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JACOBI FIELDS OF THE TANAKA-WEBSTER CONNECTION ON SASAKIAN MANIFOLDS

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Abstract

We build a variational theory of geodesics of the Tanaka-Webster connection ∇ on a strictly pseudoconvex CR manifold M. Given a contact form θ on M such that (M, θ) has nonpositive pseudohermitian sectional curvature $(k_\theta(\sigma) \le 0)$ we show that (M, θ) has no horizontally conjugate points. Moreover, if (M, θ) is a Sasakian manifold such that $k_\theta(\sigma) \ge k_0 > 0$ then we show that the distance between any two consecutive conjugate points on a lengthy geodesic of ∇ is at most $\pi/(2\sqrt{k_0})$. We obtain the first and second variation formulae for the Riemannian length of a curve in M and show that in general geodesics of ∇ admitting horizontally conjugate points do not realize the Riemannian distance.

1. Introduction

Sasakian manifolds possess a rich geometric structure (cf. [5], p. 73–80) and are perhaps the closest odd dimensional analog of Kählerian manifolds. In particular the concept of holomorphic sectional curvature admits a Sasakian counterpart, the so called φ-sectional curvature $H(X)$ (cf. [5], p. 94) and it is a natural problem (as well as in Kählerian geometry, cf. e.g. [17], p. 171, and p. 368–373) to investigate how restrictions on $H(X)$ influence upon the topology of the manifold. An array of findings in this direction are described in [5], p. 77–80. For instance, by a result of M. Harada, [11], for any compact regular Sasakian manifold M satisfying the inequality $h > k^2$ the fundamental group $\pi_1(M)$ is cyclic. Here $h = \inf\{H(X) : X \in T_x(M), \|X\| = 1, x \in M\}$ and it is also assumed that the least upper bound of the sectional curvature of M is $1/k^2$. Moreover, if additionally M has minimal diameter π then M is isometric to the standard sphere S^{2n+1}, cf. [12], p. 200.

In the present paper we embrace a different point of view, that of pseudohermitian geometry (cf. [27]). To describe it we need to introduce a few basic objects (cf. [5], p. 19–28). Let M be a $(2n+1)$-dimensional C^∞ manifold and (φ, ξ, η, g) a *contact metric structure* i.e. φ is an endomorphism of the tangent bundle, ξ is a tangent vector field, η is a differential 1-form, and g is a Riemannian metric on M such that

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$$\varphi^2 = -I + \eta \otimes \xi, \quad \varphi(\xi) = 0, \quad \eta(\xi) = 1,$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in T(M),$$

and $\Omega = d\eta$ (the *contact condition*) where $\Omega(X, Y) = g(X, \varphi Y)$. Any contact Riemannian manifold $(M, (\varphi, \xi, \eta, g))$ admits a natural almost CR structure

$$T_{1,0}(M) = \{X - iJX : X \in \mathrm{Ker}(\eta)\}$$

$(i = \sqrt{-1})$ i.e. it satisfies (2) below. By a result of S. Ianuş, [14], if (φ, ξ, η) is *normal* (i.e. $[\varphi, \varphi] + 2(d\eta) \otimes \xi = 0$) then $T_{1,0}(M)$ is integrable, i.e. it obeys to (3) in Section 2. Cf. [5], p. 57–61, for the geometric interpretation of normality, as related to the classical embeddability theorem for real analytic CR structures (cf. [1]). Integrability of $T_{1,0}(M)$ is required in the construction of the Tanaka-Webster connection of (M, η), cf. [25], [27] and definitions in Section 2 (although many results in pseudohermitian geometry are known to carry over to arbitrary contact Riemannian manifolds, cf. [26] and more recently [2], [6]). A manifold carrying a contact metric structure (φ, ξ, η, g) whose underlying contact structure (φ, ξ, η) is normal is a *Sasakian manifold* (and g is a *Sasakian metric*). The main tool in the Riemannian approach to the study of Sasakian geometry is the availability of a variational theory of geodesics of the Levi-Civita connection of (M, g) (cf. e.g. [12], 194–197). In this paper we start the elaboration of a similar theory regarding the geodesics of the Tanaka-Webster connection ∇ of (M, η) and give a few applications (cf. Theorems 6–7 and 13 below). Our motivation is twofold. First, we aim to study the topology of Sasakian manifolds under restrictions on the curvature of ∇ and conjecture that Carnot-Carathéodory complete Sasakian manifolds whose pseudohermitian Ricci tensor ρ satisfies $\rho(X, X) \geq (2n - 1)k_0\|X\|^2$ for some $k_0 > 0$ and any $X \in \mathrm{Ker}(\eta)$ must be compact. Second, the relationship between the sub-Riemannian geodesics of the sub-Riemannain manifold $(M, \mathrm{Ker}(\eta), g)$ and the geodesics of ∇ (emphasized by our Corollary 1) together with R. S. Strichartz's arguments (cf. [23], p. 245 and 261–262) clearly indicates that a variational theory of geodesics of ∇ is the key requirement in bringing results such as those in [24] or [22] into the realm of subelliptic theory. In [3] one obtains a pseudohermitian version of the Bochner formula (cf. e.g. [4], p. 131) implying a lower bound on the first nonzero eigenvalue λ_1 of the sublaplacian Δ_b of a compact Sasakian manifold

$$-\lambda_1 \geq 2nk/(2n - 1) \tag{1}$$

(a CR analog to the Lichnerowicz theorem, [19]). It is likely that a theory of geodesics of ∇ may be employed to show that equality in (1) implies that M is CR isomorphic to a sphere S^{2n+1} (the CR analog to Obata's result, [22]).

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2. Sub-Riemannian geometry on CR manifolds

Let M be an orientable $(2n+1)$-dimensional C^∞ manifold. A *CR structure* on M is a complex distribution $T_{1,0}(M) \subset T(M) \otimes \mathbf{C}$, of complex rank n, such that

$$T_{1,0}(M) \cap T_{0,1}(M) = (0) \tag{2}$$

and

$$Z, W \in T_{1,0}(M) \Rightarrow [Z, W] \in T_{1,0}(M) \tag{3}$$

(the *formal integrability property*). Here $T_{0,1}(M) = \overline{T_{1,0}(M)}$ (overbars denote complex conjugates). The integer n is the *CR dimension*. The pair $(M, T_{1,0}(M))$ is a *CR manifold* (of *hypersurface type*). Let $H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$ be the *Levi distribution*. It carries the complex structure $J : H(M) \to H(M)$ given by $J(Z + \overline{Z}) = i(Z - \overline{Z})$ $(i = \sqrt{-1})$. Let $H(M)^\perp \subset T^*(M)$ the conormal bundle, i.e. $H(M)^\perp_x = \{\omega \in T^*_x(M) : \mathrm{Ker}(\omega) \supseteq H(M)_x\}$, $x \in M$. A *pseudohermitian structure* on M is a globally defined nowhere zero cross-section θ in $H(M)^\perp$. Pseudohermitian structures exist as the orientability assumption implies that $H(M)^\perp \approx M \times \mathbf{R}$ (a diffeomorphism) i.e. $H(M)^\perp$ is a trivial line bundle. For a review of the main notions of CR and pseudohermitian geometry one may see [8].

Let $(M, T_{1,0}(M))$ be a CR manifold, of CR dimension n. Let θ be a pseudohermitian structure on M. The *Levi form* is

$$L_\theta(Z, \overline{W}) = -i(d\theta)(Z, \overline{W}), \quad Z, W \in T_{1,0}(M).$$

M is *nondegnerate* if L_θ is nondegenerate for some θ. Two pseudohermitian structures θ and $\hat{\theta}$ are related by

$$\hat{\theta} = f\theta \tag{4}$$

for some C^∞ function $f : M \to \mathbf{R}\setminus\{0\}$. Since $L_{\hat{\theta}} = fL_\theta$ nondegeneracy of M is a *CR invariant* notion, i.e. it is invariant under a transformation (4) of the pseudohermitian structure. The whole setting bears an obvious analogy to conformal geometry (a fact already exploited by many authors, cf. e.g. [10], [25]–[27]). If M is nondegenerate then any pseudohermitian structure θ on M is actually a *contact form*, i.e. $\theta \wedge (d\theta)^n$ is a volume form on M. By a fundamental result of N. Tanaka and S. Webster (cf. *op. cit.*) on any nondegenerate CR manifold on which a contact form θ has been fixed there is a canonical linear connection ∇ (the *Tanaka-Webster connection* of (M, θ)) compatible to the Levi distribution and its complex structure, as well as to the Levi form. Precisely, let T be the globally defined nowhere zero tangent vector field on M, transverse to $H(M)$, uniquely determined by $\theta(T) = 1$ and $T \rfloor d\theta = 0$ (the *characteristic direction* of $d\theta$). Let

$$G_\theta(X, Y) = (d\theta)(X, JY), \quad X, Y \in H(M),$$

(the *real* Levi form) and consider the semi-Riemannian metric g_θ on M given by

$$g_\theta(X, Y) = G_\theta(X, Y), \quad g_\theta(X, T) = 0, \quad g_\theta(T, T) = 1,$$

for any $X, Y \in H(M)$ (the *Webster metric* of (M, θ)). Let us extend J to an endomorphism of the tangent bundle by setting $JT = 0$. Then there is a unique linear connection ∇ on M such that i) $H(M)$ is parallel with respect to ∇, ii) $\nabla g_\theta = 0$, $\nabla J = 0$, and iii) the torsion T_∇ of ∇ is *pure*, i.e.

$$T_\nabla(Z, W) = T_\nabla(\bar{Z}, \overline{W}) = 0, \quad T_\nabla(Z, \overline{W}) = 2iL_\theta(Z, \overline{W})T, \tag{5}$$

for any $Z, W \in T_{1,0}(M)$, and

$$\tau \circ J + J \circ \tau = 0, \tag{6}$$

where $\tau(X) = T_\nabla(T, X)$ for any $X \in T(M)$ (the *pseudohermitian torsion* of ∇). The Tanaka-Webster connection is a pseudohermitian analog to both the Levi-Civita connection in Riemannian geometry and the Chern connection in Hermitian geometry.

A CR manifold M is *strictly pseudoconvex* if L_θ is positive definite for some θ. If this is the case then the Webster metric g_θ is a Riemannian metric on M and if we set $\varphi = J$, $\xi = -T$, $\eta = -\theta$ and $g = g_\theta$ then (φ, ξ, η, g) is a contact metric structure on M. Also (φ, ξ, η, g) is normal if and only if $\tau = 0$. If this is the case g_θ is a Sasakian metric and (M, θ) is a Sasakian manifold.

We proceed by recalling a few concepts from *sub-Riemannian geometry* (cf. e.g. R. S. Strichartz, [23]) on a strictly pseudoconvex CR manifold. Let $(M, T_{1,0}(M))$ be a strictly pseudoconvex CR manifold, of CR dimension n. Let θ be a contact form on M such that the Levi form G_θ is positive definite. The Levi distribution $H(M)$ is *bracket generating* i.e. the vector fields which are sections of $H(M)$ together with all brackets span $T_x(M)$ at each point $x \in M$, merely as a consequence of the nondegeneracy of the given CR structure. Indeed, let ∇ be the Tanaka-Webster connection of (M, θ) and let $\{T_\alpha : 1 \leq \alpha \leq n\}$ be a local frame of $T_{1,0}(M)$, defined on the open set $U \subseteq M$. By the purity property (5)

$$\Gamma^{\bar{\gamma}}_{\alpha\bar{\beta}} T_{\bar{\gamma}} - \Gamma^{\gamma}_{\bar{\beta}\alpha} T_\gamma - [T_\alpha, T_{\bar{\beta}}] = 2ig_{\alpha\bar{\beta}} T, \tag{7}$$

where Γ^A_{BC} are the coefficients of ∇ with respect to $\{T_\alpha\}$

$$\nabla_{T_B} T_C = \Gamma^A_{BC} T_A$$

and $g_{\alpha\bar{\beta}} = L_\theta(T_\alpha, T_{\bar{\beta}})$. Our conventions as to the range of indices are $A, B, C, \ldots \in \{0, 1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$ and $\alpha, \beta, \gamma, \ldots \in \{1, \ldots, n\}$ (where $T_0 = T$). Note that $\{T_\alpha, T_{\bar{\alpha}}, T\}$ is a local frame of $T(M) \otimes \mathbf{C}$ on U. If $T_\alpha = X_\alpha - iJX_\alpha$ are the real and imaginary parts of T_α then (7) shows that $\{X_\alpha, JX_\alpha\}$ together with their brackets span the whole of $T_x(M)$, for any $x \in U$. Actually more has been proved. Given $x \in M$ and $v \in H(M)_x \backslash \{0\}$ there is an open neighborhood $U \subseteq M$ of x and a local frame $\{T_\alpha\}$ of $T_{1,0}(M)$ on U such that $T_1(x) = v - iJ_x v$, hence v is a 2-*step bracket generator* so that $H(M)$ satisfies the *strong bracket generating hypothesis* (cf. the terminology in [23], p. 224).

Let $x \in M$ and $g(x) : T^*_x(M) \to H(M)_x$ determined by

$$G_{\theta,x}(v, g(x)\xi) = \xi(v), \quad v \in H(M)_x,\ \xi \in T_x^*(M).$$

Note that the kernel of g is precisely the conormal bundle $H(M)^\perp$. In other words G_θ is a *sub-Riemannian metric* on $H(M)$ and g its alternative description (cf. also (2.1) in [23], p. 225). If $\hat{\theta} = e^u \theta$ is another contact form such that $G_{\hat{\theta}}$ is positive definite ($u \in C^\infty(M)$) then $\hat{g} = e^{-u} g$. Clearly if the Levi form L_θ is only nondegenerate then $(M, H(M), G_\theta)$ is a *sub-Lorentzian manifold*, cf. the terminology in [23], p. 224.

Let $\gamma : I \to M$ be a piecewise C^1 curve (where $I \subseteq \mathbf{R}$ is an interval). Then γ is a *lengthy curve* if $\dot{\gamma}(t) \in H(M)_{\gamma(t)}$ for every $t \in I$ such that $\dot{\gamma}(t)$ is defined. For instance, any geodesic of ∇ (i.e. any C^1 curve $\gamma(t)$ such that $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$) of initial data (x, v), $v \in H(M)_x$, is lengthy (as a consequence of $\nabla g_\theta = 0$ and $\nabla T = 0$). A piecewise C^1 curve $\xi : I \to T^*(M)$ is a *cotangent lift* of γ if $\xi(t) \in T^*_{\gamma(t)}(M)$ and $g(\gamma(t))\xi(t) = \dot{\gamma}(t)$ for every t (where defined). Clearly cotangent lifts of a given lengthy curve γ exist (cf. also Proposition 1 below). Also, cotangent lifts of γ are uniquely determined modulo sections of the conormal bundle $H(M)^\perp$ along γ. That is, if $\eta : I \to T^*(M)$ is another cotangent lift of γ then $\eta(t) - \xi(t) \in H(M)^\perp_{\gamma(t)}$ for every t. The *length* of a lengthy curve $\gamma : I \to M$ is given by

$$L(\gamma) = \int_I \{\xi(t)[g(\gamma(t))\xi(t)]\}^{1/2}\, dt.$$

The definition doesn't depend upon the choice of cotangent lift ξ of γ. The *Carnot-Carathéodory distance* $\rho(x, y)$ among $x, y \in M$ is the infimum of the lengths of all lengthy curves joining x and y. That ρ is indeed a distance function on M follows from a theorem of W. L. Chow, [7], according to which any two points $x, y \in M$ may be joined by a lengthy curve (provided that M is connected).

Let g_θ be the Webster metric of (M, θ). Then g_θ is a *contraction* of the sub-Riemannian metric G_θ (G_θ is an *expansion* of g_θ), cf. [23], p. 230. Let d be the distance function corresponding to the Webster metric. The length $L(\gamma)$ of a lengthy curve γ is precisely its length with respect to g_θ hence

$$d(x, y) \le \rho(x, y), \quad x, y \in M. \tag{8}$$

While ρ and d are known to be inequivalent distance functions, they do determine the same topology. For further details on Carnot-Carathéodory metrics see J. Mitchell, [21].

Let $(U, x^1, \ldots, x^{2n+1})$ be a system of local coordinates on M and let us set $G_{ij} = g_\theta(\partial_i, \partial_j)$ (where ∂_i is short for $\partial/\partial x^i$) and $[G^{ij}] = [G_{ij}]^{-1}$. Using

$$G_\theta(X, g\, dx^i) = (dx^i)(X), \quad X \in H(M),$$

for $X = \partial_k - \theta_k T$ (where $\theta_i = \theta(\partial_i)$) leads to

$$g^{ij}(G_{jk} - \theta_j \theta_k) = \delta^i_k - \theta_k T^i \tag{9}$$

where $g\, dx^i = g^{ij}\partial_j$ and $T = T^i \partial_i$. On the other hand $g^{ij}\theta_j = \theta(g\, dx^i) = 0$ so that (9) yields

$$g^{ij} = G^{ij} - T^i T^j. \tag{10}$$

As an application we introduce a *canonical* cotangent lift of a given lengthy curve on M.

Proposition 1. *Let $\gamma : I \to M$ be a lengthy curve and let $\xi : I \to T^*(M)$ be given by $\xi(t)T_{\gamma(t)} = 1$ and $\xi(t)X = g_\theta(\dot\gamma, X)$, for any $X \in H(M)_{\gamma(t)}$. Then ξ is a cotangent lift of γ.*

Proof. Let $x^i(t)$ be the components of γ with respect to the chosen local coordinate system. By the very definition of ξ

$$\xi_j = G_{ij}\frac{dx^i}{dt} + \theta_j. \tag{11}$$

Hence

$$\begin{aligned} g\xi = \xi_j g^{ij}\partial_i &= g^{ij}\left(G_{jk}\frac{dx^k}{dt} + \theta_j\right)\partial_i = g^{ij}G_{jk}\frac{dx^k}{dt}\partial_i \\ &= (G^{ij} - T^iT^j)G_{jk}\frac{dx^k}{dt}\partial_i = (\delta^i_k - T^i\theta_k)\frac{dx^k}{dt}\partial_i \\ &= \dot\gamma(t) - \theta(\dot\gamma(t))T = \dot\gamma(t). \end{aligned}$$

We recall (cf. [23], p. 233) that a *sub-Riemannian geodesic* is a C^2 curve $\gamma(t)$ in M satisfying the Hamilton-Jacobi equations associated to the Hamiltonian function $H(x,\xi) = \frac{1}{2}g^{ij}(x)\xi_i\xi_j$ that is

$$\frac{dx^i}{dt} = g^{ij}(\gamma(t))\xi_j(t), \tag{12}$$

$$\frac{d\xi_k}{dt} = -\frac{1}{2}\frac{\partial g^{ij}}{\partial x^k}(\gamma(t))\xi_i(t)\xi_j(t), \tag{13}$$

for some cotangent lift $\xi(t) \in T^*(M)$ of $\gamma(t)$. Our purpose is to show that

Theorem 1. *Let M be a strictly pseudoconvex CR manifold and θ a contact form on M such that G_θ is positive definite. A C^2 curve $\gamma(t) \in M$, $|t| < \varepsilon$, is a sub-Riemannian geodesic of $(M, H(M), G_\theta)$ if and only if $\gamma(t)$ is a solution to*

$$\nabla_{\dot\gamma}\dot\gamma = -2b(t)J\dot\gamma, \quad b'(t) = A(\dot\gamma,\dot\gamma), \quad |t| < \varepsilon, \tag{14}$$

with $\dot\gamma(0) \in H(M)_{\gamma(0)}$, for some C^2 function $b : (-\varepsilon, \varepsilon) \to \mathbf{R}$. Here $A(X,Y) = g_\theta(\tau X, Y)$ is the pseudohermitian torsion of (M,θ).

