a) + n(r\sigma^{2m}, \theta, \delta_0, f^{(k)} = b)\} + B \log r,$$

where $A = \frac{m+1}{m} \frac{c}{(1-\kappa)^2} \log \frac{80e}{1-\kappa}$, $B = \left[2 \frac{c}{(1-\kappa)^2} \log \frac{80e}{1-\kappa} + \log \frac{1}{|D|} \right] \Big/ m \log \sigma + 1$.

Now, we are in the position to prove the Theorem 1.

Proof of Theorem 1. Suppose that the Theorem 1 does not hold. Then for any $\theta \in [0, 2\pi)$, we have a $0 < \beta_\theta < \pi/2$ and two finite complex number a_θ, b_θ ($b_\theta \ne 0$) such that

$$\limsup_{r \to \infty} \frac{n(r, \theta, \beta_\theta, f = \infty) + n(r, \theta, \beta_\theta, f = a_\theta) + n(r, \theta, \beta_\theta, f^{(k)} = b_\theta)}{\log^{\lambda-1} r} \le K_\theta < \infty. \tag{5}$$

Because $[0, 2\pi]$ is compact and $[0, 2\pi] \subset \bigcup \left\{ \left(\theta - \frac{\beta_\theta}{2}, \theta - \frac{\beta_\theta}{2} \right), \theta \in [0, 2\pi) \right\}$, then we can choose finitely many $\left(\theta_i - \frac{\beta_{\theta_i}}{2}, \theta_i - \frac{\beta_{\theta_i}}{2}\right)$ $(i = 1, 2, \dots, T)$, such that $[0, 2\pi] \subset \bigcup\left\{\left(\theta - \frac{\beta_\theta}{2}, \theta - \frac{\beta_\theta}{2}\right), i = 1, 2, \dots, T\right\}$. For any $\Omega(\theta_i, \beta_{\theta_i})$, put $\Omega_i = \Omega\left(\theta_i, \frac{\beta_{\theta_i}}{2}\right)$. By Lemma 2, we have

$$S(r, \Omega_i, f) \le A_{\theta_i}\{n(r\sigma^{2m}, \theta_i, \beta_{\theta_i}, f = \infty) + n(r\sigma^{2m}, \theta_i, \beta_{\theta_i}, f = a) + n(r\sigma^{2m}, \theta_i, \beta_{\theta_i}, f^{(k)} = b)\} + B_{\theta_i} \log r,$$

Put $A = \max_{1 \le i \le T}\{A_{\theta_i}\}$ and $B = \max_{1 \le i \le T}\{B_{\theta_i}\}$. The above expression sum from $i = 1$ to T, we have

$$S(r, f) \le A \sum_{i=1}^{T}\{n(r\sigma^{2m}, \theta_i, \beta_{\theta_i}, f = \infty) + n(r\sigma^{2m}, \theta_i, \beta_{\theta_i}, f = a) + n(r\sigma^{2m}, \theta_i, \beta_{\theta_i}, f^{(k)} = b)\} + BT \log r,$$

Hence

$$\begin{aligned}
T(r, f) &= \int_0^r \frac{S(t, f)}{t}\, dt \\
&\le \int_1^r \frac{A \sum_{i=1}^{T}\{n(t\sigma^{2m}, \theta_i, \beta_{\theta_i}, f = \infty) + n(t\sigma^{2m}, \theta_i, \beta_{\theta_i}, f = a) + n(t\sigma^{2m}, \theta_i, \beta_{\theta_i}, f^{(k)} = b)\}}{t}\, dt \\
&\quad + \int_1^r \frac{BT \log t}{t}\, dt + \int_0^1 \frac{S(t, f)}{t}\, dt, \\
&\le A \sum_{i=1}^{T}\{n(r\sigma^{2m}, \theta_i, \beta_{\theta_i}, f = \infty) + n(r\sigma^{2m}, \theta_i, \beta_{\theta_i}, f = a) \\
&\quad + n(r\sigma^{2m}, \theta_i, \beta_{\theta_i}, f^{(k)} = b)\} \log r\sigma^{2m} + BT \log^2 r + T(1, f)
\end{aligned}$$

So, by using (5), we have

$$\begin{aligned}
&\limsup_{r\to\infty} \frac{T(r, f)}{\log^\lambda r} \\
&\le \limsup_{r\to\infty} \frac{A \sum_{i=1}^{T}\{n(r\sigma^{2m}, \theta_i, \beta_{\theta_i}, f = \infty) + n(r\sigma^{2m}, \theta_i, \beta_{\theta_i}, f = a) + n(r\sigma^{2m}, \theta_i, \beta_{\theta_i}, f^{(k)} = b)\} \log r\sigma^{2m}}{\log^\lambda r} \\
&\le A \sum_{i=1}^{T} K_{\theta_i} < \infty.
\end{aligned}$$

This contradicts (4) and the Theorem follows.

References

[1] H. Chen, The singular directions of a meromorphic functions of order zero corresponding to Hayman's inequality, Acta Math. Sinica **30** (1987), 234–237.

[2] T. Y. Chern, A note on Hayman direction for meromorphic functions of finite logarithmic order, Southeast Asian Bulletin of Mathematics **24** (2000), 521–523.

[3] C. T. Chuang, Singular directions of meromorphic functions, Science Press, Beijing, 1982.

[4] H. Guo, J. Zheng and T. W. Ng, On a new singular direction of meromorphic functions, Bull. Austral. Math. Soc. **69** (2004), 277–287.

[5] W. Hayman, Meromorphic functions, Clarendon Press, Oxford, 1964.

[6] P. Hu, A generalization of Q. D. Chang and Le Yang's theorem, Acta Sci. Natur. Univ. Shandong **25** (1990), 137–147.

[7] Y. Lu and G. Zhang, On Nevanlinna direction of meromorphic function, Science in China, Ser. A. **3** (1983), 215–224.

[8] M. Tsuji, Potential theory in modern function theory, Maruzen Co. LTD, Tokyo, 1959.

[9] L. Yang, Meromorphic functions and their derivatives, J. London Math. Soc. **25** (1982), 288–296.

[10] L. Yang, Value distribution theory, Science Press, Beijing, 1982 (in Chinese), Springer-Verlag, 1993 (in English).

[11] Y. Yang and M. Liu, On the Borel direction of K-quasimeromorphic mappings, Acta Math. Scientia, Ser. B. **24** (2004), 75–82.

Zhao-Jun Wu
School of Mathematics
South China Normal University
Guangzhou, 510631
P.R. China