According to the terminology in [23], p. 237, the canonical cotangent lift $\xi(t)$ of a given lengthy curve $\gamma(t)$ is the one determined by the orthogonality requirement

$$V_j(\xi)\Gamma^j(\xi, v) = 0, \tag{15}$$

for any $v \in H(M)^\perp_{\gamma(t)}$ and any $|t| < \varepsilon$, where

$$V_k(\xi) = \frac{d\xi_k}{dt} + \frac{1}{2}\frac{\partial g^{ij}}{\partial x^k}\xi_i\xi_j,$$

$$\Gamma^i(\xi, v) = \Gamma^{ijk}\xi_j v_k, \quad \Gamma^{ijk} = \frac{1}{2}\left(g^{\ell j}\frac{\partial g^{ik}}{\partial x^\ell} + g^{\ell k}\frac{\partial g^{ij}}{\partial x^\ell} - g^{\ell i}\frac{\partial g^{jk}}{\partial x^\ell}\right).$$

Let $\gamma(t)$ be a lengthy curve and $\xi_0(t)$ the cotangent lift of $\gamma(t)$ furnished by Proposition 1. Then any other cotangent lift $\xi(t)$ is given by

$$\xi(t) = \xi_0(t) + a(t)\theta_{\gamma(t)}, \quad |t| < \varepsilon, \tag{16}$$

for some $a : (-\varepsilon, \varepsilon) \to \mathbf{R}$. We shall need the following result (a replica of Lemma 4.4 in [23], p. 237)

Lemma 1. *The unique cotangent lift $\xi(t)$ of $\gamma(t)$ satisfying the orthogonality condition* (15) *is given by* (16) *where*

$$a(t) = -\frac{1}{2}|\dot\gamma(t)|^{-2} g_\theta(\nabla_{\dot\gamma}\dot\gamma, J\dot\gamma) - 1, \quad |t| < \varepsilon.$$

Proof. By (11) and (16)

$$\begin{aligned} V_k(\xi) = {} & V_k(\xi_0) + a'(t)\theta_k + a(t)\frac{\partial\theta_k}{\partial x^\ell}\frac{dx^\ell}{dt} \\ & + \frac{1}{2}\frac{\partial g^{ij}}{\partial x^k}[a(t)(\xi_i^0\theta_j + \xi_j^0\theta_i) + a(t)^2\theta_i\theta_j] \end{aligned}$$

(where $\xi_0 = \xi_i^0\, dx^i$) and using

$$\frac{\partial g^{ij}}{\partial x^k}\theta_i\theta_j = 0$$

we obtain

$$V_i(\xi) = V_i(\xi_0) + a'(t)\theta_i + 2a(t)(d\theta)(\dot\gamma, \partial_i). \tag{17}$$

Note that $\Gamma^i(\xi, v) = \Gamma^i(\xi_0, v)$ and $\Gamma^{ijk}\theta_j v_k = 0$, for any $v \in H(M)^\perp_{\gamma(t)}$. Let us contract (17) with $\Gamma^i(\xi, v)$ and use (15) and $\Gamma^i(\xi_0, v)\theta_i = 0$. This ought to determine $a(t)$. Indeed

$$V_i(\xi_0)\Gamma^i(\xi_0, v) + 2a(t)(d\theta)(\dot\gamma, \Gamma(\xi_0, v)) = 0, \tag{18}$$

where $\Gamma(\xi, v) = \Gamma^i(\xi, v)\partial_i$. On the other hand, a calculation based on (10)–(11) shows that

$$V_k(\xi_0) = G_{k\ell}\left(\frac{d^2x^\ell}{dt^2} + \left|\begin{matrix}\ell \\ ij\end{matrix}\right|\frac{dx^i}{dt}\frac{dx^j}{dt}\right) + 2(d\theta)(\dot\gamma, \partial_k),$$

$$\left|\begin{matrix}\ell \\ ij\end{matrix}\right| = G^{\ell k}|ij, k|, \quad |ij, k| = \frac{1}{2}\left(\frac{\partial G_{ik}}{\partial x^j} + \frac{\partial G_{jk}}{\partial x^i} - \frac{\partial G_{ij}}{\partial x^k}\right),$$

hence

$$V_i(\xi_0) = G_{ij}(D_{\dot\gamma}\dot\gamma)^j + 2(d\theta)(\dot\gamma, \partial_i), \tag{19}$$

where D is the Levi-Civita connection of (M, g_θ). Then (18)–(19) yield

$$g_\theta(D_{\dot\gamma}\dot\gamma, \Gamma(\xi_0, v)) + 2(a(t)+1)(d\theta)(\dot\gamma, \Gamma(\xi_0, v)) = 0,$$

for any $v \in H(M)^\perp_{\gamma(t)}$. Yet $H(M)^\perp$ is the span of θ hence

$$g_\theta(\Gamma(\xi_0, \theta), D_{\dot\gamma}\dot\gamma + 2(a(t)+1)J\dot\gamma) = 0$$

and

$$\Gamma^i(\xi_0, \theta) = -G^{ij}(d\theta)(\dot\gamma, \partial_j),$$

(because of $T \rfloor d\theta = 0$) yields

$$2(a(t)+1)|\dot\gamma(t)|^2 + g_\theta(D_{\dot\gamma}\dot\gamma, J\dot\gamma) = 0. \tag{20}$$

Lemma 1 is proved. At this point we may prove Theorem 1. Let $\gamma(t) \in M$ be a sub-Riemannian geodesic of $(M, H(M), G_\theta)$. Then there is a cotangent lift $\xi(t) \in T^*(M)$ of $\gamma(t)$ (given by (16) for some $a : (-\varepsilon, \varepsilon) \to \mathbf{R}$) such that $V(\xi) = 0$ (where $V(\xi) = V^i(\xi)\partial_i$). In particular the orthogonality condition (15) is identically satisfied, hence $a(t)$ is determined according to Lemma 1. Using (17) and (19) the sub-Riemannian geodesics equations are

$$G_{ij}(D_{\dot\gamma}\dot\gamma)^j + a'(t)\theta_i + 2(a(t)+1)(d\theta)(\dot\gamma, \partial_i) = 0$$

or

$$D_{\dot\gamma}\dot\gamma + a'(t)T + 2(a(t)+1)J\dot\gamma = 0. \tag{21}$$

We recall (cf. e.g. [10]) that $D = \nabla - (d\theta + A) \otimes T$ on $H(M) \otimes H(M)$ hence (by the uniqueness of the direct sum decomposition $T(M) = H(M) \oplus \mathbf{R}T$) the equations (21) become

$$\nabla_{\dot\gamma}\dot\gamma + 2(a(t)+1)J\dot\gamma = 0, \quad a'(t) = A(\dot\gamma, \dot\gamma),$$

(and we set $b = a + 1$). Theorem 1 is proved.

Corollary 1. *Let M be a strictly pseudoconvex CR manifold and θ a contact form on M with vanishing pseudohermitian torsion ($\tau = 0$). Then any lengthy geodesic of the Tanaka-Webster connection ∇ of (M, θ) is a sub-Riemannian geodesic of $(M, H(M), G_\theta)$. Viceversa, if every lengthy geodesic $\gamma(t)$ of ∇ is a sub-Riemannian geodesic then $\tau = 0$.*

Indeed, if $\nabla_{\dot\gamma}\dot\gamma = 0$ then the equations (14) (with $b = 0$) are identically satisfied.

Proposition 2. *Let $\gamma(t) \in M$ be a sub-Riemannian geodesic and $s = \phi(t)$ a C^2 diffeomorphism. If $\gamma(t) = \bar\gamma(\phi(t))$ then $\bar\gamma(s)$ is a sub-Riemannian geodesic if and only if ϕ is affine, i.e. $\phi(t) = \alpha t + \beta$, for some $\alpha, \beta \in \mathbf{R}$. In particular, every sub-Riemannian geodesic may be reparametrized by arc length $\phi(t) = \int_0^t |\dot\gamma(u)|\, du$.*

Proof. Set $k = |\dot{\gamma}(0)|^2 > 0$. By taking the inner product of the first equation in (14) by $\dot{\gamma}(t)$ it follows that $d|\dot{\gamma}(t)|^2/dt = 0$, hence $|\dot{\gamma}(t)|^2 = k$, $|t| < \varepsilon$. Throughout the proof an overbar indicates the similar quantities associated to $\bar{\gamma}(s)$. In particular $\bar{k} = \phi'(0)^{-2}k$. Locally

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = -2(a+1)J^i_j\frac{dx^j}{dt}. \tag{22}$$

On the other hand, using (20) and

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = \phi''(t)\frac{d^2\bar{x}^i}{ds^2} + \phi'(t)^2\left(\frac{d^2\bar{x}^i}{ds^2} + \Gamma^i_{jk}\frac{d\bar{x}^j}{ds}\frac{d\bar{x}^k}{ds}\right)$$

we obtain

$$k(a+1) = \bar{k}(\bar{a}+1)\phi'(t)^3.$$

Then (22) may be written

$$k\phi''(t)\frac{d\bar{\gamma}}{ds} + 2(\bar{a}+1)\phi'(t)^2[\bar{k}\phi'(t)^2 - k]J\frac{d\bar{\gamma}}{ds} = 0$$

hence $\phi''(t) = 0$. Proposition 2 is proved.

Let $S^1 \to C(M) \xrightarrow{\pi} M$ be the canonical circle bundle over M (cf. e.g. [8], p. 104). Let Σ be the tangent to the S^1-action. Next, let us consider the 1-form σ on $C(M)$ given by

$$\sigma = \frac{1}{n+2}\left\{dr + \pi^*\left(i\omega^\alpha_\alpha - \frac{i}{2}g^{\alpha\bar{\beta}}\,dg_{\alpha\bar{\beta}} - \frac{R}{4(n+1)}\theta\right)\right\},$$

where r is a local fibre coordinate on $C(M)$ (so that locally $\Sigma = \partial/\partial r$) and $R = g^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}}$ is the pseudohermitian scalar curvature of (M,θ). Then σ is a connection 1-form in $S^1 \to C(M) \to M$. Given a tangent vector $v \in T_x(M)$ and a point $z \in \pi^{-1}(x)$ we denote by $v^\uparrow$ its horizontal lift with respect to σ, i.e. the unique tangent vector $v^\uparrow \in \mathrm{Ker}(\sigma_z)$ such that $(d_z\pi)v^\uparrow = v$. The *Fefferman metric* of (M,θ) is the Lorentz metric on $C(M)$ given by

$$F_\theta = \pi^*\tilde{G}_\theta + 2(\pi^*\theta)\odot\sigma,$$

where $\tilde{G}_\theta = G_\theta$ on $H(M)\otimes H(M)$ and $\tilde{G}_\theta(X,T) = 0$, for any $X \in T(M)$. Also $\odot$ is the symmetric tensor product. We close this section by demonstrating the following geometric interpretation of sub-Riemannian geodesics (of a strictly pseudoconvex CR manifold).

Theorem 2. *Let M be a strictly pseudoconvex CR manifold, θ a contact form on M such that G_θ is positive definite, and F_θ the Fefferman metric of (M,θ). For any geodesic $z:(-\varepsilon,\varepsilon)\to C(M)$ of F_θ if the projection $\gamma(t) = \pi(z(t))$ is lengthy then $\gamma:(-\varepsilon,\varepsilon)\to M$ is a sub-Riemannian geodesic of $(M,H(M),G_\theta)$. Viceversa, let $\gamma(t)\in M$ be a sub-Riemannian geodesic. Then any solution $z(t)\in C(M)$ to the ODE*

$$\dot{z}(t) = \dot{\gamma}(t)^{\uparrow} + ((n+2)/2)b(t)\Sigma_{z(t)}, \tag{23}$$

where $b(t) = a(t) + 1$ *is given by* (20), *is a geodesic of* F_θ.

Here $\dot{\gamma}(t)^{\uparrow} \in \mathrm{Ker}(\sigma_{z(t)})$ and $(d_{z(t)}\pi)\dot{\gamma}(t)^{\uparrow} = \dot{\gamma}(t)$. To prove Theorem 2 we shall need the following

Lemma 2. *For any* $X, Y \in H(M)$

$$\nabla^{C(M)}_{X^{\uparrow}} Y^{\uparrow} = (\nabla_X Y)^{\uparrow} - (d\theta)(X,Y)T^{\uparrow} - (A(X,Y) + (d\sigma)(X^{\uparrow}, Y^{\uparrow}))\hat{\Sigma},$$
$$\nabla^{C(M)}_{X^{\uparrow}} T^{\uparrow} = (\tau X + \phi X)^{\uparrow},$$
$$\nabla^{C(M)}_{T^{\uparrow}} X^{\uparrow} = (\nabla_T X + \phi X)^{\uparrow} + 2(d\sigma)(X^{\uparrow}, T^{\uparrow})\hat{\Sigma},$$
$$\nabla^{C(M)}_{X^{\uparrow}} \hat{\Sigma} = \nabla^{C(M)}_{\hat{\Sigma}} X^{\uparrow} = (JX)^{\uparrow},$$
$$\nabla^{C(M)}_{T^{\uparrow}} T^{\uparrow} = V^{\uparrow}, \quad \nabla^{C(M)}_{\hat{\Sigma}} \hat{\Sigma} = 0,$$
$$\nabla^{C(M)}_{\hat{\Sigma}} T^{\uparrow} = \nabla^{C(M)}_{T^{\uparrow}} \hat{\Sigma} = 0,$$

where $\phi : H(M) \to H(M)$ *is given by* $G_\theta(\phi X, Y) = (d\sigma)(X^{\uparrow}, Y^{\uparrow})$, *and* $V \in H(M)$ *is given by* $G_\theta(V, Y) = 2(d\sigma)(T^{\uparrow}, Y^{\uparrow})$. *Also* $\hat{\Sigma} = ((n+2)/2)\Sigma$.

This relates the Levi-Civita connection $\nabla^{C(M)}$ of F_θ to the Tanaka-Webster connection of (M, θ). Cf. [9] for a proof of Lemma 2.

Proof of Theorem 2. Let $z(t) \in C(M)$ be a geodesic of $\nabla^{C(M)}$ and $\gamma(t) = \pi(z(t))$. Assume that $\dot{\gamma}(t) \in H(M)_{\gamma(t)}$. Note that $\dot{z}(t) - \dot{\gamma}(t)^{\uparrow} \in \mathrm{Ker}(d_{z(t)}\pi)$ hence $\dot{z}(t)$ is given by (23), for some $b : (-\varepsilon, \varepsilon) \to \mathbf{R}$. Then (by Lemma 2)

$$\begin{aligned} 0 = \nabla^{C(M)}_{\dot{z}} \dot{z} &= \nabla^{C(M)}_{\dot{\gamma}^{\uparrow}} \dot{\gamma}^{\uparrow} + b'(t)\hat{\Sigma} + 2b(t)(J\dot{\gamma})^{\uparrow} \\ &= (\nabla_{\dot{\gamma}}\dot{\gamma})^{\uparrow} + [b'(t) - A(\dot{\gamma}, \dot{\gamma})]\hat{\Sigma} + 2b(t)(J\dot{\gamma})^{\uparrow} \end{aligned}$$

hence (by $T(C(M)) = \mathrm{Ker}(\sigma) \oplus \mathbf{R}\Sigma$) $\gamma(t)$ satisfies the equations (14), i.e. $\gamma(t)$ is a sub-Riemannian geodesic. The converse is obvious.

3. Jacobi fields on CR manifolds

Let M be a strictly pseudoconvex CR manifold endowed with a contact form θ such that G_θ is positive definite. Let ∇ be the Tanaka-Webster connection of (M, θ). Let $\gamma(t) \in M$ be a geodesic of ∇, parametrized by arc length. A *Jacobi field* along γ is vector field X on M satisfying to the second order ODE

$$\nabla^2_{\dot{\gamma}} X + \nabla_{\dot{\gamma}} T_\nabla(X, \dot{\gamma}) + R(X, \dot{\gamma})\dot{\gamma} = 0. \tag{24}$$

Let J_γ be the real linear space of all Jacobi fields of (M, ∇). Then J_γ is $(4n+2)$-dimensional (cf. Prop. 1.1 in [17], Vol. II, p. 63). We denote by $\hat{\gamma}$ the vector field along γ defined by $\hat{\gamma}_{\gamma(t)} = t\dot{\gamma}(t)$ for every value of the parameter t. Note that $\dot{\gamma}, \hat{\gamma} \in J_\gamma$. We establish

Theorem 3. *Every Jacobi field X along a lengthy geodesic γ of ∇ can be uniquely decomposed in the following form*

$$X = a\dot{\gamma} + b\hat{\gamma} + Y \tag{25}$$

where $a, b \in \mathbf{R}$ and Y is a Jacobi field along γ such that

$$g_\theta(Y, \dot{\gamma})_{\gamma(t)} = -\int_0^t \theta(X)_{\gamma(s)} A(\dot{\gamma}, \dot{\gamma})_{\gamma(s)}\, ds. \tag{26}$$

In particular, if i) $X_{\gamma(t)} \in H(M)_{\gamma(t)}$ *for every t, or* ii) (M, θ) *is a Sasakian manifold (i.e. $\tau = 0$), then Y is perpendicular to γ.*

We need the following

Lemma 3. *For any Jacobi field $X \in J_\gamma$*

$$\frac{d}{dt}\{g_\theta(X, \dot{\gamma})\} + \theta(X)_{\gamma(t)} A(\dot{\gamma}, \dot{\gamma})_{\gamma(t)} = \text{const.}$$

Proof. Let us take the inner product of the Jacobi equation (24) by $\dot{\gamma}$ and use the skew symmetry of $g_\theta(R(X, Y)Z, W)$ in the arguments (Z, W) (a consequence of $\nabla g_\theta = 0$) so that to get

$$\frac{d^2}{dt^2}\{g_\theta(X, \dot{\gamma})\} + \frac{d}{dt}\{g_\theta(T_\nabla(X, \dot{\gamma}), \dot{\gamma})\} = 0.$$

On the other hand, let us set $X_H = X - \theta(X)T$ (so that $X_H \in H(M)$). Then

$$g_\theta(T_\nabla(X, \dot{\gamma}), \dot{\gamma}) = -2\Omega(X_H, \dot{\gamma}) g_\theta(T, \dot{\gamma}) + \theta(X) g_\theta(\tau(\dot{\gamma}), \dot{\gamma})$$

or (as γ is lengthy)

$$g_\theta(T_\nabla(X, \dot{\gamma}), \dot{\gamma}) = \theta(X) A(\dot{\gamma}, \dot{\gamma}).$$

Lemma 3 is proved. Throughout the section we adopt the notation $X' = \nabla_{\dot{\gamma}} X$ and $X'' = \nabla_{\dot{\gamma}}^2 X$.

Proof of Theorem 3. We set by definition

$$a = g_\theta(X, \dot{\gamma})_{\gamma(0)}, \quad b = g_\theta(X', \dot{\gamma})_{\gamma(0)} + \theta(X)_{\gamma(0)} A(\dot{\gamma}, \dot{\gamma})_{\gamma(0)},$$

and $Y = X - a\dot{\gamma} - b\hat{\gamma}$. Clearly $Y \in J_\gamma$. Then, by Lemma 3

$$\frac{d}{dt}\{g_\theta(Y, \dot{\gamma})\} + \theta(Y) A(\dot{\gamma}, \dot{\gamma}) = \alpha,$$

for some $\alpha \in \mathbf{R}$. Next we integrate from 0 to t

$$g_\theta(Y, \dot{\gamma})_{\gamma(t)} - g_\theta(Y, \dot{\gamma})_{\gamma(0)} + \int_0^t \theta(Y)_{\gamma(s)} A(\dot{\gamma}, \dot{\gamma})_{\gamma(s)}\, ds = \alpha t$$

and substitute Y from (25) (and use $\dot{\gamma}, \hat{\gamma} \in H(M)$). Differentiating the resulting relation with respect to t at $t = 0$ gives $\alpha = 0$. Hence

$$g_\theta(Y, \dot\gamma) + \int_0^t \theta(X)_{\gamma(s)} A(\dot\gamma, \dot\gamma)_{\gamma(s)}\, ds = 0.$$

The existence statement in Theorem 3 is proved. We need the following terminology. Given $X \in J_\gamma$ a Jacobi field $Y \in J_\gamma$ satisfying (26) is said to be *slant at* $\gamma(t)$ *relative to* X. Also Y is *slant* if it is slant at any point of γ. To check the uniqueness statement let $X = a'\dot\gamma + b'\hat\gamma + Z$ be another decomposition of X, where $a', b' \in \mathbf{R}$ and $Z \in J_\gamma$ is slant (relative to X). Then

$$(a + bt)\dot\gamma(t) + Y_{\gamma(t)} = (a' + b't)\dot\gamma(t) + Z_{\gamma(t)}$$

and taking the inner product with $\dot\gamma(t)$ yields $a + bt = a' + b't$, i.e. $a = a'$, $b = b'$ and $Y_{\gamma(t)} = Z_{\gamma(t)}$. Q.e.d.

Corollary 2. *Suppose a Jacobi field $X \in J_\gamma$ is slant at $\gamma(r)$ and at $\gamma(s)$ relative to itself, for some $r \neq s$. Then X is slant. In particular, if* i) *$X_{\gamma(t)} \in H(M)_{\gamma(t)}$ for every t, or* ii) *(M, θ) is a Sasakian manifold, and X is perpendicular to γ at two points, it is perpendicular to γ at every point of γ.*

Proof. By Theorem 3 we may decompose $X = a\dot\gamma + b\hat\gamma + Y$, where $Y \in J_\gamma$ is slant (relative to X). Taking the inner product of $X_{\gamma(r)} = (a + br)\dot\gamma(r) + Y_{\gamma(r)}$ with $\dot\gamma(r)$ gives $a + br = 0$. Similarly $a + bs = 0$ hence (as $r \neq s$) $a = b = 0$, so that $X = Y$. Q.e.d.

4. CR manifolds without conjugate points

Two points x and y on a lengthy geodesic $\gamma(t)$ are *horizontally conjugate* if there is a Jacobi field $X \in J_\gamma$ such that $X_{\gamma(t)} \in H(M)_{\gamma(t)}$ for every t and $X_x = X_y = 0$. As T_∇ is pure, the Jacobi equation (24) may also be written

$$X'' - 2\Omega(X', \dot\gamma)T + \theta(X')\tau(\dot\gamma) + \theta(X)(\nabla_{\dot\gamma}\tau)\dot\gamma + R(X, \dot\gamma)\dot\gamma = 0. \tag{27}$$

Given $X \in J_\gamma$ one has (by (27))

$$\begin{aligned}\frac{d}{dt}\{g_\theta(X', X)\} &= g_\theta(X'', X) + g_\theta(X', X') \\ &= |X'|^2 + 2\theta(X)\Omega(X', \dot\gamma) - \theta(X')A(\dot\gamma, X) \\ &\quad - \theta(X)g_\theta(\nabla_{\dot\gamma}\tau\dot\gamma, X) - g_\theta(R(X, \dot\gamma)\dot\gamma, X).\end{aligned}$$

On the other hand (again by $\nabla g_\theta = 0$)

$$\begin{aligned}&\theta(X')A(\dot\gamma, X) + \theta(X)g_\theta(\nabla_{\dot\gamma}\tau\dot\gamma, X) \\ &\quad = \theta(X')A(\dot\gamma, X) + \theta(X)\frac{d}{dt}\{A(\dot\gamma, X)\} - \theta(X)A(\dot\gamma, X') \\ &\quad = \frac{d}{dt}\{\theta(X)A(\dot\gamma, X)\} - \theta(X)A(\dot\gamma, X')\end{aligned}$$

hence

$$\frac{d}{dt}\{g_\theta(X', X) + \theta(X)A(\dot{\gamma}, X)\} = |X'|^2 - g_\theta(R(X, \dot{\gamma})\dot{\gamma}, X) + \theta(X)[A(\dot{\gamma}, X') + 2\Omega(X', \dot{\gamma})]. \tag{28}$$

S. Webster (cf. [27]) has introduced a notion of pseudohermitian sectional curvature by setting

$$k_\theta(\sigma) = \frac{1}{4} G_\theta(X, X)^{-2} g_{\theta,x}(R_x(X, J_x X)J_x X, X), \tag{29}$$

for any holomorphic 2-plane σ (i.e. a 2-plane $\sigma \subset H(M)_x$ such that $J_x(\sigma) = \sigma$), where $\{X, J_x X\}$ is a basis of σ. The coefficient 1/4 makes the sphere $\iota : S^{2n+1} \subset \mathbf{C}^{n+1}$ (endowed with the contact form $\theta_0 = \iota^*\left[\frac{i}{2}(\bar{\partial} - \partial)|z|^2\right]$) have constant curvature $+1$. Clearly, this is a pseudohermitian analog to the notion of holomorphic sectional curvature in Hermitian geometry. On the other hand, for any 2-plane $\sigma \subset T_x(M)$ one may set

$$k_\theta(\sigma) = \frac{1}{4} g_{\theta,x}(R_x(X, Y)Y, X)$$

where $\{X, Y\}$ is a $g_{\theta,x}$-orthonormal basis of σ. Cf. [17], Vol. I, p. 200, the definition of $k_\theta(\sigma)$ doesn't depend upon the choice of orthonormal basis in σ because the curvature $R(X, Y, Z, W) = g_\theta(R(Z, W)Y, X)$ of the Tanaka-Webster connection is skew symmetric in both pairs (X, Y) and (Z, W). We refer to k_θ as the *pseudohermitian sectional curvature* of (M, θ). *A posteriori* the restriction (29) of k_θ to holomorphic 2-planes is referred to as the *holomorphic pseudohermitian sectional curvature* of (M, θ). As an application of (28) we may establish

THEOREM 4. *If (M, θ) has nonpositive pseudohermitian sectional curvature then (M, θ) has no horizontally conjugate points.*

We need

LEMMA 4. *For every Jacobi field $X \in J_\gamma$*

$$\frac{d}{dt}\{\theta(X)\} - 2\Omega(X, \dot{\gamma})_{\gamma(t)} = c = \text{const.}$$

To prove Lemma 4 one merely takes the inner product of (27) by T.

Proof of Theorem 4. The proof is by contradiction. If there is a lengthy geodesic $\gamma(t) \in M$ (parametrized by arc length) and a Jacobi field $X \in J_\gamma$ such that $X_{\gamma(a)} = X_{\gamma(b)} = 0$ for two values a and b of the parameter then we may integrate in (28) so that to obtain

$$(30)\qquad \int_a^b \{|X'|^2 - g_\theta(R(X,\dot\gamma)\dot\gamma, X) + \theta(X)[A(\dot\gamma, X') + 2\Omega(X',\dot\gamma)]\}\, dt = 0.$$

On the other hand

$$\begin{aligned}\theta(X)\Omega(X',\dot\gamma) &= \theta(X)\frac{d}{dt}\{\Omega(X,\dot\gamma)\}\\ &= \frac{d}{dt}\{\theta(X)\Omega(X,\dot\gamma)\} - \Omega(X,\dot\gamma)\theta(X').\end{aligned}$$

Then (by Lemma 4)

$$\begin{aligned}2\int_a^b \theta(X)\Omega(X',\dot\gamma)\, dt &= -2\int_a^b \Omega(X,\dot\gamma)\frac{d}{dt}\{\theta(X)\}\, dt\\ &= c\int_a^b \frac{d}{dt}\{\theta(X)\}\, dt - \int_a^b \theta(X')^2\, dt = -\int_a^b \theta(X')^2\, dt\end{aligned}$$

hence (30) becomes

$$\int_a^b \{|X'|^2 - g_\theta(R(X,\dot\gamma)\dot\gamma, X) + \theta(X)A(\dot\gamma, X') - \theta(X')^2\}\, dt = 0.$$

Finally, if $X \in H(M)$ then $X' \in H(M)$ and then (under the assumptions of Theorem 4) $X' = 0$, a contradiction.

5. Jacobi fields on CR manifolds of constant pseudohermitian sectional curvature

As well known (cf. Example 2.1 in [17], Vol. II, p. 71) one may determine a basis of J_γ for any elliptic space form (a Riemannian manifold of positive constant sectional curvature). Similarly, we shall prove

Proposition 3. *Let M be a strictly pseudoconvex CR manifold of CR dimension n, θ a contact form with G_θ positive definite and constant pseudohermitian sectional curvature. Let $\gamma(t) \in M$ be a lengthy geodesic of the Tanaka-Webster connection ∇ of (M,θ), parametrized by arc length. For each $v \in T_{\gamma(0)}(M)$ we let $E(v)$ be the space of all vector fields X along γ defined by $X_{\gamma(t)} = (at+b)Y_{\gamma(t)}$, where $a, b \in \mathbf{R}$, $\nabla_{\dot\gamma}Y = 0$, $Y_{\gamma(0)} = v$. Assume that (M,θ) has parallel pseudohermitian torsion, i.e. $\nabla\tau = 0$. Then $T \in J_\gamma$. Let $\{v_1,\dots,v_{2n-2}\} \subset H(M)_{\gamma(0)}$ such that $\{\dot\gamma(0), J_{\gamma(0)}\dot\gamma(0), v_1,\dots,v_{2n-2}\}$ is a $G_{\theta,\gamma(0)}$-orthonormal basis of $H(M)_{\gamma(0)}$. Then*

$$E(\dot\gamma(0)) \oplus E(v_1) \oplus \cdots \oplus E(v_{2n-2}) \subseteq \mathscr{H}_\gamma := J_\gamma \cap \Gamma^\infty(\gamma^{-1}H(M))$$

if and only if

$$A_{\gamma(0)}(\dot\gamma(0),\dot\gamma(0)) = 0, \quad A_{\gamma(0)}(v_i,\dot\gamma(0)) = 0, \quad 1 \le i \le 2n-2,$$

where $\gamma^{-1}H(M)$ is the pullback of $H(M)$ by γ. If additionally (M,θ) has vanishing pseudohermitian torsion (i.e. (M,θ) is Tanaka-Webster flat) then $E(T_{\gamma(0)}) \subset J_\gamma$.

The proof of Proposition 3 requires the explicit form of the curvature tensor of the Tanaka-Webster connection of (M, θ) when $k_\theta = \text{const}$. This is provided by

Theorem 5. *Let M be a strictly pseudoconvex CR manifold and θ a contact form on M such that G_θ is positive definite and $k_\theta(\sigma) = c$, for some $c \in \mathbf{R}$ and any 2-plane $\sigma \subset T_x(M)$, $x \in M$. Then $c = 0$ and the curvature of the Tanaka-Webster connection of (M, θ) is given by*

$$R(X, Y)Z = \Omega(Z, Y)\tau(X) - \Omega(Z, X)\tau(Y) + A(Z, Y)JX - A(Z, X)JY, \tag{31}$$

for any $X, Y, Z \in T(M)$. In particular, if (M, θ) has constant pseudohermitian sectional curvature and CR dimension $n \geq 2$ then the Tanaka-Webster connection of (M, θ) is flat if and only if (M, θ) has vanishing pseudohermitian torsion ($\tau = 0$).

The proof of Theorem 5 is given in Appendix A. By Theorem 5 there are no "pseudohermitian space forms" except for those of zero pseudohermitian sectional curvature and these aren't in general flat. Cf. [10] the term *pseudohermitian space form* is reserved for manifolds of constant *holomorphic* pseudohermitian sectional curvature (and then examples with arbitrary $c \in \mathbf{R}$ abound, cf. [10], Chapter 1).

Proof of Proposition 3. By (31)

$$R(X, \dot\gamma)\dot\gamma = \Omega(X, \dot\gamma)\tau(\dot\gamma) + A(\dot\gamma, \dot\gamma)JX - A(X, \dot\gamma)J\dot\gamma$$

hence the Jacobi equation (27) becomes

$$\begin{aligned} X'' - 2\Omega(X', \dot\gamma)T + \theta(X')\tau(\dot\gamma) + \theta(X)(\nabla_{\dot\gamma}\tau)\dot\gamma \\ + \Omega(X, \dot\gamma)\tau(\dot\gamma) + A(\dot\gamma, \dot\gamma)JX - A(X, \dot\gamma)J\dot\gamma = 0. \end{aligned} \tag{32}$$

We look for solutions to (32) of the form $X_{\gamma(t)} = f(t)T_{\gamma(t)}$. The relevant equation is

$$f''(t)T + f'(t)\tau(\dot\gamma) + f(t)(\nabla_{\dot\gamma}\tau)\dot\gamma = 0$$

(by $\nabla T = 0$) or $f''(t) = 0$ and $f'(t)\tau(\dot\gamma) + f(t)(\nabla_{\dot\gamma}\tau)\dot\gamma = 0$. Therefore, if $\nabla\tau = 0$ then $T \in J_\gamma$ while if $\tau = 0$ then $T, \hat T \in J_\gamma$, where $\hat T_{\gamma(t)} = tT_{\gamma(t)}$. Next, we look for solutions to (32) of the form $X_{\gamma(t)} = f(t)Y_{\gamma(t)}$ where Y is a vector field along γ such that $\nabla_{\dot\gamma}Y = 0$, $Y_{\gamma(0)} =: v \in H(M)_{\gamma(0)}$, $|v| = 1$, and $g_{\theta,\gamma(0)}(v, J_{\gamma(0)}\dot\gamma(0)) = 0$. Substitution into (32) gives

$$f''(t)Y + f(t)[A(\dot\gamma, \dot\gamma)JY - A(Y, \dot\gamma)J\dot\gamma] = 0$$

or (by taking the inner product with Y) $f''(t) = 0$, i.e. $f(t) = at + b$, $a, b \in \mathbf{R}$. Therefore (with the notations in Proposition 3) $E(v_i) \subset J_\gamma \cap \Gamma^\infty(\gamma^{-1}H(M))$ if and only if $A_{\gamma(0)}(\dot\gamma(0), \dot\gamma(0)) = 0$ and $A_{\gamma(0)}(v_i, \dot\gamma(0)) = 0$. Also, to start with, $E(\dot\gamma(0))$ (the space spanned by $\dot\gamma$ and $\hat\gamma$) consists of Jacobi fields lying in $H(M)$. As $\{\dot\gamma(t), J_{\gamma(t)}\dot\gamma(t), Y_{1,\gamma(t)}, \ldots, Y_{2n-2,\gamma(t)}\}$ is an orthonormal basis of $H(M)_{\gamma(t)}$ (where Y_i is the unique solution to $(\nabla_{\dot\gamma}Y)_{\gamma(t)} = 0$, $Y_{\gamma(0)} = v_i$) it follows that the sum $E(\dot\gamma(0)) + E(v_1) + \cdots + E(v_{2n-2})$ is direct. Q.e.d.

Let $(M, (\varphi, \xi, \eta, g))$ be a contact Riemannian manifold. Let $X \in T_x(M)$ be a unit tangent vector orthogonal to ξ and $\sigma \subset T_x(M)$ the 2-plane spanned by $\{X, \varphi X\}$ (a *φ-holomorphic plane*). We recall (cf. e.g. [5], p. 94) that the *φ-sectional curvature* is the restriction of the sectional curvature k of (M, g) to the φ-holomorphic planes. Let us set $H(X) = k(\sigma)$. A Sasakian manifold of constant φ-sectional curvature $H(X) = c$, $c \in \mathbf{R}$, is a *Sasakian space form*. Compact Sasakian space forms have been classified in [16]. By a result in [5], p. 97, the Riemannian curvature R^D of a Sasakian space form M (of φ-sectional curvature c) is given by

$$\begin{aligned}(33)\quad R^D(X, Y)Z = {} & \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ & + \frac{c-1}{4}\{\eta(Z)[\eta(X)Y - \eta(Y)X] \\ & \qquad + [g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\xi \\ & \qquad + \Omega(Z, Y)\varphi X - \Omega(Z, X)\varphi Y + 2\Omega(X, Y)\varphi Z\}\end{aligned}$$

for any $X, Y, Z \in T(M)$. Given a strictly pseudoconvex CR manifold M and a contact form θ we recall (cf. e.g. (1.59) in [10]) that

$$D = \nabla + (\Omega - A) \otimes T + \tau \otimes \theta + 2\theta \odot J. \tag{34}$$

A calculation based on (34) leads to

$$\begin{aligned}R^D(X, Y)Z = {} & R(X, Y)Z + (LX \wedge LY)Z - 2\Omega(X, Y)JZ \\ & - g_\theta(S(X, Y), Z)T + \theta(Z)S(X, Y) \\ & - 2g_\theta((\theta \wedge \mathcal{O})(X, Y), Z)T + 2\theta(Z)(\theta \wedge \mathcal{O})(X, Y)\end{aligned}$$

for any $X, Y, Z \in T(M)$, relating the Riemannian curvature R^D of (M, g_θ) to the curvature R of the Tanaka-Webster connection. Here

$$L = \tau + J, \quad \mathcal{O} = \tau^2 + 2J\tau - I,$$

and $(X \wedge Y)Z = g_\theta(X,Z)Y - g_\theta(Y,Z)X$. Also $S(X, Y) = (\nabla_X \tau)Y - (\nabla_Y \tau)X$. Let us assume that (M, θ) is a Sasakian manifold ($\tau = 0$) whose Tanaka-Webster connection is flat ($R = 0$). Then $S = 0$, $L = J$ and $\mathcal{O} = -I$ hence

$$\begin{aligned}R^D(X, Y)Z = {} & (JX \wedge JY)Z - 2\Omega(X, Y)JZ \\ & + 2g_\theta((\theta \wedge I)(X, Y), Z)T - 2\theta(Z)(\theta \wedge I)(X, Y)\end{aligned}$$

and a comparison to (33) shows that

Proposition 4. *Let (M, θ) be a Sasakian manifold. Then its Tanaka-Webster connection is flat if and only if $(M, (J, -T, -\theta, g_\theta))$ is a Sasakian space form of φ-sectional curvature $c = -3$.*

By Lemma 8 below the dimension of $\mathscr{H}_\gamma$ is at most $4n$. On a Sasakian space form we may determine $4n-1$ independent vectors in $\mathscr{H}_\gamma$. Indeed, by combining Propositions 3 and 4 we obtain

Corollary 3. *Let (M,θ) be a Sasakian space form of φ-sectional curvature $c=-3$ and $\gamma(t)\in M$ a lengthy geodesic of the Tanaka-Webster connection ∇, parametrized by arc length. Let $\{v_1,\dots,v_{2n-2}\}\subset H(M)_{\gamma(0)}$ such that $\{\dot\gamma(0), J_{\gamma(0)}\dot\gamma(0), v_1,\dots,v_{2n-2}\}$ is a $G_{\theta,\gamma(0)}$-orthonormal basis of $H(M)_{\gamma(0)}$. Let X_i be the vector field along γ determined by*

$$\nabla_{\dot\gamma(t)}X_i=0,\quad X_i(\gamma(0))=v_i,$$

for $1\le i\le 2n-2$. Then $\mathscr{S}=\{\dot\gamma,\hat\gamma,J\dot\gamma,X_i,\hat X_i : 1\le i\le 2n-2\}$ is a free system in $\mathscr{H}_\gamma$ while $\mathscr{S}\cup\{T,\hat T\}$ is free in J_γ. Here if Y is a vector field along $\gamma(t)$ we set $\hat Y_{\gamma(t)}=tY_{\gamma(t)}$ for every t.

6. Conjugate points on Sasakian manifolds

Let (M,θ) be a Sasakian manifold and $\gamma:[a,b]\to M$ a geodesic of the Tanaka-Webster connection ∇, parametrized by arc length. Given a piecewise differentiable vector field X along γ we set

$$I_a^b(X)=\int_a^b\{g_\theta(\nabla_{\dot\gamma}X,\nabla_{\dot\gamma}X)-g_\theta(R(X,\dot\gamma)\dot\gamma,X)\}_{\gamma(t)}\,dt$$

where R is the curvature of ∇. We shall prove the following

Proposition 5. *Let (M,θ) be a Sasakian manifold and $\gamma(t)\in M$, $a\le t\le b$, a lengthy geodesic of ∇, parametrized by arc length, such that $\gamma(a)$ has no conjugate point along γ. Let $Y\in\mathscr{H}_\gamma$ be a horizontal Jacobi field along γ such that $Y_{\gamma(a)}=0$ and Y is perpendicular to γ. Let X be a piecewise differentiable vector field along γ such that $X_{\gamma(a)}=0$ and X is perpendicular to γ. If $X_{\gamma(b)}=Y_{\gamma(b)}$ then*

$$I_a^b(X)\ge I_a^b(Y)\tag{35}$$

and the equality holds if and only if $X=Y$.

Proof. Let $J_{\gamma,a}$ be the space of all Jacobi fields $Z\in J_\gamma$ such that $Z_{\gamma(a)}=0$. By Prop. 1.1 in [17], Vol. II, p. 63, $J_{\gamma,a}$ has dimension $2n+1$. Moreover, let $J_{\gamma,a,\perp}$ be the space of all $Z\in J_{\gamma,a}$ such that $g_\theta(Z,\dot\gamma)_{\gamma(t)}=0$, for every t. Then by Theorem 3 it follows that $J_{\gamma,a,\perp}$ has dimension $2n$. We shall need the following

Lemma 5. *For every Sasakian manifold (M,θ) the characteristic direction T of (M,θ) is a Jacobi field along any geodesic $\gamma:[a,b]\to M$ of ∇. Also, if T_a is the vector field along γ given by $T_{a,\gamma(t)}=(t-a)T_{\gamma(t)}$, $a\le t\le b$, and γ is lengthy then $T_a\in J_{\gamma,a,\perp}$ and $T_{a,\gamma(t)}\ne 0$, $a<t\le b$.*

Proof. Let $\mathscr{J}_\gamma$ be the Jacobi operator. Then

$$\mathscr{J}_\gamma T=T''-2\Omega(T',\dot\gamma)T+R(T,\dot\gamma)\dot\gamma=R(T,\dot\gamma)\dot\gamma$$

as $T' = \nabla_{\dot\gamma} T = 0$. On the other hand, on any nondegenerate CR manifold with $S = 0$ (i.e. $S(X, Y) \equiv (\nabla_X \tau)Y - (\nabla_Y \tau)X = 0$, for any $X, Y \in T(M)$) the curvature of the Tanaka-Webster connection satisfies

$$R(T, X)X = 0, \quad X \in T(M), \tag{36}$$

hence $\mathcal{J}_\gamma T = 0$. As $R(T, T) = 0$ and $R(X, Y)T = 0$ it suffices to check (36) for $X \in H(M)$, i.e. locally $X = Z^\alpha T_\alpha + Z^{\bar\alpha} T_{\bar\alpha}$. Then

$$R(T, X)X = \{R_\beta{}^\gamma{}_{0\alpha} Z^\alpha Z^\beta + R_\beta{}^\gamma{}_{0\bar\alpha} Z^{\bar\alpha} Z^\beta\} T_\gamma + R_{\bar\beta}{}^{\bar\gamma}{}_{0\alpha} Z^\alpha Z^{\bar\beta} + R_{\bar\beta}{}^{\bar\gamma}{}_{0\bar\alpha} Z^{\bar\alpha} Z^{\bar\beta}\} T_{\bar\gamma}$$

and (by (1.85)–(1.86) in [10], section 1.4)

$$R^\gamma_{\beta 0\alpha} = g^{\gamma\bar\lambda} g_{\alpha\bar\mu} S^{\bar\mu}_{\beta\bar\lambda}, \quad R^\gamma_{\beta 0\bar\alpha} = g_{\lambda\bar\alpha} g^{\gamma\bar\mu} S^\lambda_{\beta\bar\mu}.$$

To complete the proof of Lemma 5 let $u(t) = t - a$. Then (by $T \rfloor \Omega = 0$ and (36))

$$\mathcal{J}_\gamma T_a = u'' T - 2u'\Omega(T, \dot\gamma)T + uR(T, \dot\gamma)\dot\gamma = 0.$$

Lemma 6. *Let (M, θ) be a Sasakian manifold and $\gamma(t) \in M$ a geodesic of ∇. If $X \in J_\gamma$ then $X_H \equiv X - \theta(X)T$ satisfies the second order ODE*

$$\nabla^2_{\dot\gamma} X_H + R(X_H, \dot\gamma)\dot\gamma = 0. \tag{37}$$

Proof.

$$0 = \mathcal{J}_\gamma X = \nabla^2_{\dot\gamma} X_H + \theta(X'')T - 2\Omega(\nabla_{\dot\gamma} X_H, \dot\gamma)T + R(X_H, \dot\gamma)\dot\gamma$$

hence (by the uniqueness of the direct sum decomposition $T(M) = H(M) \oplus \mathbf{R}T$) X_H satisfies (37).

Let us go back to the proof of Proposition 5. Let us complete T_a to a linear basis $\{T_a, Y_2, \dots, Y_{2n}\}$ of $J_{\gamma,a,\perp}$ and set $Y_1 = T_a$ for simplicity. Then $Y = a^i Y_i$ for some $a^i \in \mathbf{R}$, $1 \le i \le 2n$. Let us observe that for each $a < t \le b$ the tangent vectors

$$\{T_{a,\gamma(t)}, Y^H_{2,\gamma(t)}, \dots, Y^H_{2n,\gamma(t)}\} \subset [\mathbf{R}\dot\gamma(t)]^\perp \subset T_{\gamma(t)}(M)$$

are linearly independent, where $Y^H_j := Y_j - \theta(Y_j)T$, $2 \le j \le 2n$. Indeed

$$0 = \alpha T_{a,\gamma(t)} + \sum_{j=2}^{2n} \alpha^j Y^H_{j,\gamma(t)} = \left\{\alpha - \sum_{j=2}^{2n} \frac{\alpha^j}{t - a} \theta(Y_j)_{\gamma(t)}\right\} T_{a,\gamma(t)} + \sum_{j=2}^{2n} \alpha^j Y_{j,\gamma(t)}$$

implies $\alpha^j = 0$, and then $\alpha = 0$, because $\{Y_{i,\gamma(t)} : 1 \le i \le 2n\}$ are linearly independent, for any $a < t \le b$. At their turn, the vectors $Y_{i,\gamma(t)}$ are independent because $\gamma(a)$ has no conjugate point along γ. The proof is by contradiction. Assume that

$$\lambda^i Y_{i,\gamma(t_0)} = 0, \tag{38}$$

for some $a < t_0 \le b$ and some $\lambda = (\lambda^1, \dots, \lambda^{2n}) \in \mathbf{R}^{2n} \setminus \{0\}$. Let us set $Z_0 = \lambda^i Y_i \in J_\gamma$. Then

$$\lambda \neq 0 \Rightarrow Z_0 \neq 0,$$

$$Z_0 \in J_{\gamma,a} \Rightarrow Z_{0,\gamma(a)} = 0, \quad (38) \Rightarrow Z_{0,\gamma(t_0)} = 0,$$

hence $\gamma(a)$ and $\gamma(t_0)$ are conjugate along γ, a contradiction. Yet $[\mathbf{R}\dot{\gamma}(t)]^{\perp}$ has dimension $2n$ hence

$$X_{\gamma(t)} = f(t)T_{a,\gamma(t)} + \sum_{j=2}^{2n} f^j(t) Y^H_{j,\gamma(t)},$$

for some piecewise differentiable functions $f(t)$, $f^j(t)$. We set $f^1 = f$, $Z_1 = T_a$ and $Z_j = Y_j^H$, $2 \le j \le 2n$, for simplicity. Then

$$|X'|^2 = \left|\frac{df^i}{dt} Z_i\right|^2 + |f^i Z_i'|^2 + 2g_\theta\left(\frac{df^i}{dt} Z_i, f^j Z_j'\right). \tag{39}$$

Also (by (36) and Lemma 6)

$$\begin{aligned}-g_\theta(R(X,\dot{\gamma})\dot{\gamma}, X) &= -f^i g_\theta(R(Z_i,\dot{\gamma})\dot{\gamma}, X) \\ &= -\sum_{j=2}^{2n} f^j g_\theta(R(Z_j,\dot{\gamma})\dot{\gamma}, X) = \sum_{j=2}^{2n} f^j g_\theta(Z_j'', X)\end{aligned}$$

or (as $T_a'' = 0$)

$$-g_\theta(R(X,\dot{\gamma})\dot{\gamma}, X) = g_\theta(f^i Z_i'', f^j Z_j). \tag{40}$$

Finally, note that

$$\begin{aligned}g_\theta\left(\frac{df^i}{dt} Z_i, f^j Z_j'\right) + |f^i Z_i'|^2 + g_\theta(f^i Z_i, f^j Z_j'') \\ = \frac{d}{dt} g_\theta(f^i Z_i, f^j Z_j') - g_\theta\left(f^i Z_i, \frac{df^j}{dt} Z_j'\right).\end{aligned} \tag{41}$$

Summing up (by (39)–(41))

$$\begin{aligned}&|X'|^2 - g_\theta(R(X,\dot{\gamma})\dot{\gamma}, X) \\ &= \frac{d}{dt} g_\theta(f^i Z_i, f^j Z_j') - g_\theta\left(f^i Z_i, \frac{df^j}{dt} Z_j'\right) + g_\theta\left(\frac{df^i}{dt} Z_i, f^j Z_j'\right) + \left|\frac{df^i}{dt} Z_i\right|^2.\end{aligned} \tag{42}$$

Lemma 7. *Let (M,θ) be a Sasakian manifold and $\gamma(t) \in M$ a geodesic of ∇. If X and Y are solutions to $\nabla^2_{\dot{\gamma}} Z + R(Z,\dot{\gamma})\dot{\gamma} = 0$ then*

$$\frac{d}{dt}\{g_\theta(X, Y') - g_\theta(X', Y)\} = 0. \tag{43}$$

In particular, if $X_{\gamma(a)} = 0$ and $Y_{\gamma(a)} = 0$ at some point $\gamma(a)$ of γ then

$$g_\theta(X, Y') - g_\theta(X', Y) = 0.$$

Proof. As $\tau = 0$ the 4-tensor $R(X, Y, Z, W)$ possesses the symmetry property $R(X, Y, Z, W) = R(Z, W, X, Y)$ (cf. (92) in Appendix A) one may subtract the identities

$$\frac{d}{dt} g_\theta(X, Y') = g_\theta(X', Y') - g_\theta(X, R(Y, \dot\gamma)\dot\gamma),$$

$$\frac{d}{dt} g_\theta(X', Y) = g_\theta(X', Y') - g_\theta(R(X, \dot\gamma)\dot\gamma, Y)$$

so that to obtain (43). Q.e.d.

By Lemma 6 the fields Z_j, $2 \le j \le 2n$ satisfy $\nabla_{\dot\gamma}^2 Z_j + R(Z_j, \dot\gamma)\dot\gamma = 0$. Then we may apply Lemma 7 to conclude that

$$g_\theta\left(\frac{df^i}{dt} Z_i, f^j Z_j'\right) - g_\theta\left(f^i Z_i, \frac{df^j}{dt} Z_j'\right) = f^i \frac{df^j}{dt}\{g_\theta(Z_j, Z_i') - g_\theta(Z_j', Z_i)\} = 0$$

so that (42) becomes

$$|X'|^2 - g_\theta(R(X, \dot\gamma)\dot\gamma, X) = \frac{d}{dt} g_\theta(f^i Z_i, f^j Z_j') + \left|\frac{df^i}{dt} Z_i\right|^2$$

and integration gives

$$I_a^b(X) = g_\theta(f^i Z_i, f^j Z_j')_{\gamma(b)} + \int_a^b \left|\frac{df^i}{dt} Z_i\right|^2 dt. \tag{44}$$

We wish to apply (44) to the vector field $X = Y$. If this is the case the functions f^j are $f^1(t) = a^1 + (1/(t-a)) \sum_{j=2}^{2n} a^j \theta(Y_j)_{\gamma(t)} = 0$ (because of $Y_{\gamma(t)} \in H(M)_{\gamma(t)}$) and $f^j = a^j$ (so that $df^j/dt = 0$) for $2 \le j \le 2n$. Then (by (44))

$$I_a^b(Y) = g_\theta\left(\sum_{i=2}^{2n} a^i Z_i, \sum_{j=2}^{2n} a^j Z_j'\right)_{\gamma(b)}. \tag{45}$$

As $X_{\gamma(b)} = Y_{\gamma(b)}$ it follows that $f^1(b) = 0$ and $f^j(b) = a^j$, $2 \le j \le 2n$, so that by subtracting (44) and (45) we get

$$I_a^b(X) - I_a^b(Y) = \int_a^b \left|\frac{df^i}{dt} Z_i\right|^2 dt \ge 0$$

and (35) is proved. The equality $I_a^b(X) = I_a^b(Y)$ yields $df^i/dt = 0$, i.e. $f^1(t) = f^1(b) = 0$ and $f^j(t) = f^j(b) = a^j$, $2 \le j \le 2n$, hence

$$X_{\gamma(t)} = \sum_{j=2}^{2n} a^j Y_{j,\gamma(t)}^H = \sum_{j=2}^{2n} a^j \{Y_{j,\gamma(t)} - \theta(Y_j)_{\gamma(t)} T_{\gamma(t)}\}$$

$$= \sum_{j=2}^{2n} a^j Y_{j,\gamma(t)} + (t-a) a^1 T_{\gamma(t)} = a^i Y_{i,\gamma(t)} = Y_{\gamma(t)}. \qquad \text{Q.e.d.}$$

Setting $Y = 0$ in Proposition 5 leads to

Corollary 4. *Let (M, θ) be a Sasakian manifold and $\gamma : [a, b] \to M$ a lengthy geodesic of the Tanaka-Webster connection, parametrized by arc length and such that $\gamma(a)$ has no conjugate point along γ. If X is a piecewise differentiable vector field along γ such that $X_{\gamma(a)} = X_{\gamma(b)} = 0$ and X is perpendicular to γ then $I_a^b(X) \geq 0$ and equality holds if and only if $X = 0$.*

Corollary 4 admits the following application

Theorem 6. *Let (M, θ) be a Sasakian manifold and ∇ its Tanaka-Webster connection. Assume that the pseudohermitian sectional curvature satisfies $k_\theta(\sigma) \geq k_0 > 0$, for any 2-plane $\sigma \subset T_x(M)$, $x \in M$. Then for any lengthy geodesic $\gamma(t) \in M$ of ∇, parametrized by arc length, the distance between two consecutive conjugate points of γ is less equal than $\pi/(2\sqrt{k_0})$.*

Proof. Let $\gamma : [a, c] \to M$ be a geodesic of ∇, parametrized by arc length, such that $\gamma(c)$ is the first conjugate point of $\gamma(a)$ along γ. Let $b \in (a, c)$ and let Y be a unit vector field along γ such that $(\nabla_{\dot\gamma} Y)_{\gamma(t)} = 0$ and Y is perpendicular to γ. Let $f(t)$ be a nonzero smooth function such that $f(a) = f(b) = 0$. Then we may apply Corollary 4 to the vector field $X = fY$ so that

$$\begin{aligned} 0 \leq I_a^b(X) &= \int_a^b \{f'(t)^2 |Y|^2 - f(t)^2 g_\theta(R(Y, \dot\gamma)\dot\gamma, Y)\}\, dt \\ &= \int_a^b \{f'(t)^2 - 4f(t)^2 k_\theta(\sigma)\}\, dt \leq \int_a^b \{f'(t)^2 - 4k_0 f(t)^2\}\, dt \end{aligned}$$

where $\sigma \subset T_{\gamma(t)}(M)$ is the 2-plane spanned by $\{Y_{\gamma(t)}, \dot\gamma(t)\}$. Finally, we may choose $f(t) = \sin[\pi(t-a)/(b-a)]$ and use $\int_0^\pi \cos^2 x\, dx = \int_0^\pi \sin^2 x\, dx = \pi/2$. We get $b - a \leq \pi/\sqrt{4k_0}$ and let $b \to c$. Q.e.d.

We may establish the following more general version of Theorem 6

Theorem 7. *Let (M, θ) be a Sasakian manifold of CR dimension n such that the Ricci tensor ρ of the Tanaka-Webster connection ∇ satisfies*

$$\rho(X, X) \geq (2n-1)k_0 g_\theta(X, X), \quad X \in H(M),$$

for some constant $k_0 > 0$. Then for any geodesic γ of ∇, parametrized by arc length, the distance between any two consecutive conjugate points of γ is less than $\pi/\sqrt{k_0}$.

Remark. The assumption on ρ in Theorem 7 involves but the pseudohermitian Ricci curvature. Indeed (cf. (1.98) in [10], section 1.4)

$$\mathrm{Ric}(T_\alpha, T_{\bar\beta}) = g_{\alpha\bar\beta} - \frac{1}{2} R_{\alpha\bar\beta},$$

$$R_{\alpha\beta} = i(n-1)A_{\alpha\beta}, \quad R_{0\beta} = S^{\bar\alpha}_{\bar\alpha\beta}, \quad R_{\alpha 0} = R_{00} = 0,$$

hence (by $\tau = 0$) $\rho(X, X) = 2R_{\alpha\bar{\beta}} Z^\alpha Z^{\bar{\beta}}$, for any $X = Z^\alpha T_\alpha + Z^{\bar{\alpha}} T_{\bar{\alpha}} \in H(M)$. Here Ric is the Ricci tensor of the Riemannian manifold (M, g_θ) (whose symmetry yields $R_{\alpha\bar{\beta}} = R_{\bar{\beta}\alpha}$). Note that $S = 0$ alone implies $T \rfloor \rho = 0$. Also, if (M, g_θ) is Ricci flat then (M, θ) is pseudo-Einstein (of pseudohermitian scalar curvature $R = 2$), in the sense of [18].

Proof of Theorem 7. Let $\gamma(t) \in M$ as in the proof of Theorem 6. Let $\{Y_1, \dots, Y_{2n-1}\}$ be parallel (i.e. $(\nabla_{\dot{\gamma}} Y_i)_{\gamma(t)} = 0$) vector fields such that $Y_i \in H(M)$ and $\{\dot{\gamma}(t), Y_{1,\gamma(t)}, \dots, Y_{2n-1,\gamma(t)}\}$ is an orthonormal basis of $H(M)_{\gamma(t)}$ for every t. Let $f(t)$ be a nonzero smooth function such that $f(a) = f(b) = 0$ and let us set $X_i = f Y_i$. Then (by Corollary 4)

$$\begin{aligned} 0 \le \sum_{i=1}^{2n-1} I_a^b(X_i) &= \sum_{i=1}^{2n-1} \int_a^b \{f'(t)^2 |Y_i|^2 - f(t)^2 g_\theta(R(Y_i, \dot{\gamma})\dot{\gamma}, Y_i)\}\, dt \\ &= \int_a^b \{(2n-1) f'(t)^2 - f(t)^2 \rho(\dot{\gamma}, \dot{\gamma})\}\, dt \\ &\le (2n-1) \int_a^b \{f'(t)^2 - k_0 f(t)^2\}\, dt \end{aligned}$$

and the proof may be completed as that of Theorem 6.

Remark. The assumption in Theorem 7 is weaker than that in Theorem 6. Indeed, let $X \in H(M)$, $X \neq 0$, and $V = |V|^{-1} V$. Let $\{X_j : 1 \le j \le 2n\}$ be a local orthonormal frame of $H(M)$ and $\sigma_j \subset T_x(M)$ the 2-plane spanned by $\{Y_{j,x}, X_x\}$, where $Y_j := X_j - g_\theta(V, X_j) V$. Then $k_\theta(\sigma_j) = \frac{1}{4} g_\theta(R(V_j, V)V, V_j)_x$ where $V_j = |Y_j|^{-1} Y_j$ and $k_\theta(\sigma_j) \ge k_0/4$ yields

$$\rho(X, X)_x = 4|X|_x^2 \sum_{j=1}^{2n} k_\theta(\sigma_j) |Y_j|_x^2 \ge (2n-1) k_0 |X|_x^2.$$

As another application of Proposition 5 we establish

Theorem 8. *Let (M, θ) be a Sasakian manifold, of CR dimension n. Let $\gamma : [a, b] \to M$ be a lengthy geodesic of the Tanaka-Webster connection ∇, parametrized by arch length. Assume that* i) *there is $c \in (a, b)$ such that the points $\gamma(a)$ and $\gamma(c)$ are horizontally conjugate along γ and* ii) *for any $\delta > 0$ such that $[c - \delta, c + \delta] \subset (a, b)$ one has $\dim_{\mathbf{R}} \mathscr{H}_{\gamma_\delta} = 4n$, where γ_δ is the restriction of γ to $[c - \delta, c + \delta]$. Then there is a piecewise differentiable horizontal vector field X along γ such that* 1) *X is perpendicular to $\dot{\gamma}$ and $J\dot{\gamma}$,* 2) *$X_{\gamma(a)} = X_{\gamma(b)} = 0$, and* 3) *$I_a^b(X) < 0$.*

In general, we have

LEMMA 8. *Let* (M, θ) *be a Sasakian manifold of CR dimension* n *and* $\gamma(t) \in M$ *a lengthy geodesic of* ∇, *parametrized by arch length. Then*

$$2n + 1 \le \dim_{\mathbf{R}} \mathscr{H}_\gamma \le 4n.$$

Hence the hypothesis in Theorem 8 is that $\mathscr{H}_{\gamma_\delta}$ has maximal dimension. We shall prove Lemma 8 later on. As to the converse of Theorem 8, Corollary 4 guarantees only that the existence of a piecewise differentiable vector field X as above implies that there is some point $\gamma(c)$ conjugate to $\gamma(a)$ along γ.

Proof of Theorem 8. Let $a < c < b$ such that $\gamma(a)$ and $\gamma(c)$ are horizontally conjugate and let $Y \in \mathscr{H}_\gamma$ such that $Y_{\gamma(a)} = Y_{\gamma(c)} = 0$. By Corollary 2 (as (M, θ) is Sasakian) Y is perpendicular to γ. Let (U, x^i) be a normal (with respect to ∇) coordinate neighborhood with origin at $\gamma(c)$. By Theorem 8.7 in [17], Vol. I, p. 149, there is $R > 0$ such that for any $0 < r < R$ the open set

$$U(\gamma(c); r) \equiv \left\{ y \in U : \sum_{i=1}^{2n+1} x^i(y)^2 < r^2 \right\}$$

is convex[1] and each point of $U(\gamma(c); r)$ has a normal coordinate neighborhood containing $U(\gamma(c); r)$. By continuity there is $\delta > 0$ such that $\gamma(t) \in U(\gamma(c); r)$ for any $c - \delta \le t \le c + \delta$. Let γ_δ denote the restriction of γ to the interval $[c - \delta, c + \delta]$. We need the following

LEMMA 9. *The points* $\gamma(c \pm \delta)$ *are not conjugate along* γ_δ.

The proof is by contradiction. If $\gamma(c + \delta)$ is conjugate to $\gamma(c - \delta)$ along γ_δ then (by Theorem 1.4 in [17], Vol. II, p. 67) there is $v \in T_{\gamma(c-\delta)}(M)$ such that $\exp_{\gamma(c-\delta)} v = \gamma(c + \delta)$ and the linear map

$$d_v \exp_{\gamma(c-\delta)} : T_v(T_{\gamma(c-\delta)}(M)) \to T_{\gamma(c+\delta)}(M)$$

is singular, i.e. $\mathrm{Ker}(d_v \exp_{\gamma(c-\delta)}) \neq 0$. Yet $\gamma(c - \delta) \in U(\gamma(c); r)$ hence there is a normal (relative to ∇) coordinate neighborhood V with origin at $\gamma(c - \delta)$ such that $V \supseteq U(\gamma(c); r)$. In particular $\exp_{\gamma(c-\delta)} : V \to M$ is a diffeomorphism on its image, so that $d_v \exp_{\gamma(c-\delta)}$ is a linear isomorphism, a contradiction. Lemma 9 is proved.

Let us go back to the proof of Theorem 8. The linear map

$$\Phi : J_{\gamma_\delta} \to T_{\gamma(c-\delta)}(M) \oplus T_{\gamma(c+\delta)}(M), \quad Z \mapsto (Z_{\gamma(c-\delta)}, Z_{\gamma(c+\delta)}),$$

is a monomorphism. Indeed $\mathrm{Ker}(\Phi) = 0$, otherwise $\gamma(c \pm \delta)$ would be conjugate (in contradiction with Lemma 9). Both spaces are $(4n + 2)$-dimensional so that Φ is an epimorphism, as well. By hypothesis $\mathscr{H}_{\gamma_\delta}$ is $4n$-dimensional hence Φ descends to an isomorphism

$$\mathscr{H}_{\gamma_\delta} \approx H(M)_{\gamma(c-\delta)} \oplus H(M)_{\gamma(c+\delta)}.$$

[1] That is any two points of $U(\gamma(c); r)$ may be joined by a geodesic of ∇ lying in $U(\gamma(c); r)$.

Let then $Z \in \mathscr{H}_{\gamma_\delta}$ be a horizontal Jacobi field such that

$$Z_{\gamma(c-\delta)} = Y_{\gamma(c-\delta)}, \quad Z_{\gamma(c+\delta)} = 0.$$

We set

$$X = \begin{cases} Y & \text{on } \gamma|_{[a, c-\delta]}, \\ Z & \text{on } \gamma_\delta, \\ 0 & \text{on } \gamma|_{[c+\delta, b]}. \end{cases}$$

By the very definition X is horizontal, i.e. $X_{\gamma(t)} \in H(M)_{\gamma(t)}$ for every t. Moreover (by $\mathscr{J}_\gamma Y = 0$ and $\theta(Y) = 0$)

$$\begin{aligned} I_a^c(Y) &= \int_a^c \{|\nabla_{\dot\gamma} Y|^2 - g_\theta(R(Y,\dot\gamma)\dot\gamma, Y)\}\, dt \\ &= \int_a^c \{|\nabla_{\dot\gamma} Y|^2 + g_\theta(\nabla_{\dot\gamma}^2 Y, Y)\}\, dt \\ &= g_\theta(\nabla_{\dot\gamma} Y, Y)_{\gamma(c)} - g_\theta(\nabla_{\dot\gamma} Y, Y)_{\gamma(a)} = 0 \end{aligned}$$

i.e. $I_a^{c-\delta}(Y) = -I_{c-\delta}^c(Y)$. Hence

$$I_a^b(X) = I_a^{c-\delta}(Y) + I_{c-\delta}^{c+\delta}(Z) = -I_{c-\delta}^c(Y) + I_{c-\delta}^{c+\delta}(Z).$$

Finally, let us consider the vector field along γ_δ

$$W = \begin{cases} Y & \text{on } \gamma|_{[c-\delta, c]}, \\ 0, & \text{on } \gamma|_{[c, c+\delta]}. \end{cases}$$

Note that $W_{\gamma(c+\delta)} = 0$, $W_{\gamma(c-\delta)} = Z_{\gamma(c-\delta)}$ and W is perpendicular to γ. Thus we may apply Proposition 5 to W and to $Z \in \mathscr{H}_{\gamma_\delta}$ to conclude that $I_{c-\delta}^c(Y) = I_{c-\delta}^{c+\delta}(W) \geq I_{c-\delta}^{c+\delta}(Z)$. Consequently $I_a^b(X) < 0$. Let us show that X is orthogonal to $J\dot\gamma$. By Lemma 4 (as $Y \in J_\gamma$)

$$\theta(Y')_{\gamma(t)} - 2\Omega(Y,\dot\gamma)_{\gamma(t)} = \text{const.} = \theta(Y')_{\gamma(a)} - 2\Omega(Y,\dot\gamma)_{\gamma(a)}$$

hence (as $Y_{\gamma(a)} = 0$ and $Y_{\gamma(t)} \in H(M)_{\gamma(t)} \Rightarrow Y'_{\gamma(t)} \in H(M)_{\gamma(t)}$)

$$2\Omega(Y,\dot\gamma)_{\gamma(t)} = \theta(Y')_{\gamma(t)} - \theta(Y')_{\gamma(a)} = 0$$

for any $a \leq t \leq c-\delta$. Similarly (as $Z_{\gamma(c+\delta)} = 0$ and Z is horizontal) $\Omega(Z,\dot\gamma)_{\gamma(t)} = 0$ for any $c-\delta \leq t \leq c+\delta$. Therefore $\Omega(X,\dot\gamma)_{\gamma(t)} = 0$ for every t. Theorem 8 is proved.

It remains that we prove Lemma 8. Let $\gamma(t) \in M$, $|t| < \varepsilon$, be a lengthy geodesic of ∇. Let $X \in \mathscr{H}_\gamma$ and $\{Y_j : 1 \leq j \leq 4n+2\}$ a linear basis in J_γ. Then $X = c^j Y_j = c^j Y_j^H + c^j\theta(Y_j)T$ (where $Y_j^H \equiv Y_j - \theta(Y_j)T$) for some $c^j \in \mathbf{R}$. As $X_{\gamma(t)} \in H(M)_{\gamma(t)}$ one has i) $c^j\theta(Y_j)_{\gamma(t)} = 0$ on one hand, and ii) $c^j f_j^a(\gamma(t)) = f^a(\gamma(t))$, $1 \leq a \leq 2n$, on the other, where $X = f^a X_a$, $Y_j^H = f_j^a X_a$ and $\{X_a : 1 \leq a \leq 2n\}$ is a local frame of $H(M)$. One may think of (i)–(ii) as a linear system in the unknowns c^j. Let $r(t)$ be its rank. Then $\dim_{\mathbf{R}} \mathscr{H}_\gamma = 4n + 2 - r(t) \geq 2n + 1$. To prove the remaining inequality in Lemma 8 it suffices to observe that $\mathscr{H}_\gamma$ is contained in the space of all solutions to $X'' + R(X, \dot\gamma)\dot\gamma = 0$ obeying $X_{\gamma(0)} \in H(M)_{\gamma(0)}$ and $X'_{\gamma(0)} \in H(M)_{\gamma(0)}$, which is $4n$-dimensional.

7. The first variation of the length integral

Let M be a strictly pseudoconvex CR manifold and $y, z \in M$. Let Γ be the set of all piecewise differentiable curves $\gamma : [a, b] \to M$ parametrized proportionally to arc length, such that $\gamma(a) = y$ and $\gamma(b) = z$. As usual, for each $\gamma \in \Gamma$ we let $T_\gamma(\Gamma)$ be the space of all piecewise differentiable vector fields along γ such that $X_y = X_z = 0$. Given $X \in T_\gamma(\Gamma)$ let $\gamma^s : [a, b] \to M$, $|s| < \varepsilon$, be a family of curves such that i) $\gamma^s \in \Gamma$, $|s| < \varepsilon$, ii) $\gamma^0 = \gamma$, iii) there is a partition $a = t_0 < t_1 < \cdots < t_k = b$ such that the map $(t, s) \mapsto \gamma^s(t)$ is differentiable on each rectangle $[t_j, t_{j+1}] \times (-\varepsilon, \varepsilon)$, $0 \leq j \leq k - 1$, and iv) for each fixed $t \in [a, b]$ the tangent vector to

$$\sigma_t : (-\varepsilon, \varepsilon) \to M, \quad \sigma_t(s) = \gamma^s(t), \quad |s| < \varepsilon,$$

at the point $\gamma(t)$ is $X_{\gamma(t)}$. We set as usual

$$(d_\gamma L)X = \frac{d}{ds}\{L(\gamma^s)\}_{s=0}.$$

Here $L(\gamma^s)$ is the Riemannian length of γ^s with respect to the Webster metric g_θ (so that γ^s need not be lengthy to start with). One scope of this section is to establish the following

Theorem 9. *Let $\gamma^s : [a, b] \to M$, $|s| < \varepsilon$, be a 1-parameter family of curves such that $(t, s) \mapsto \gamma^s(t)$ is differentiable on $[a, b] \times (-\varepsilon, \varepsilon)$ and each γ^s is parametrized proportionally to arc length. Let us set $\gamma = \gamma^0$. Then*

$$\frac{d}{ds}\{L(\gamma^s)\}_{s=0} = \frac{1}{r}\Big\{ g_\theta(X, \dot\gamma)_{\gamma(b)} - g_\theta(X, \dot\gamma)_{\gamma(a)} - \int_a^b [g_\theta(X, \nabla_{\dot\gamma}\dot\gamma) - g_\theta(T_\nabla(X, \dot\gamma), \dot\gamma)]_{\gamma(t)}\, dt \Big\} \tag{46}$$

where $X_{\gamma(t)} = \dot\sigma_t(0)$, $a \leq t \leq b$, and $r = |\dot\gamma(t)|$ is the common length of all tangent vectors along γ.

This will be shortly seen to imply

Theorem 10. *Let $\gamma \in \Gamma$ and $X \in T_\gamma(\Gamma)$. Let $a = c_0 < c_1 < \cdots < c_h < c_{h+1} = b$ be a partition such that γ is differentiable on each $[c_j, c_{j+1}]$, $0 \leq j \leq h$. Then*

$$(d_\gamma L)X = \frac{1}{r}\left\{\sum_{j=1}^{h} g_{\theta,\gamma(c_j)}(X_{\gamma(c_j)}, \dot\gamma(c_j^-) - \dot\gamma(c_j^+)) - \int_a^b [g_\theta(X, \nabla_{\dot\gamma}\dot\gamma) - g_\theta(T_\nabla(X,\dot\gamma),\dot\gamma)]_{\gamma(t)}\, dt\right\} \tag{47}$$

where $\dot\gamma(c_j^\pm) = \lim_{t\to c_j^\pm}\dot\gamma(t)$.

Consequently, we shall prove

Corollary 5. *A lengthy curve* $\gamma \in \Gamma$ *is a geodesic of the Tanaka-Webster connection if and only if*

$$(d_\gamma L)X = \frac{1}{r}\int_a^b \theta(X)_{\gamma(t)} A(\dot\gamma,\dot\gamma)_{\gamma(t)}\, dt \tag{48}$$

for all $X \in T_\gamma(\Gamma)$. *In particular, if* (M,θ) *is a Sasakian manifold then lengthy geodesics belonging to* Γ *are the critical points of* L *on* Γ.

The remainder of this section is devoted to the proofs of the results above. We adopt the principal bundle approach in [17], Vol. II, p. 80–83. The proof is a *verbatim* transcription of the arguments there, except for the presence of torsion terms.

Let $\pi : O(M, g_\theta) \to M$ be the $O(2n+1)$-bundle of g_θ-orthonormal frames tangent to M. Let $Q = [a,b]\times(-\varepsilon,\varepsilon)$. Let $f : Q \to O(M,g_\theta)$ be a parametrized surface in $O(M,g_\theta)$ such that i) $\pi(f(t,s)) = \gamma^s(t)$, $(t,s)\in Q$, and ii) $f^0 : [a,b] \to O(M,g_\theta)$, $f^0(t) = f(t,0)$, $a \le t \le b$, is a horizontal curve. Precisely, the Tanaka-Webster connection ∇ of (M,θ) induces an infinitesimal connection in the principal bundle $\mathrm{GL}(2n+1,\mathbf{R}) \to L(M) \to M$ (of all linear frames tangent to M) descending (because of $\nabla g_\theta = 0$) to a connection H in $O(2n+1) \to O(M,g_\theta) \to M$. The requirement is that $(df^0/dt)(t) \in H_{f^0(t)}$, $a \le t \le b$.

Let $\mathbf{S},\mathbf{T} \in \mathscr{X}(Q)$ be given by $\mathbf{S} = \partial/\partial s$ and $\mathbf{T} = \partial/\partial t$. Let

$$\mu \in \Gamma^\infty(T^*(O(M,g_\theta)) \otimes \mathbf{R}^{2n+1}), \quad \Theta = D\mu,$$
$$\omega \in \Gamma^\infty(T^*(O(M,g_\theta)) \otimes \mathbf{o}(2n+1)), \quad \Omega = D\omega,$$

be respectively the canonical 1-form, the torsion 2-form, the connection 1-form, and the curvature 2-form of H on $O(M,g_\theta)$. We denote by

$$\mu^* = f^*\mu, \quad \Theta^* = f^*\Theta, \quad \omega^* = f^*\omega, \quad \Omega^* = f^*\Omega,$$

the pullback of these forms to the rectangle Q. We claim that

$$[\mathbf{S},\mathbf{T}] = 0, \tag{49}$$

$$\omega^*(\mathbf{T})_{(t,0)} = 0, \quad a \le t \le b. \tag{50}$$

Indeed (49) is obvious. To check (50) one needs to be a bit pedantic and introduce the injections

$$\alpha^s : [a,b] \to Q, \quad \beta_t : (-\varepsilon, \varepsilon) \to Q,$$
$$\alpha^s(t) = \beta_t(s) = (t,s), \quad a \le t \le b, \ |s| < \varepsilon,$$

so that $f^0 = f \circ \alpha^0$. Then

$$H_{f^0(t)} \ni \frac{df^0}{dt}(t) = (d_{(t,0)}f)(d_t\alpha^0)\frac{d}{dt}\bigg|_t = (d_{(t,0)}f)\mathbf{T}_{(t,0)},$$
$$\omega^*(\mathbf{T})_{(t,0)} = \omega_{f(t,0)}((d_{(t,0)}f)\mathbf{T}_{(t,0)}) = 0.$$

Next, we claim that

$$\mathbf{S}(\mu^*(\mathbf{T})) = \mathbf{T}(\mu^*(\mathbf{S})) + \omega^*(\mathbf{T}) \cdot \mu^*(\mathbf{S}) - \omega^*(\mathbf{S}) \cdot \mu^*(\mathbf{T}) + 2\Theta^*(\mathbf{S},\mathbf{T}), \tag{51}$$

$$\mathbf{S}(\omega^*(\mathbf{T})) = \mathbf{T}(\omega^*(\mathbf{S})) + \omega^*(\mathbf{T})\omega^*(\mathbf{S}) - \omega^*(\mathbf{S})\omega^*(\mathbf{T}) + 2\Omega^*(\mathbf{S},\mathbf{T}). \tag{52}$$

The identities (51)–(52) follow from Prop. 3.11 in [17], Vol. I, p. 36, our identity (49), and the first and second structure equations for a linear connection (cf. e.g. Theor. 2.4 in [17], Vol. I, p. 120). Let us consider the C^∞ function $F : Q \to [0,+\infty)$ given by

$$F(t,s) = \langle \mu^*(\mathbf{T})_{(t,s)}, \mu^*(\mathbf{T})_{(t,s)} \rangle^{1/2}, \quad (t,s) \in Q.$$

Here $\langle \xi, \eta \rangle$ is the Euclidean scalar product of $\xi, \eta \in \mathbf{R}^{2n+1}$. Note that

$$\begin{aligned}\mu^*(\mathbf{T})_{(t,s)} &= \mu_{f(t,s)}((d_{(t,s)}f)\mathbf{T}_{(t,s)}) \\ &= f(t,s)^{-1}(d_{f(t,s)}\pi)(d_{(t,s)}f)\mathbf{T}_{(t,s)} = f(t,s)^{-1}d_t(\pi \circ f \circ \alpha^s)\frac{d}{dt}\bigg|_t\end{aligned}$$

i.e.

$$\mu^*(\mathbf{T})_{(t,s)} = f(t,s)^{-1}\dot\gamma^s(t). \tag{53}$$

Yet $f(t,s) \in O(M, g_\theta)$, i.e. $f(t,s)$ is a linear isometry of $(\mathbf{R}^{2n+1}, \langle\ ,\ \rangle)$ onto $(T_{\gamma^s(t)}(M), g_{\theta,\gamma^s(t)})$, so that

$$F(t,s) = g_{\theta,\gamma^s(t)}(\dot\gamma^s(t), \dot\gamma^s(t))^{1/2}$$

and then

$$L(\gamma^s) = \int_a^b F(t,s)\,dt.$$

As γ^s is parametrized proportionally to arc length $F(t,s)$ doesn't depend on t. In particular

$$F(t,0) = r. \tag{54}$$

We claim that

$$\mathbf{S}(F) = \frac{1}{r}\{\langle \mathbf{T}(\mu^*(\mathbf{S})), \mu^*(\mathbf{T})\rangle + 2\langle \Theta^*(\mathbf{S},\mathbf{T}), \mu^*(\mathbf{T})\rangle\} \tag{55}$$

at all points $(t,0) \in Q$. Indeed, by (51)

$$\begin{aligned}2F\mathbf{S}(F) &= \mathbf{S}(F^2) = \mathbf{S}(\langle \mu^*(\mathbf{T}), \mu^*(\mathbf{T})\rangle) = 2\langle \mathbf{S}(\mu^*(\mathbf{T})), \mu^*(\mathbf{T})\rangle \\ &= 2\langle \mathbf{T}(\mu^*(\mathbf{S})), \mu^*(\mathbf{T})\rangle + 2\langle \omega^*(\mathbf{T})\cdot \mu^*(\mathbf{S}), \mu^*(\mathbf{T})\rangle \\ &\quad - 2\langle \omega^*(\mathbf{S})\cdot \mu^*(\mathbf{T}), \mu^*(\mathbf{T})\rangle + 4\langle \Theta^*(\mathbf{S},\mathbf{T}), \mu^*(\mathbf{T})\rangle.\end{aligned}$$

On the other hand ω is $\mathbf{o}(2n+1)$-valued (where $\mathbf{o}(2n+1)$ is the Lie algebra of $O(2n+1)$), i.e. $\omega^*(\mathbf{S})_{(t,s)} : \mathbf{R}^{2n+1} \to \mathbf{R}^{2n+1}$ is skew symmetric, hence the last-but-one term vanishes. Therefore (55) follows from (50) and (54). We may compute now the first variation of the length integral

$$\begin{aligned}\frac{d}{dt}\{L(\gamma^s)\}_{s=0} &= \int_a^b \mathbf{S}(F)_{(t,0)}\,dt \quad (\text{by (55)}) \\ &= \frac{1}{r}\int_a^b \{\langle \mathbf{T}(\mu^*(\mathbf{S})), \mu^*(\mathbf{T})\rangle_{(t,0)} + 2\langle \Theta^*(\mathbf{S},\mathbf{T}), \mu^*(\mathbf{T})\rangle_{(t,0)}\}\,dt.\end{aligned}$$

On the other hand

$$\begin{aligned}\mu^*(\mathbf{S})_{(t,0)} &= \mu_{f^0(t)}((d_{(t,0)}f)\mathbf{S}_{(t,0)}) \\ &= f(t,0)^{-1} d_0(\pi\circ f\circ\beta_t)\frac{d}{ds}\bigg|_0 = f(t,0)^{-1}\frac{d\sigma_t}{ds}(0)\end{aligned}$$

i.e.

$$\mu^*(\mathbf{S})_{(t,0)} = f^0(t)^{-1}X_{\gamma(t)}. \tag{56}$$

Note that given $u \in C^\infty(Q)$ one has $\mathbf{T}(u)_{(t,0)} = (u\circ\alpha^0)'(t)$. Then

$$\begin{aligned}\mathbf{T}(\mu^*(\mathbf{T}))_{(t,0)} &= \lim_{h\to 0}\frac{1}{h}\{\mu^*(\mathbf{T})_{(t+h,0)} - \mu^*(\mathbf{T})_{(t,0)}\} \quad (\text{by (53)}) \\ &= \lim_{h\to 0}\frac{1}{h}\{f^0(t+h)^{-1}\dot\gamma(t+h) - f^0(t)^{-1}\dot\gamma(t)\}.\end{aligned}$$

Yet, as f^0 is an horizontal curve

$$f^0(t+h)^{-1}\dot\gamma(t+h) = f^0(t)^{-1}\tau_t^{t+h}\dot\gamma(t+h),$$

where $\tau_t^{t+h} : T_{\gamma(t+h)}(M) \to T_{\gamma(t)}(M)$ is the parallel displacement operator along γ from $\gamma(t+h)$ to $\gamma(t)$. Hence

$$\mathbf{T}(\mu^*(\mathbf{T}))_{(t,0)} = f^0(t)^{-1}\left(\lim_{h\to 0}\frac{1}{h}\{\tau_t^{t+h}\dot\gamma(t+h) - \dot\gamma(t)\}\right)$$

i.e.

$$\mathbf{T}(\mu^*(\mathbf{T}))_{(t,0)} = f^0(t)^{-1}(\nabla_{\dot\gamma}\dot\gamma)_{\gamma(t)}. \tag{57}$$

To compute the torsion term we recall (cf. [17], Vol. I, p. 132)

$$T_{\nabla,x}(X,Y) = 2v(\Theta_v(X^*,Y^*)),$$

for any $X, Y \in T_x(M)$, where v is a linear frame at x and $X^*, Y^* \in T_v(L(M))$ project respectively on X, Y. Note that $(d_{(t,0)}f)\mathbf{S}_{(t,0)}$ and $(d_{(t,0)}f)\mathbf{T}_{(t,0)}$ project on $X_{\gamma(t)}$ and $\dot{\gamma}(t)$, respectively. Then

$$2\Theta^*(\mathbf{S},\mathbf{T})_{(t,0)} = f^0(t)^{-1} T_\nabla(X,\dot{\gamma})_{\gamma(t)}. \tag{58}$$

Finally (by (53) and (56)–(58))

$$\begin{aligned}\frac{d}{dt}\{L(\gamma^s)\}_{s=0} &= \frac{1}{r}\int_a^b \{\mathbf{T}(\langle \mu^*(\mathbf{S}), \mu^*(\mathbf{T})\rangle) \\ &\quad - \langle \mu^*(\mathbf{S}), \mathbf{T}(\mu^*(\mathbf{T}))\rangle + 2\langle \Theta^*(\mathbf{S},\mathbf{T}), \mu^*(\mathbf{T})\}_{(t,0)}\, dt \\ &= \frac{1}{r}\{\langle \mu^*(\mathbf{S}), \mu^*(\mathbf{T})\rangle_{(b,0)} - \langle \mu^*(\mathbf{S}), \mu^*(\mathbf{T})\rangle_{(a,0)}\} \\ &\quad - \frac{1}{r}\int_a^b \{g_\theta(X, \nabla_{\dot{\gamma}}\dot{\gamma}) - g_\theta(T_\nabla(X,\dot{\gamma}),\dot{\gamma})\}_{\gamma(t)}\, dt\end{aligned}$$

and (46) is proved.

Proof of Theorem 10. Let $c_j = t_0^{(j)} < t_1^{(j)} < \cdots < t_{k_j}^{(j)} = c_{j+1}$ be a partition of $[c_j, c_{j+1}]$ such that X is differentiable along the restriction of γ at each $[t_i^{(j)}, t_{i+1}^{(j)}]$, $0 \le i \le k_j - 1$. Moreover, let $\{\gamma^s\}_{|s|<\varepsilon}$ be a family of curves $\gamma^s \in \Gamma$ such that $\gamma^0 = \gamma$, the map $(t,s) \mapsto \gamma^s(t)$ is differentiable on $[c_j, c_{j+1}] \times (-\varepsilon, \varepsilon)$ for every $0 \le j \le h$, and $X_{\gamma(t)} = (d\sigma_t/ds)(0)$ for every t (with $\sigma_t(s) = \gamma^s(t)$). Let γ_j^s (respectively γ_{ji}^s) be the restriction of γ^s (respectively of γ_j^s) to $[c_j, c_{j+1}]$ (respectively to $[t_i^{(j)}, t_{i+1}^{(j)}]$). We may apply Theorem 9 (to the interval $[t_i^{(j)}, t_{i+1}^{(j)}]$ rather than $[a,b]$) so that to get

$$\frac{d}{ds}\{L(\gamma_{ji}^s)\}_{s=0} = \frac{1}{r}\left\{g_\theta(X,\dot{\gamma})_{\gamma(t_{i+1}^{(j)})} - g_\theta(X,\dot{\gamma})_{\gamma(t_i^{(j)})} - \int_{t_i^{(j)}}^{t_{i+1}^{(j)}} F(X,\dot{\gamma})\, dt\right\}$$

where $F(X,\dot{\gamma})$ is short for $g_\theta(X, \nabla_{\dot{\gamma}}\dot{\gamma})_{\gamma(t)} - g_\theta(T_\nabla(X,\dot{\gamma}),\dot{\gamma})_{\gamma(t)}$. Let us take the sum over $0 \le i \le k_j - 1$. The lengths $L(\gamma_{ji}^s)$ ad up to $L(\gamma_j^s)$. Taking into account that at the points $\gamma(c_j)$ only the lateral limits of $\dot{\gamma}$ are actually defined, we obtain

$$\begin{aligned}\frac{d}{ds}\{L(\gamma_j^s)\}_{s=0} = \frac{1}{r}\Big\{&g_{\theta,\gamma(c_{j+1})}(X_{\gamma(c_{j+1})}, \dot{\gamma}(c_{j+1}^-)) \\ &- g_{\theta,\gamma(c_j)}(X_{\gamma(c_j)}, \dot{\gamma}(c_j^+)) - \int_{c_j}^{c_{j+1}} F(X,\dot{\gamma})\, dt\Big\}\end{aligned}$$

and taking the sum over $0 \le j \le h$ leads to (47) (as $X_{\gamma(c_0)} = 0$ and $X_{\gamma(c_{h+1})} = 0$). Q.e.d.

Proof of Corollary 5. Let $\gamma(t) \in M$ be a lengthy curve such that $\gamma \in \Gamma$. If γ is a geodesic of ∇ then $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ implies (by Theorem 9)

$$(d_\gamma L)X = \frac{1}{r}\int_a^b g_\theta(T_\nabla(X,\dot{\gamma}),\dot{\gamma})_{\gamma(t)}\, dt$$

for any $X \in T_\gamma(\Gamma)$ and then

$$T_\nabla(X, \dot{\gamma}) = -2\Omega(X_H, \dot{\gamma})T + \theta(X)\tau(\dot{\gamma}), \quad g_\theta(T, \dot{\gamma}) = 0,$$

yield (48). Viceversa, let $\gamma \in \Gamma$ be a lengthy curve such that (48) holds. There is a partition $a = c_0 < c_1 < \cdots < c_{h+1} = b$ such that γ is differentiable in $[c_j, c_{j+1}]$, $0 \le j \le h$. Let f be a continuous function defined along γ such that $f(\gamma(c_j)) = 0$ for $1 \le j \le h$ and $f(\gamma(t)) > 0$ elsewhere. We may apply (47) in Theorem 10 to the vector field $X = f\nabla_{\dot{\gamma}}\dot{\gamma}$ so that to get

$$(59) \qquad (d_\gamma L)X = -\frac{1}{r}\int_a^b f\{|\nabla_{\dot{\gamma}}\dot{\gamma}|^2 - g_\theta(T_\nabla(\nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}), \dot{\gamma})\}\, dt.$$

As γ is lengthy and $H(M)$ is parallel with respect to ∇ one has $\nabla_{\dot{\gamma}}\dot{\gamma} \in H(M)$ hence (by (48)) $(d_\gamma L)(f\nabla_{\dot{\gamma}}\dot{\gamma}) = 0$ and

$$g_\theta(T_\nabla(\nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}), \dot{\gamma}) = -2\Omega(\nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma})g_\theta(T, \dot{\gamma}) = 0$$

so that (by (59)) it must be $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ whenever $\nabla_{\dot{\gamma}}\dot{\gamma}$ makes sense, i.e. γ is a broken geodesic of ∇. It remains that we prove differentiability of γ at the points c_j, $1 \le j \le h$. Let $j \in \{1, \dots, h\}$ be a fixed index and let us consider a vector field $X_j \in T_\gamma(\Gamma)$ such that $X_{j,\gamma(c_j)} = \dot{\gamma}(c_j^-) - \dot{\gamma}(c_j^+)$ and $X_{j,\gamma(c_k)} = 0$ for any $k \in \{1, \dots h\} \setminus \{j\}$. Then (by (47)–(48)) one has $|\dot{\gamma}(c_j^-) - \dot{\gamma}(c_j^+)|^2 = 0$. Q.e.d.

Remark. The following alternative proof of Theorem 9 is also available. Since (M, g_θ) is a Riemannian manifold and $L(\gamma^s)$ is the Riemannian length of γ^s we have (cf. Theorem 5.1 in [17], Vol. II, p. 80)

$$(60) \qquad \frac{d}{ds}\{L(\gamma^s)\}_{s=0} = \frac{1}{r}\{g_\theta(X, \dot{\gamma})_{\gamma(b)} - g_\theta(X, \dot{\gamma})_{\gamma(a)}\} - \frac{1}{r}\int_a^b g_\theta(X, D_{\dot{\gamma}}\dot{\gamma})_{\gamma(t)}\, dt$$

where D is the Levi-Civita connection of (M, g_θ). On the other hand (cf. e.g. [10], section 1.3) D is related to the Tanaka-Webster connection of (M, θ) by $D = \nabla + (\Omega - A) \otimes T + \tau \otimes \theta + 2\theta \odot J$ hence

$$\begin{aligned} g_\theta(X, D_{\dot{\gamma}}\dot{\gamma}) &= g_\theta(X, \nabla_{\dot{\gamma}}\dot{\gamma}) - \theta(X)A(\dot{\gamma}, \dot{\gamma}) + \theta(\dot{\gamma})A(X, \dot{\gamma}) + 2\theta(\dot{\gamma})\Omega(X, \dot{\gamma}) \\ &= g_\theta(X, \nabla_{\dot{\gamma}}\dot{\gamma}) - g_\theta(T_\nabla(X, \dot{\gamma}), \dot{\gamma}) \end{aligned}$$

so that (60) yields (46). Q.e.d.

8. The second variation of the length integral

We introduce the Hessian I of L at a geodesic $\gamma \in \Gamma$ as follows. Given $X \in T_\gamma(\Gamma)$ let us consider a 1-parameter family of curves $\{\gamma^s\}_{|s|<\varepsilon}$ as in the definition of $(d_\gamma L)X$. Let $I(X, X)$ be given by

$$I(X, X) = \frac{d^2}{ds^2}\{L(\gamma^s)\}_{s=0}$$

and define $I(X, Y)$ by polarization. By analogy to Riemannian geometry (cf. e.g. [17], Vol. II, p. 81) $I(X, Y)$ is referred to as the *index form*. The scope of this section is to establish

THEOREM 11. *Let (M, θ) be a Sasakian manifold. If $\gamma \in \Gamma$ is a lengthy geodesic of the Tanaka-Webster connection ∇ of (M, θ) and $X, Y \in T_\gamma(\Gamma)$ then*

$$I(X, Y) = \frac{1}{r}\int_a^b \{g_\theta(\nabla_{\dot\gamma} X^\perp, \nabla_{\dot\gamma} Y^\perp) - g_\theta(R(X^\perp, \dot\gamma)\dot\gamma, Y^\perp) - 2\Omega(X^\perp, \dot\gamma)\theta(\nabla_{\dot\gamma} Y^\perp) - 2[\theta(\nabla_{\dot\gamma} X^\perp) - 2\Omega(X^\perp, \dot\gamma)]\Omega(Y^\perp, \dot\gamma)\}\, dt \tag{61}$$

where $X^\perp = X - (1/r^2)g_\theta(X, \dot\gamma)\dot\gamma$.

We shall need the following reformulation of Theorem 11

THEOREM 12. *Let (M, θ), γ and X, Y be as in Theorem* 11. *Then*

$$I(X, Y) = -\frac{1}{r}\int_a^b \{g_\theta(\mathscr{J}_\gamma X^\perp, Y^\perp) + 2[\theta(\nabla_{\dot\gamma} X^\perp) - 2\Omega(X^\perp, \dot\gamma)]\Omega(Y^\perp, \dot\gamma)\}\, dt + \frac{1}{r}\sum_{j=1}^{h} g_{\theta,\gamma(t_j)}((\nabla_{\dot\gamma} X^\perp)^-_{\gamma(t_j)} - (\nabla_{\dot\gamma} X^\perp)^+_{\gamma(t_j)}, Y^\perp_{\gamma(t_j)}) \tag{62}$$

where $\mathscr{J}_\gamma X \equiv \nabla^2_{\dot\gamma} X - 2\Omega(X', \dot\gamma)T + R(X, \dot\gamma)\dot\gamma$ is the Jacobi operator and $a = t_0 < t_1 < \cdots < t_h < t_{h+1} = b$ is a partition of $[a, b]$ such that X is differentiable in each interval $[t_j, t_{j+1}]$, $0 \le j \le h$, and $(\nabla_{\dot\gamma} X^\perp)^\pm_{\gamma(t_j)} = \lim_{t \to t_j^\pm} (\nabla_{\dot\gamma} X^\perp)_{\gamma(t)}$.

This will be seen to imply

COROLLARY 6. *Let (M, θ) be a Sasakian manifold, $\gamma \in \Gamma$ a lengthy geodesic of the Tanaka-Webster connection of (M, θ), and $X \in T_\gamma(\Gamma)$. Then $X^\perp$ is a Jacobi field if and only if there is $\alpha(X) \in \mathbf{R}$ such that*

$$\frac{d}{dt}\{\theta(X^\perp) \circ \gamma\}(t) - 2\Omega(X^\perp, \dot\gamma)_{\gamma(t)} = \alpha(X) \tag{63}$$

for any $a \le t \le b$, and

$$I(X, Y) = -(2/r)\alpha(X)\int_a^b \Omega(Y^\perp, \dot\gamma)_{\gamma(t)}\, dt, \tag{64}$$

for any $Y \in T_\gamma(\Gamma)$.

Proof of Theorem 11. We adopt the notations and conventions in the proof of Theorem 9. As a byproduct of the proof of (55) we have the identity

$$\frac{1}{2}\mathbf{S}(F^2) = \langle \mathbf{T}(\mu^*(\mathbf{S})), \mu^*(\mathbf{T})\rangle + \langle \omega^*(\mathbf{T}) \cdot \mu^*(\mathbf{S}), \mu^*(\mathbf{T})\rangle + 2\langle \Theta^*(\mathbf{S}, \mathbf{T}), \mu^*(\mathbf{T})\rangle. \tag{65}$$

Applying $\mathbf{S}$ we get

$$\begin{aligned}\frac{1}{2}\mathbf{S}^2(F^2) = {} & \langle \mathbf{ST}(\mu^*(\mathbf{S})), \mu^*(\mathbf{T})\rangle + \langle \mathbf{T}(\mu^*(\mathbf{S})), \mathbf{S}(\mu^*(\mathbf{T}))\rangle \\ & + \langle \mathbf{S}(\omega^*(\mathbf{T})) \cdot \mu^*(\mathbf{S}), \mu^*(\mathbf{T})\rangle + \langle \omega^*(\mathbf{T}) \cdot \mathbf{S}(\mu^*(\mathbf{S})), \mu^*(\mathbf{T})\rangle \\ & + \langle \omega^*(\mathbf{T}) \cdot \mu^*(\mathbf{S}), \mathbf{S}(\mu^*(\mathbf{T}))\rangle + 2\langle \mathbf{S}(\Theta^*(\mathbf{S}, \mathbf{T})), \mu^*(\mathbf{T})\rangle \\ & + 2\langle \Theta^*(\mathbf{S}, \mathbf{T}), \mathbf{S}(\mu^*(\mathbf{T}))\rangle.\end{aligned}$$

When calculated at points of the form $(t, 0) \in Q$ the 4th and 5th terms vanish (by (50)). We proceed by calculating the remaining terms (at $(t, 0)$). By (49)

$$\begin{aligned}1^{\text{st}} \text{ term} &= \langle \mathbf{ST}(\mu^*(\mathbf{S})), \mu^*(\mathbf{T})\rangle = \langle \mathbf{TS}(\mu^*(\mathbf{S})), \mu^*(\mathbf{T})\rangle \\ &= \mathbf{T}(\langle \mathbf{S}(\mu^*(\mathbf{S})), \mu^*(\mathbf{T})\rangle) - \langle \mathbf{S}(\mu^*(\mathbf{S})), \mathbf{T}(\mu^*(\mathbf{T}))\rangle.\end{aligned}$$

Yet $\gamma \in \Gamma$ is a geodesic hence (by (57)) $\mathbf{T}(\mu^*(\mathbf{T}))_{(t,0)} = 0$. Hence

$$1^{\text{st}} \text{ term} = \mathbf{T}(\langle \mathbf{S}(\mu^*(\mathbf{S})), \mu^*(\mathbf{T})\rangle)_{(t,0)}.$$

Next (by (51))

$$\begin{aligned}2^{\text{nd}} \text{ term} = {} & \langle \mathbf{T}(\mu^*(\mathbf{S})), \mathbf{S}(\mu^*(\mathbf{T}))\rangle = \langle \mathbf{T}(\mu^*(\mathbf{S})), \mathbf{T}(\mu^*(\mathbf{S}))\rangle \\ & + \langle \mathbf{T}(\mu^*(\mathbf{S})), \omega^*(\mathbf{T}) \cdot \mu^*(\mathbf{S})\rangle - \langle \mathbf{T}(\mu^*(\mathbf{S})), \omega^*(\mathbf{S}) \cdot \mu^*(\mathbf{T})\rangle \\ & + 2\langle \mathbf{T}(\mu^*(\mathbf{S})), \Theta^*(\mathbf{S}, \mathbf{T})\rangle.\end{aligned}$$

Again terms are evaluated at $(t, 0)$ hence $\omega^*(\mathbf{T}) = 0$ (by (50)). On the other hand $\omega^*(\mathbf{S})$ is $\mathbf{o}(2n+1)$-valued hence

$$\begin{aligned}2^{\text{nd}} \text{ term} = {} & \langle \mathbf{T}(\mu^*(\mathbf{S})), \mathbf{T}(\mu^*(\mathbf{S}))\rangle + \langle \omega^*(\mathbf{S}) \cdot \mathbf{T}(\mu^*(\mathbf{S})), \mu^*(\mathbf{T})\rangle \\ & + 2\langle \mathbf{T}(\mu^*(\mathbf{S})), \Theta^*(\mathbf{S}, \mathbf{T})\rangle\end{aligned}$$

at each $(t, 0) \in Q$. Next (by (52))

$$\begin{aligned}3^{\text{rd}} \text{ term} = {} & \langle \mathbf{S}(\omega^*(\mathbf{T})) \cdot \mu^*(\mathbf{S}), \mu^*(\mathbf{T})\rangle = \langle \mathbf{T}(\omega^*(\mathbf{S})) \cdot \mu^*(\mathbf{S}), \mu^*(\mathbf{T})\rangle \\ & + \langle \omega^*(\mathbf{T})\omega^*(\mathbf{S}) \cdot \mu^*(\mathbf{S}), \mu^*(\mathbf{T})\rangle - \langle \omega^*(\mathbf{T})\omega^*(\mathbf{S}) \cdot \mu^*(\mathbf{S}), \mu^*(\mathbf{T})\rangle \\ & + 2\langle \Omega^*(\mathbf{S}, \mathbf{T}) \cdot \mu^*(\mathbf{S}), \mu^*(\mathbf{T})\rangle\end{aligned}$$

or (by (50))

$$3^{\text{rd}} \text{ term} = \langle \mathbf{T}(\omega^*(\mathbf{S})) \cdot \mu^*(\mathbf{S}), \mu^*(\mathbf{T})\rangle + 2\langle \Omega^*(\mathbf{S}, \mathbf{T}) \cdot \mu^*(\mathbf{S}), \mu^*(\mathbf{T})\rangle$$

at each $(t, 0) \in Q$. Finally (by (51))

$$7^{\text{th}}\ \text{term} = 2\langle \Theta^*(\mathbf{S},\mathbf{T}), \mathbf{S}(\mu^*(\mathbf{T}))\rangle$$
$$= 2\langle \Theta^*(\mathbf{S},\mathbf{T}), \mathbf{T}(\mu^*(\mathbf{S}))\rangle + 2\langle \Theta^*(\mathbf{S},\mathbf{T}), \omega^*(\mathbf{T})\cdot \mu^*(\mathbf{S})\rangle$$
$$- 2\langle \Theta^*(\mathbf{S},\mathbf{T}), \omega^*(\mathbf{S})\cdot \mu^*(\mathbf{T})\rangle + 4|\Theta^*(\mathbf{S},\mathbf{T})|^2$$

or (by (50) and the fact that $\omega^*(\mathbf{S})$ is skew)

$$7^{\text{th}}\ \text{term} = 2\langle \Theta^*(\mathbf{S},\mathbf{T}), \mathbf{T}(\mu^*(\mathbf{S}))\rangle + 2\langle \omega^*(\mathbf{S})\cdot\Theta^*(\mathbf{S},\mathbf{T}), \mu^*(\mathbf{T})\rangle + 4|\Theta^*(\mathbf{S},\mathbf{T})|^2$$

at each $(t,0)\in Q$. Summing up the various expressions and noting that (again by (57))

$$\mathbf{T}(\langle \mathbf{S}(\mu^*(\mathbf{S})), \mu^*(\mathbf{T})\rangle) + \langle \omega^*(\mathbf{S})\cdot \mathbf{T}(\mu^*(\mathbf{S})), \mu^*(\mathbf{T})\rangle + \langle \mathbf{T}(\omega^*(\mathbf{S}))\cdot \mu^*(\mathbf{S}), \mu^*(\mathbf{T})\rangle$$
$$= \mathbf{T}(\langle \mathbf{S}(\mu^*(\mathbf{S})) + \omega^*(\mathbf{S})\cdot\mu^*(\mathbf{S}), \mu^*(\mathbf{T})\rangle)$$

we obtain

$$\frac{1}{2}\mathbf{S}^2(F^2) = |\mathbf{T}(\mu^*(\mathbf{S}))|^2 + 2\langle \Omega^*(\mathbf{S},\mathbf{T})\cdot\mu^*(\mathbf{S}), \mu^*(\mathbf{T})\rangle$$
$$+ \mathbf{T}(\langle \mathbf{S}(\mu^*(\mathbf{S})) + \omega^*(\mathbf{S})\cdot\mu^*(\mathbf{S}), \mu^*(\mathbf{T})\rangle)$$
$$+ 4\langle \Theta^*(\mathbf{S},\mathbf{T}), \mathbf{T}(\mu^*(\mathbf{S}))\rangle + 2\langle \mathbf{S}(\Theta^*(\mathbf{S},\mathbf{T})), \mu^*(\mathbf{T})\rangle$$
$$+ 2\langle \omega^*(\mathbf{S})\cdot\Theta^*(\mathbf{S},\mathbf{T}), \mu^*(\mathbf{T})\rangle + 4|\Theta^*(\mathbf{S},\mathbf{T})|^2 \tag{66}$$

at each $(t,0)\in Q$. Since $F\mathbf{S}^2(F) = \frac{1}{2}\mathbf{S}^2(F^2) - \mathbf{S}(F)^2$ we get (by (55) and (66))

$$r\mathbf{S}^2(F) = |\mathbf{T}(\mu^*(\mathbf{S}))|^2 + 2\langle \Omega^*(\mathbf{S},\mathbf{T})\cdot\mu^*(\mathbf{S}), \mu^*(\mathbf{T})\rangle$$
$$+ \mathbf{T}(\langle \mathbf{S}(\mu^*(\mathbf{S})) + \omega^*(\mathbf{S})\cdot\mu^*(\mathbf{S}), \mu^*(\mathbf{T})\rangle)$$
$$+ 4\langle \Theta^*(\mathbf{S},\mathbf{T}), \mathbf{T}(\mu^*(\mathbf{S}))\rangle + 2\langle \mathbf{S}(\Theta^*(\mathbf{S},\mathbf{T})), \mu^*(\mathbf{T})\rangle$$
$$+ 2\langle \omega^*(\mathbf{S})\cdot\Theta^*(\mathbf{S},\mathbf{T}), \mu^*(\mathbf{T})\rangle + 4|\Theta^*(\mathbf{S},\mathbf{T})|^2$$
$$- \frac{1}{r^2}\{\langle \mathbf{T}(\mu^*(\mathbf{S})), \mu^*(\mathbf{T})\rangle^2 + 4\langle \Theta^*(\mathbf{S},\mathbf{T}), \mu^*(\mathbf{T})\rangle^2$$
$$+ 4\langle \mathbf{T}(\mu^*(\mathbf{S})), \mu^*(\mathbf{T})\rangle\langle \Theta^*(\mathbf{S},\mathbf{T}), \mu^*(\mathbf{T})\rangle\} \tag{67}$$

at any $(t,0)\in Q$. Moreover (by (56))

$$\mathbf{T}(\mu^*(\mathbf{S}))_{(t,0)} = \frac{d}{dt}\{\mu^*(\mathbf{S})\circ\alpha^0\}(t)$$
$$= \lim_{h\to 0}\frac{1}{h}\{\mu^*(\mathbf{S})_{(t+h,0)} - \mu^*(\mathbf{S})_{(t,0)}\}$$
$$= \lim_{h\to 0}\frac{1}{h}\{f^0(t+h)^{-1}X_{\gamma(t+h)} - f^0(t)^{-1}X_{\gamma(t)}\}$$

(as $f^0 : [a,b]\to O(M,g_\theta)$ is a horizontal curve)

$$= \lim_{h\to 0} \frac{1}{h}\{f^0(t)^{-1}\tau_t^{t+h}X_{\gamma(t+h)} - f^0(t)^{-1}X_{\gamma(t)}\}$$
$$= f^0(t)^{-1}\left(\lim_{h\to 0}\frac{1}{h}\{\tau_t^{t+h}X_{\gamma(t+h)} - X_{\gamma(t)}\}\right)$$

that is

$$\mathbf{T}(\mu^*(\mathbf{S}))_{(t,0)} = f^0(t)^{-1}(\nabla_{\dot\gamma}X)_{\gamma(t)}. \tag{68}$$

Consequently (by (53) and (68))

$$\begin{aligned} &|\mathbf{T}(\mu^*(\mathbf{S}))|^2 + 4\langle\Theta^*(\mathbf{S},\mathbf{T}),\mathbf{T}(\mu^*(\mathbf{S}))\rangle \\ &\quad - \frac{1}{r^2}\{\langle\mathbf{T}(\mu^*(\mathbf{S})),\mu^*(\mathbf{T})\rangle^2 + 4\langle\Theta^*(\mathbf{S},\mathbf{T}),\mu^*(\mathbf{T})\rangle^2 \\ &\qquad + 4\langle\mathbf{T}(\mu^*(\mathbf{S})),\mu^*(\mathbf{T})\rangle\langle\Theta^*(\mathbf{S},\mathbf{T}),\mu^*(\mathbf{T})\rangle\} \\ &= |\nabla_{\dot\gamma}X|^2 + 2g_\theta(T_\nabla(X,\dot\gamma),\nabla_{\dot\gamma}X) - \frac{1}{r^2}\{g_\theta(\nabla_{\dot\gamma}X,\dot\gamma)^2 \\ &\quad + g_\theta(T_\nabla(X,\dot\gamma),\dot\gamma)^2 + 2g_\theta(\nabla_{\dot\gamma}X,\dot\gamma)g_\theta(T_\nabla(X,\dot\gamma),\dot\gamma)\} \\ &= |\nabla_{\dot\gamma}X^\perp|^2 + 2g_\theta(T_\nabla(X^\perp,\dot\gamma),\nabla_{\dot\gamma}X) \\ &\quad - \frac{1}{r^2}\{g_\theta(T_\nabla(X^\perp,\dot\gamma),\dot\gamma)^2 + 2g_\theta(\nabla_{\dot\gamma}X,\dot\gamma)g_\theta(T_\nabla(X^\perp,\dot\gamma),\dot\gamma)\} \end{aligned} \tag{69}$$

and (by (53) and (56))

$$\begin{aligned} \text{the curvature term} &= 2\langle\Omega^*(\mathbf{S},\mathbf{T})\cdot\mu^*(\mathbf{S}),\mu^*(\mathbf{T})\rangle \\ &= g_\theta(R(X,\dot\gamma)X,\dot\gamma)_{\gamma(t)} = -g_\theta(R(X^\perp,\dot\gamma)\dot\gamma,X^\perp). \end{aligned} \tag{70}$$

On the other hand $\pi(f(a,s)) = y$ and $\pi(f(b,s)) = z$ imply that $(d_{(a,s)}f)\mathbf{S}_{(a,s)}$ and $(d_{(b,s)}f)\mathbf{S}_{(b,s)}$ are vertical hence

$$\mu^*(\mathbf{S})_{(a,s)} = 0, \quad \mu^*(\mathbf{S})_{(b,s)} = 0. \tag{71}$$

Next, we wish to compute $\mathbf{S}(\mu^*(\mathbf{S}))_{(t,0)}$. To do so we need to further specialize the choice of $f(t,s)$. Precisely, let $v \in \pi^{-1}(\gamma(a))$ be a fixed orthonormal frame and let

$$f(t,s) = \sigma_t^\uparrow(s), \quad a \le t \le b,\ |s| < \varepsilon, \tag{72}$$

where $\sigma_t^\uparrow : (-\varepsilon,\varepsilon) \to O(M,g_\theta)$ is the unique horizontal lift of $\sigma_t : (-\varepsilon,\varepsilon) \to M$ issuing at $\sigma_t(0) = \gamma^\uparrow(t)$. Also $\gamma^\uparrow : [a,b] \to O(M,g_\theta)$ is the horizontal lift of $\gamma : [a,b] \to M$ determined by $\gamma^\uparrow(a) = v$. Therefore $f^0 = \gamma^\uparrow$ is a horizontal curve, as required by the previous part of the proof. In addition (72) possesses the property that for each t the curve $s \mapsto f(t,s)$ is horizontal, as well. Then

$$\mathbf{S}(\mu^*(\mathbf{S}))_{(t,0)} = \frac{d}{ds}\{\mu^*(\mathbf{S})\circ\beta_t\}(0)$$
$$= \lim_{s\to 0}\frac{1}{s}\{f(t,s)^{-1}\dot{\sigma}_t(s) - f(t,0)^{-1}\dot{\sigma}_t(0)\}$$

(as $f_t : (-\varepsilon,\varepsilon)\to O(M,g_\theta)$, $f_t(s) = f(t,s)$, $|s|<\varepsilon$, is horizontal)

$$= \lim_{s\to 0}\frac{1}{s}\{f(t,0)^{-1}\tau^s\dot{\sigma}_t(s) - f(t,0)^{-1}\dot{\sigma}_t(0)\}$$
$$= f(t,0)^{-1}\left(\lim_{s\to 0}\frac{1}{s}\{\tau^s\dot{\sigma}_t(s) - \dot{\sigma}_t(0)\}\right)$$

where $\tau^s : T_{\sigma_t(s)}(M)\to T_{\sigma_t(0)}(M)$ is the parallel displacement along σ_t from $\sigma_t(s)$ to $\sigma_t(0)$, i.e.

$$\mathbf{S}(\mu^*(\mathbf{S}))_{(t,0)} = f(t,0)^{-1}(\nabla_{\dot{\sigma}_t}\dot{\sigma}_t)_{\gamma(t)}. \tag{73}$$

By (73), $\dot{\sigma}_a(s) = 0$ and $\dot{\sigma}_b(s) = 0$ (as $\sigma_a(s) = \gamma^s(a) = y = \text{const.}$ and $\sigma_b(s) = \gamma^s(b) = z = \text{const.}$) it follows that

$$\mathbf{S}(\mu^*(\mathbf{S}))_{(a,0)} = 0, \quad \mathbf{S}(\mu^*(\mathbf{S}))_{(b,0)} = 0. \tag{74}$$

Using (71) and (74) we may conclude that

$$\int_a^b \mathbf{T}(\langle\mathbf{S}(\mu^*(\mathbf{S})) + \omega^*(\mathbf{S})\cdot\mu^*(\mathbf{S}),\mu^*(\mathbf{T})\rangle)_{(t,0)}\,dt \tag{75}$$
$$= \langle\mathbf{S}(\mu^*(\mathbf{S})) + \omega^*(\mathbf{S})\cdot\mu^*(\mathbf{S}),\mu^*(\mathbf{T})\rangle_{(b,0)}$$
$$- \langle\mathbf{S}(\mu^*(\mathbf{S})) + \omega^*(\mathbf{S})\cdot\mu^*(\mathbf{S}),\mu^*(\mathbf{T})\rangle_{(a,0)} = 0.$$

Similarly

$$2\mathbf{S}(\Theta^*(\mathbf{S},\mathbf{T}))_{(t,0)} = 2\lim_{s\to 0}\frac{1}{s}\{\Theta^*(\mathbf{S},\mathbf{T})_{(t,s)} - \Theta^*(\mathbf{S},\mathbf{T})_{(t,0)}\}$$
$$= \lim_{s\to 0}\frac{1}{s}\{f(t,s)^{-1}T_\nabla(\dot{\sigma}_t(s),\dot{\gamma}^s(t)) - f(t,0)^{-1}T_\nabla(\dot{\sigma}_t(0),\dot{\gamma}^0(t))\}$$
$$= f(t,0)^{-1}\left(\lim_{s\to 0}\frac{1}{s}\{\tau^s V_{\sigma_t(s)} - V_{\sigma_t(0)}\}\right)$$

where V is the vector field defined at each $a(t,s) = \sigma_t(s)$ by

$$V_{a(t,s)} = T_{\nabla,a(t,s)}(\dot{\sigma}_t(s),\dot{\gamma}^s(t)), \quad a\le t\le b,\ |s|<\varepsilon.$$

Let us assume from now on that $\tau = 0$, i.e. (M,θ) is Sasakian. Then

$$V_{a(t,s)} = -2\Omega_{a(t,s)}(\dot{\sigma}_t(s),\dot{\gamma}^s(t))T_{a(t,s)}$$

and $\nabla T = 0$ yields

$$\tau^s V_{a(t,s)} = -2\Omega_{a(t,s)}(\dot{\sigma}_t(s),\dot{\gamma}^s(t))T_{\gamma(t)}.$$

Finally

$$(76)\qquad 2\langle \mathbf{S}(\Theta^*(\mathbf{S},\mathbf{T})),\mu^*(\mathbf{T})\rangle_{(t,0)}$$
$$= \lim_{s\to 0}\frac{1}{s}\{g_{\theta,a(t,s)}(\tau^s V_{a(t,s)},\dot\gamma(t)) - g_{\theta,a(t,0)}(V_{a(t,0)},\dot\gamma(t))\} = 0,$$

as $\dot\gamma(t)\in H(M)_{\gamma(t)}$. It remains that we compute the term $2\langle\omega^*(\mathbf{S})\cdot\Theta^*(\mathbf{S},\mathbf{T}), \mu^*(\mathbf{T})\rangle$. As $f(t,s)$ is a linear frame at $a(t,s)$

$$f(t,s) = (a(t,s),\{X_{i,a(t,s)} : 1\le i\le 2n+1\}),$$

where $X_i \in T_{a(t,s)}(M)$. Let (U,x^i) be a local coordinate system on M and let us set $X_i = X_i^j\partial/\partial x^j$. Let $(\Pi^{-1}(U),\tilde x^i, g_j^i)$ be the naturally induced local coordinates on $L(M)$, where $\Pi : L(M)\to M$ is the projection. Then $g_j^i(f(t,s)) = X_j^i(a(t,s))$. As ω is the connection 1-form of a linear connection

$$\omega = \omega_i^j \otimes E_j^i$$

where ω_j^i are scalar 1-forms on $L(M)$ and $\{E_j^i : 1\le i,j\le 2n+1\}$ is the basis of the Lie algebra $\mathbf{gl}(2n+1)$ given by $E_j^i = [\delta_\ell^i\delta_j^k]_{1\le k,\ell\le 2n+1}$. Let $\{e_1,\dots,e_{2n+1}\}$ be the canonical linear basis of $\mathbf{R}^{2n+1}$. Then

$$\mu^*(\mathbf{T})_{(t,0)} = f(t,0)^{-1}\dot\gamma(t) = \frac{dx^i}{dt}f(t,0)^{-1}\frac{\partial}{\partial x^i}\bigg|_{\gamma(t)} = \frac{dx^i}{dt}Y_i^j e_j$$

where $[Y_j^i] = [X_j^i]^{-1}$. Therefore

$$\omega^*(\mathbf{S})_{(t,0)}\cdot\mu^*(\mathbf{T})_{(t,0)} = \frac{dx^k}{dt}Y_k^i(f^*\omega_i^j)(\mathbf{S})_{(t,0)}e_j$$

(because of $E_j^i e_k = \delta_k^i e_j$). On the other hand (by Prop. 1.1 in [18], Vol. I, p. 64) $\omega^*(\mathbf{S})_{(t,0)} = A$ where the left invariant vector field $A\in\mathbf{gl}(2n+1)$ is given by

$$(77)\qquad A^*_{f(t,0)} = (d_{(t,0)}f)\mathbf{S}_{(t,0)} - \ell_{f(t,0)}\dot\sigma_t(0)$$

and $\ell_u : T_{\Pi(u)}(M)\to H_u$ is the inverse of $d_u\Pi : H_u\to T_{\Pi(u)}(M)$, $u\in L(M)$ (the horizontal lift operator with respect to H). Here A^* is the fundamental vector field associated to A, i.e.

$$A^*_{f(t,0)} = (d_e L_{f(t,0)})A_e$$

where $L_u : \mathrm{GL}(2n+1)\to L(M)$, $u\in L(M)$, is given by $L_u(g) = ug$ for any $g\in\mathrm{GL}(2n+1)$, and $e\in\mathrm{GL}(2n+1)$ is the unit matrix. If $A = A_i^j E_j^i$ then $A_j^i = (f^*\omega_j^i)(\mathbf{S})_{(t,0)}$. Let (g_j^i) be the natural coordinates on $\mathrm{GL}(2n+1)$ so that $L_{f(t,0)}$ is locally given by

$$L^i(g) = \tilde x^i,\quad L_j^i(g) = X_k^i g_j^k,$$

and then $(d_e L_{f(t,0)})(\partial/\partial g_j^i)_e = X_i^k(\partial/\partial g_j^k)_{f(t,0)}$. Next (cf. [18], Vol. I, p. 143)

$$\ell\frac{\partial}{\partial x^j} = \partial_j - (\Gamma_{jk}^i\circ\Pi)g_\ell^k\frac{\partial}{\partial g_\ell^i}$$

(where $\partial_i = \partial/\partial\tilde x^i$) and (77) lead to

$$A^k_\ell X^i_k \frac{\partial}{\partial g^i_\ell}\bigg|_{f(t,0)} = (d_{(t,0)}f)\mathbf{S}_{(t,0)} - X^j(\gamma(t))\left\{\partial_j - (\Gamma^i_{jk}\circ\Pi)g^k_\ell \frac{\partial}{\partial g^i_\ell}\right\}_{f(t,0)}$$

or (by applying this identity to the coordinate functions g^i_ℓ)

$$A^k_\ell X^i_k = \mathbf{S}_{(t,0)}(g^i_\ell \circ f) + X^j(\gamma(t))\Gamma^i_{jk}(\gamma(t))X^k_\ell. \tag{78}$$

If $f^i_j = g^i_j \circ f$ then

$$\mathbf{S}_{(t,0)}(f^i_j) = \frac{d}{ds}\{f^i_j \circ \beta_t\}(0) = \frac{\partial f^i_j}{\partial s}(t,0)$$

Therefore (by (78))

$$A^k_\ell = Y^k_i\left\{\frac{\partial f^i_\ell}{\partial s}(t,0) + X^j(\gamma(t))\Gamma^i_{jm}(\gamma(t))X^m_\ell\right\}. \tag{79}$$

So far we got (by (79))

$$\omega^*(\mathbf{S})_{(t,0)}\cdot\mu^*(\mathbf{T})_{(t,0)} = Y^\ell_k \frac{dx^k}{dt}\left\{\frac{\partial f^i_\ell}{\partial s}(t,0) + X^j(\gamma(t))\Gamma^i_{jm}(\gamma(t))X^m_\ell\right\} f(t,0)^{-1}\frac{\partial}{\partial x^i}\bigg|_{\gamma(t)}.$$

Let us observe that

$$\frac{\partial f^i_j}{\partial s}(t,0) = \frac{\partial X^i_j}{\partial x^k}(\gamma(t))\frac{\partial a^k}{\partial s}(t,0) = \frac{\partial X^i_j}{\partial x^k}(\gamma(t))X^k(\gamma(t))$$

hence

$$\frac{\partial f^i_j}{\partial s}(t,0) + X^k(\gamma(t))\Gamma^i_{k\ell}(\gamma(t))X^\ell_j = (\nabla_X X_j)^i_{\gamma(t)}$$

and we may conclude that

$$\omega^*(\mathbf{S})_{(t,0)}\cdot\mu^*(\mathbf{T})_{(t,0)} = Y^j_k\frac{dx^k}{dt} f(t,0)^{-1}(\nabla_X X_j)_{\gamma(t)} = 0. \tag{80}$$

Indeed

$$\begin{aligned}(\nabla_X X_i)_{\gamma(t)} = (\nabla_{\dot\sigma_t}X_i)_{\sigma_t(0)} &= \lim_{s\to 0}\frac{1}{s}\{\tau^s X_{i,\sigma_t(s)} - X_{i,\sigma_t(0)}\}\\ &= \lim_{s\to 0}\frac{1}{s}\{\tau^s f(t,s)e_i - f(t,0)e_i\} = 0\end{aligned}$$

because f_t is horizontal (yielding $\tau^s f(t,s) = f(t,0)$). By (69)–(70), (75)–(76) and (80) the identity (67) may be written

$$\begin{aligned}\frac{d^2}{ds^2}\{L(\gamma^s)\}_{s=0} = \frac{1}{r}\int_a^b \Big\{&|\nabla_{\dot\gamma}X^\perp|^2 - g_\theta(R(X^\perp,\dot\gamma)\dot\gamma, X^\perp)\\ &+ 2g_\theta(T_\nabla(X^\perp,\dot\gamma),\nabla_{\dot\gamma}X) + |T_\nabla(X^\perp,\dot\gamma)|^2\\ &-\frac{1}{r^2}[g_\theta(T_\nabla(X^\perp,\dot\gamma),\dot\gamma)^2 + 2g_\theta(\nabla_{\dot\gamma}X,\dot\gamma)g_\theta(T_\nabla(X^\perp,\dot\gamma),\dot\gamma)]\Big\}\, dt\end{aligned}$$

or (by $T_\nabla(X^\perp,\dot\gamma) = -2\Omega(X^\perp,\dot\gamma)T$ and $\theta(\dot\gamma) = 0$)

$$I(X,X) = \frac{1}{r}\int_a^b \{|\nabla_{\dot\gamma} X^\perp|^2 - g_\theta(R(X^\perp,\dot\gamma)\dot\gamma, X^\perp) + 4\Omega(X^\perp,\dot\gamma)^2 - 4\Omega(X^\perp,\dot\gamma)\theta(\nabla_{\dot\gamma} X)\}\,dt. \tag{81}$$

Finally, by polarization $I(X,Y) = \frac{1}{2}\{I(X+Y,X+Y) - I(X,X) - I(Y,Y)\}$ the identity (81) leads to (61).

Proof of Theorem 12. As $\nabla g_\theta = 0$

$$\begin{aligned}
&\int_{t_j}^{t_{j+1}} \{g_\theta(\nabla_{\dot\gamma} X^\perp, \nabla_{\dot\gamma} Y^\perp) - 2\Omega(X^\perp,\dot\gamma)\theta(\nabla_{\dot\gamma} Y^\perp)\}\,dt \\
&= \int_{t_j}^{t_{j+1}} \left\{\frac{d}{dt}[g_\theta(\nabla_{\dot\gamma} X^\perp, Y^\perp) - 2\Omega(X^\perp,\dot\gamma)\theta(Y^\perp)] \right. \\
&\qquad \left. - g_\theta(\nabla_{\dot\gamma}^2 X^\perp - 2\Omega(\nabla_{\dot\gamma} X^\perp,\dot\gamma)T, Y^\perp)\right\} dt \\
&= g_{\theta,\gamma(t_{j+1})}((\nabla_{\dot\gamma} X^\perp)^-_{\gamma(t_{j+1})}, Y^\perp_{\gamma(t_{j+1})}) - g_{\theta,\gamma(t_j)}((\nabla_{\dot\gamma} X^\perp)^+_{\gamma(t_j)}, Y^\perp_{\gamma(t_j)}) \\
&\quad - 2\Omega(X^\perp,\dot\gamma)_{\gamma(t_{j+1})}\theta(Y^\perp)_{\gamma(t_{j+1})} + 2\Omega(X^\perp,\dot\gamma)_{\gamma(t_j)}\theta(Y^\perp)_{\gamma(t_j)} \\
&\quad - \int_{t_j}^{t_{j+1}} \{g_\theta(\nabla_{\dot\gamma}^2 X^\perp - 2\Omega(\nabla_{\dot\gamma} X^\perp,\dot\gamma)T, Y^\perp)\}\,dt
\end{aligned}$$

and (61) implies (62). Q.e.d.

Proof of Corollary 6. If $X^\perp \in J_\gamma$ then $X^\perp$ is differentiable in $[a,b]$ hence the last term in (62) vanishes. Also $\mathscr{J}_\gamma X^\perp = 0$ and (62) yield

$$I(X,Y) = -\frac{2}{r}\int_a^b \{\theta(\nabla_{\dot\gamma} X^\perp) - 2\Omega(X^\perp,\dot\gamma)\}\Omega(Y^\perp,\dot\gamma)\,dt$$

which implies (by Lemma 4) both (63)–(64). Viceversa, let us assume that (63) holds for some $\alpha(X) \in \mathbf{R}$. Let f be a smooth function on M such that $f(\gamma(t_j)) = 0$ for any $0 \le j \le h$ and $f(\gamma(t)) > 0$ for any $t \in [a,b]\backslash\{t_0, t_1, \dots, t_h\}$ and let us consider the vector field $Y = f\mathscr{J}_\gamma X^\perp$. As $\mathscr{J}_\gamma X^\perp$ is orthogonal to $\dot\gamma$ the identity (64) implies

$$\int_a^b f(\gamma(t))|\mathscr{J}_\gamma X^\perp|^2\,dt = 0$$

hence $\mathscr{J}_\gamma X^\perp = 0$ in each interval $[t_j, t_{j+1}]$. To prove that $X^\perp \in J_\gamma$ it suffices (by Prop. 1.1 in [17], Vol. II, p. 63) to check that $X^\perp$ is of class C^1 at each t_j. To this end, for each fixed j we consider a vector field Y along γ such that

$$Y_{j,\gamma(t)} = \begin{cases} (\nabla_{\dot\gamma} X^\perp)^-_{\gamma(t_j)} - (\nabla_{\dot\gamma} X^\perp)^+_{\gamma(t_j)}, & \text{for } t = t_j \\ 0, & \text{for } t = t_k,\ k \neq j. \end{cases}$$

Then (by (64)) $|(\nabla_{\dot\gamma} X^\perp)^-_{\gamma(t_j)} - (\nabla_{\dot\gamma} X^\perp)^+_{\gamma(t_j)}|^2 = 0$. Q.e.d.

Remark. Let $\gamma(t) \in M$ be a lengthy C^1 curve. Then $D_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma - A(\dot\gamma, \dot\gamma)T$ hence on a Sasakian manifold γ is a geodesic of ∇ if and only if γ is a geodesic of the Riemannian manifold (M, g_θ). This observation leads to the following alternative proof of Theorem 11. Let $\gamma \in \Gamma$ and $X, Y \in T_\gamma(\Gamma)$ be as in Theorem 11. By Theorem 5.4 in [17], Vol. II, p. 81, we have

$$I(X, Y) = \frac{1}{r}\int_a^b \{g_\theta(D_{\dot\gamma} X^\perp, D_{\dot\gamma} Y^\perp) - g_\theta(R^D(X^\perp, \dot\gamma)\dot\gamma, Y^\perp)\}\, dt. \tag{82}$$

Now on one hand

$$D_{\dot\gamma} X^\perp = \nabla_{\dot\gamma} X^\perp + \Omega(\dot\gamma, X^\perp)T + \theta(X^\perp)J\dot\gamma \tag{83}$$

and on the other the identity

$$\begin{aligned} R^D(X, Y)Z = {} & R(X, Y)Z + (JX \wedge JY)Z - 2\Omega(X, Y)JZ \\ & + 2g_\theta((\theta \wedge I)(X, Y), Z)T - 2\theta(Z)(\theta \wedge I)(X, Y), \quad X, Y, Z \in \mathscr{X}(M), \end{aligned}$$

yields

$$R^D(X^\perp, \dot\gamma)\dot\gamma = R(X^\perp, \dot\gamma)\dot\gamma - 3\Omega(X^\perp, \dot\gamma)J\dot\gamma + r^2\theta(X^\perp)T. \tag{84}$$

Let us substitute from (83)–(84) into (82) and use the identity

$$\begin{aligned} & \theta(X^\perp)\Omega(\nabla_{\dot\gamma} Y^\perp, \dot\gamma) + \theta(Y^\perp)\Omega(\nabla_{\dot\gamma} X^\perp, \dot\gamma) + \Omega(\dot\gamma, X^\perp)\theta(\nabla_{\dot\gamma} Y^\perp) + \Omega(\dot\gamma, Y^\perp)\theta(\nabla_{\dot\gamma} X^\perp) \\ & \quad = \frac{d}{dt}\{\theta(X^\perp)\Omega(Y^\perp, \dot\gamma) + \theta(Y^\perp)\Omega(X^\perp, \dot\gamma)\} \\ & \qquad - 2\{\Omega(X^\perp, \dot\gamma)\theta(\nabla_{\dot\gamma} Y^\perp) + \Omega(Y^\perp, \dot\gamma)\theta(\nabla_{\dot\gamma} X^\perp)\} \end{aligned}$$

(together with $X_{\gamma(a)} = X_{\gamma(b)} = 0$) so that to derive (61). Q.e.d.

As an application of Theorems 8 and 11 we shall establish

THEOREM 13. *Let (M, θ) be a Sasakian manifold of CR dimension n and ∇ its Tanaka-Webster connection. Let $\gamma : [a, b] \to M$ be a lengthy geodesic of ∇, parametrized by arc length. If there is $c \in (a, b)$ such that the point $\gamma(c)$ is horizontally conjugate to $\gamma(a)$ and for any $\delta > 0$ with $[c - \delta, c + \delta] \subset (a, b)$ the space $\mathscr{H}_{\gamma_\delta}$ has maximal dimension $4n$ (where γ_δ is the geodesic $\gamma : [c - \delta, c + \delta] \to M$) then γ is not a minimizing geodesic joining $\gamma(a)$ and $\gamma(b)$, that is the length of γ is greater than the Riemannian distance (associated to (M, g_θ)) between $\gamma(a)$ and $\gamma(b)$.*

Proof. Let $\gamma : [a, b] \to M$ be a geodesic of the Tanaka-Webster connection of the Sasakian manifold (M, θ), obeying to the assumptions in Theorem 13.

Then (by Theorem 8) there is a piecewise differentiable vector field X along γ such that 1) X is orthogonal to $\dot{\gamma}$ and $J\dot{\gamma}$, 2) $X_{\gamma(a)} = X_{\gamma(b)} = 0$, and 3) $I_a^b(X) < 0$. Let $\{\gamma^s\}_{|s|<\varepsilon}$ be a 1-parameter family of curves as in the definition of $(d_\gamma L)X$ and $I(X, X)$. By Corollary 5 (as γ is a geodesic of ∇) one has

$$\frac{d}{ds}\{L(\gamma^s)\}_{s=0} = 0.$$

On the other hand (by Theorem 11 and $X^\perp = X$)

$$I(X, X) = I_a^b(X) + 4\int_a^b \Omega(X, \dot{\gamma})\{\Omega(X, \dot{\gamma}) - \theta(X')\}\, dt$$

hence (as X is orthogonal to $J\dot{\gamma}$)

$$\frac{d^2}{ds^2}\{L(\gamma^s)\}_{s=0} = I_a^b(X) < 0$$

so that there is $0 < \delta < \varepsilon$ such that $L(\gamma^s) < L(\gamma)$ for any $|s| < \delta$.

Remark. If there is a 1-parameter variation of γ (inducing X) by *lengthy* curves then $L(\gamma)$ is greater than the Carnot-Carathédory distance between $\gamma(a)$ and $\gamma(b)$.

9. Final comments and open problems

Manifest in R. Strichartz's paper (cf. [23]) is the absence of covariant derivatives and curvature. Motivated by our Theorem 1 we started developing a theory of geodesics of the Tanaka-Webster connection ∇ on a Sasakian manifold M, with the hope that although lengthy geodesics of ∇ form (according to Corollary 1) a smaller family than that of sub-Riemannian geodesics, the former may suffice for establishing an analog to Theorem 7.1 in [23], under the assumption that ∇ is complete (as a linear connection on M). The advantage of working within the theory of linear connections is already quite obvious (e.g. any C^1 geodesic of ∇ is automatically of class C^∞, as an integral curve of some C^∞ basic vector field, while sub-Riemannian geodesics are assumed to be of class C^2, cf. [23], p. 233, and no further regularity is to be expected *a priori*) and doesn't contradict R. Strichartz's observation that sub-Riemannian manifolds, and in particular strictly pseudoconvex CR manifolds endowed with a contact form θ, exhibit no approximate Euclidean behavior (cf. [23], p. 223). Indeed, while Riemannian curvature measures the higher order deviation of the given Riemannian manifold from the Euclidean model, the curvature of the Tanaka-Webster connection describes the pseudoconvexity properties of the given CR manifold, as understood in several complex variables analysis. The role as a possible model space played by the *tangent cone* of the metric space (M, ρ) at a point $x \in M$ (such as produced by J. Mitchell's Theorem 1 in [21], p. 36) is unclear.

Another advantage of our approach stems from the fact that the exponential map on M thought of as a sub-Riemannian manifold is never a diffeomorphism at the origin (because all sub-Riemannian geodesics issuing at $x \in M$ must have tangent vectors in $H(M)_x$) in contrast with the ordinary exponential map associated to the Tanaka-Webster connection ∇. In particular *cut points* (as introduced in [23], p. 260) do not possess the properties enjoyed by conjugate points in Riemannian geometry because (by Theorem 11.3 in [23], p. 260) given $x \in M$ cut points occur arbitrary close to x. On the contrary (by Theorem 1.4 in [17], Vol. II, p. 67) given $x \in M$ one may speak about the *first* point conjugate to x along a geodesic of ∇ emanating from x, therefore the concept of conjugate locus $C(x)$ may be defined in the usual way (cf. e.g. [20], p. 117). The systematic study of the properties of $C(x)$ on a strictly pseudoconvex CR manifold is an open problem.

Yet another concept of exponential map was introduced by D. Jerison & J. M. Lee, [15] (associated to *parabolic geodesics* i.e. solutions $\gamma(t)$ to $(\nabla_{\dot\gamma}\dot\gamma)_{\gamma(t)} = 2cT_{\gamma(t)}$ for some $c \in \mathbf{R}$). A comparison between the three exponential formalisms (in [23], [15], and the present paper) hasn't been done as yet. We conjecture that given a 2-plane $\sigma \subset T_x(M)$ its pseudohermitian sectional curvature $k_\theta(\sigma)$ measures the difference between the length of a circle in σ (with respect to $g_{\theta,x}$) and the length of its image by $\exp_x$ (the exponential mapping at x associated to ∇). Also a useful relationship among $\exp_x$ and the exponential mapping associated to the Fefferman metric F_θ on $C(M)$ should exist (and then an understanding of the singular points of the latter, cf. e.g. M. A. Javaloyes & P. Piccione, [14], should shed light on the properties of singular points of the former).

Finally, the analogy between Theorem 7.3 in [23], p. 245 (producing "approximations to unity" on Carnot-Carathéodory complete sub-Riemannian manifolds) and Lemma 2.2 in [24], p. 50 (itself a corrected version of a result by S.-T. Yau, [28]) indicates that Theorem 7.3 is the proper ingredient for proving that the sublaplacian Δ_b is essentially self-adjoint on $C_0^\infty(M)$ and the corresponding heat operator is given by a positive C^∞ kernel. These matters are relegated to a further paper.

Appendix A. Contact forms of constant pseudohermitian sectional curvature

The scope of this section is to give a proof of Theorem 5. Let (M,θ) be a nondegenerate CR manifold and θ a contact form on M. Let ∇ be the Tanaka-Webster connection of (M,θ). We recall the first Bianchi identity

$$\sum_{XYZ} R(X,Y)Z = \sum_{XYZ} \{T_\nabla(T_\nabla(X,Y),Z) + (\nabla_X T_\nabla)(Y,Z)\} \tag{85}$$

for any $X, Y, Z \in T(M)$, where $\sum_{XYZ}$ denotes the cyclic sum over X, Y, Z. Let $X, Y, Z \in H(M)$ and note that

$$T_\nabla(T_\nabla(X,Y),Z) = -2\Omega(X,Y)\tau(Z),$$
$$(\nabla_X T_\nabla)(Y,Z) = -2(\nabla_X\Omega)(Y,Z)T = 0.$$

Indeed $\nabla g_\theta = 0$ and $\nabla J = 0$ yield $\nabla\Omega = 0$. Thus (85) leads to

$$\sum_{XYZ} R(X,Y)Z = -2\sum_{XYZ} \Omega(X,Y)\tau(Z), \tag{86}$$

for any $X, Y, Z \in H(M)$. Let us define a $(1,2)$-tensor field S by setting $S(X,Y) = (\nabla_X \tau)Y - (\nabla_Y \tau)X$. Next, we set $X, Y \in H(M)$ and $Z = T$ in (85) and observe that

$$\begin{aligned} &T_\nabla(T_\nabla(X,Y),T) + T_\nabla(T_\nabla(Y,T),X) + T_\nabla(T_\nabla(T,X),Y) \\ &\quad = -T_\nabla(\tau(Y),X) + T_\nabla(\tau(X),Y) = (\text{as } \tau \text{ is H(M)} - \text{valued}) \\ &\quad = 2\{\Omega(\tau(Y),X) - \Omega(\tau(X),Y)\}T = 2g_\theta((\tau J + J\tau)X,Y)T = 0, \end{aligned}$$

(by the purity axiom) and

$$\begin{aligned} &(\nabla_X T_\nabla)(Y,T) + (\nabla_Y T_\nabla)(T,X) + (\nabla_T T_\nabla)(X,Y) \\ &\quad = -(\nabla_X \tau)Y + (\nabla_Y \tau)X - 2(\nabla_T \Omega)(X,Y)T = -S(X,Y). \end{aligned}$$

Finally (85) becomes

$$R(X,T)Y + R(T,Y)X = S(X,Y), \tag{87}$$

for any $X, Y \in H(M)$. The 4-tensor R enjoys the properties

$$R(X,Y,Z,W) = -R(Y,X,Z,W), \tag{88}$$

$$R(X,Y,Z,W) = -R(X,Y,W,Z), \tag{89}$$

for any $X, Y, Z, W \in T(M)$. Indeed (88) follows from $\nabla g_\theta = 0$ while (89) is obvious. We may use the reformulation (86)–(87) of the first Bianchi identity to compute $\sum_{YZW} R(X,Y,Z,W)$ for arbitrary vector fields. For any $X \in T(M)$ we set $X_H = X - \theta(X)T$ (so that $X_H \in H(M)$). Then

$$\begin{aligned} \sum_{YZW} R(X,Y,Z,W) &= \sum_{YZW} g_\theta(R(Z,W)Y_H, X) \\ &= \sum_{YZW} g_\theta(R(Z_H,W_H)Y_H + \theta(Y) \\ &\qquad \times [R(W_H,T)Z_H + R(T,Z_H)W_H], X) \end{aligned}$$

hence

$$\sum_{YZW} R(X,Y,Z,W) = -\sum_{YZW} \{2\Omega(Y,Z)A(W,X) + \theta(Y)g_\theta(X, S(Z_H,W_H))\} \tag{90}$$

for any $X, Y, Z, W \in T(M)$. Next, we set

$$K(X,Y,Z,W) = \sum_{YZW} R(X,Y,Z,W)$$

and compute (by (88)–(89))

$$K(X, Y, Z, W) - K(Y, Z, W, X) - K(Z, W, X, Y) + K(W, X, Y, Z)$$
$$= 2R(X, Y, Z, W) - 2R(Z, W, X, Y)$$

hence (by (90))

$$\begin{aligned}2R(X, Y, Z, W) &- 2R(Z, W, X, Y)\\ &= -\sum_{YZW}\{2\Omega(Y, Z)A(X, W) + \theta(Y)g_\theta(X, S(Z_H, W_H))\}\\ &\quad + \sum_{ZWX}\{2\Omega(Z, W)A(Y, X) + \theta(Z)g_\theta(Y, S(W_H, X_H))\}\\ &\quad + \sum_{WXY}\{2\Omega(W, X)A(Y, Z) + \theta(W)g_\theta(Z, S(X_H, Y_H))\}\\ &\quad - \sum_{XYZ}\{2\Omega(X, Y)A(Z, W) + \theta(X)g_\theta(W, S(Y_H, Z_H))\}\end{aligned}$$

or

$$\begin{aligned}(91)\quad 2R(X, Y, Z, W) &- 2R(Z, W, X, Y)\\ &= -4\Omega(Y, Z)A(X, W) + 4\Omega(Y, W)A(X, Z)\\ &\quad - 4\Omega(X, W)A(Y, Z) + 4\Omega(X, Z)A(Y, W)\\ &\quad + \theta(X)[g_\theta(Y, S(Z_H, W_H)) + g_\theta(Z, S(Y_H, W_H)) - g_\theta(W, S(Y_H, Z_H))]\\ &\quad + \theta(Y)[g_\theta(Z, S(W_H, X_H)) - g_\theta(W, S(Z_H, X_H)) - g_\theta(X, S(Z_H, W_H))]\\ &\quad + \theta(Z)[g_\theta(Y, S(W_H, X_H)) - g_\theta(X, S(W_H, Y_H)) - g_\theta(W, S(X_H, Y_H))]\\ &\quad + \theta(W)[g_\theta(Y, S(X_H, Z_H)) - g_\theta(X, S(Y_H, Z_H)) + g_\theta(Z, S(X_H, Y_H))].\end{aligned}$$

As $\nabla_X \tau$ is symmetric one has

$$g_\theta(Y, S(X, Z)) - g_\theta(X, S(Y, Z)) = g_\theta(S(X, Y), Z)$$

for any $X, Y, Z \in H(M)$, so that (91) may be written

$$\begin{aligned}(92)\quad R(X, Y, Z, W) = R(Z, W, X, Y) &- 2\Omega(Y, Z)A(X, W)\\ &+ 2\Omega(Y, W)A(X, Z) - 2\Omega(X, W)A(Y, Z)\\ &+ 2\Omega(X, Z)A(Y, W) + \theta(X)g_\theta(S(Z_H, W_H), Y)\\ &+ \theta(Y)g_\theta(S(W_H, Z_H), X) + \theta(Z)g_\theta(S(Y_H, X_H), W)\\ &+ \theta(W)g_\theta(S(X_H, Y_H), Z),\end{aligned}$$

for any $X, Y, Z, W \in T(M)$.

The properties (88)–(90) and (92) may be used to compute the full curvature of a manifold of constant pseudohermitian sectional curvature (the arguments are similar to those in the proof of Prop. 1.2 in [17], Vol. I, p. 198). Assume from now on that M is strictly pseudoconvex and G_θ positive definite. Let us set

$$R_1(X, Y, Z, W) = g_\theta(X, Z)g_\theta(Y, W) - g_\theta(Y, Z)g_\theta(W, X)$$

so that

$$R_1(X, Y, Z, W) = -R_1(Y, X, Z, W), \tag{93}$$

$$R_1(X, Y, Z, W) = -R_1(X, Y, W, Z), \tag{94}$$

$$\sum_{YZW} R_1(X, Y, Z, W) = 0. \tag{95}$$

Assume from now on that $k_\theta = c = \text{const.}$ Let us set $L = R - 4cR_1$ and observe that

$$L(X, Y, X, Y) = 0 \tag{96}$$

for any $X, Y \in T(M)$. Indeed, if X, Y are linearly dependent then (96) follows from the skew symmetry of L in the pairs (X, Y) and (Z, W), respectively. If X, Y are independent then let $\sigma \subset T_x(M)$ be the 2-plane spanned by $\{X_x, Y_x\}$, $x \in M$. Then

$$\begin{aligned} L(X, Y, X, Y)_x &= R(X, Y, X, Y)_x - 4cR_1(X, Y, X, Y)_x \\ &= 4k_\theta(\sigma)[|X|^2|Y|^2 - g_\theta(X, Y)^2]_x - 4cR_1(X, Y, X, Y)_x = 0. \end{aligned}$$

Next (by (96))

$$0 = L(X, Y + W, X, Y + W) = L(X, Y, X, W) + L(X, W, X, Y)$$

i.e.

$$L(X, Y, X, W) = -L(X, W, X, Y) \tag{97}$$

for any $X, Y, W \in T(M)$. As well known (cf. e.g. Prop. 1.1 in [18], Vol. I, p. 198) the properties (93)–(95) imply as well the symmetry property

$$R_1(X, Y, Z, W) = R_1(Z, W, X, Y). \tag{98}$$

Therefore $L(X, Y, Z, W) - L(Z, W, X, Y) = R(X, Y, Z, W) - R(Z, W, X, Y)$ hence (by (92))

$$\begin{aligned} L(X, Y, Z, W) = L(Z, W, X, Y) &+ 2\Omega(Y, W)A(X, Z) \\ &- 2\Omega(Y, Z)A(X, W) + 2\Omega(X, Z)A(Y, W) \\ &- 2\Omega(X, W)A(Y, Z) + \theta(X)g_\theta(S(Z_H, W_H), Y) \\ &+ \theta(Y)g_\theta(S(W_H, Z_H), X) + \theta(Z)g_\theta(S(Y_H, X_H), W) \\ &+ \theta(W)g_\theta(S(X_H, Y_H), Z). \end{aligned} \tag{99}$$

Applying (99) (to interchange the pairs (X, W) and (X, Y)) we get

$$\begin{aligned} L(X, W, X, Y) = L(X, Y, X, W) &+ 2\Omega(W, Y)A(X, X) \\ &- 2\Omega(W, X)A(X, Y) - 2\Omega(X, Y)A(W, X) \\ &+ \theta(X)g_\theta(S(W_H, Y_H), X) + \theta(Y)g_\theta(S(X_H, W_H), X) \\ &+ \theta(W)g_\theta(S(Y_H, X_H), X) \end{aligned}$$

hence (97) may be written

$$\begin{aligned} L(X, Y, X, W) = \Omega(W, X)A(X, Y) &\\ &+ \Omega(X, Y)A(W, X) - \Omega(W, Y)A(X, X) \\ &- \frac{1}{2}\{\theta(X)g_\theta(S(W_H, Y_H), X) + \theta(Y)g_\theta(S(X_H, W_H), X) \\ &\quad + \theta(W)g_\theta(S(Y_H, X_H), X)\}. \end{aligned} \tag{100}$$

Consequently

$$\begin{aligned} L(X+Z, Y, X+Z, W) = \Omega(W, X+Z)A(X+Z, Y) &\\ &+ \Omega(X+Z, Y)A(W, X+Z) \\ &- \Omega(W, Y)A(X+Z, X+Z) \\ &- \frac{1}{2}g_\theta(X+Z, \theta(X+Z)S(W_H, Y_H) \\ &+ \theta(Y)S(X_H+Z_H, W_H) + \theta(W)S(Y_H, X_H+Z_H)) \end{aligned}$$

or (using (100) to calculate $L(X, Y, X, W)$ and $L(Z, Y, Z, W)$)

$$\begin{aligned} &L(X, Y, Z, W) + L(Z, Y, X, W) \\ &= \Omega(X, Y)A(W, Z) + \Omega(W, X)A(Z, Y) + \Omega(W, Z)A(X, Y) \\ &\quad + \Omega(Z, Y)A(W, X) - 2\Omega(W, Y)A(X, Z) \\ &\quad - \frac{1}{2}g_\theta(X, \theta(Z)S(W_H, Y_H) + \theta(Y)S(Z_H, W_H) + \theta(W)S(Y_H, Z_H)) \\ &\quad - \frac{1}{2}g_\theta(Z, \theta(X)S(W_H, Y_H) + \theta(Y)S(X_H, W_H) + \theta(W)S(Y_H, X_H)). \end{aligned} \tag{101}$$

On the other hand, by the skew symmetry of L in the first pair of arguments and by (99) (used to interchange the pairs (Y, Z) and (X, W))

$$\begin{aligned} L(Z, Y, X, W) = -L(Y, Z, X, W) = -L(X, W, Y, Z) &\\ &+ 2\Omega(Z, X)A(Y, W) - 2\Omega(Z, W)A(Y, X) \\ &+ 2\Omega(Y, W)A(Z, X) - 2\Omega(Y, X)A(Z, W) \\ &- \theta(Y)g_\theta(S(X_H, W_H), Z) - \theta(Z)g_\theta(S(W_H, X_H), Y) \\ &- \theta(X)g_\theta(S(Z_H, Y_H), W) - \theta(W)g_\theta(S(Y_H, Z_H), X) \end{aligned}$$

so that (101) becomes

$$
\begin{aligned}
L(X, Y, Z, W) = {} & L(X, W, Y, Z) + 2\Omega(X, Z)A(Y, W) \\
& - \Omega(W, Z)A(X, Y) - \Omega(X, Y)A(Z, W) \\
& + \Omega(W, X)A(Z, Y) + \Omega(Z, Y)A(W, X) \\
& + \frac{1}{2}\theta(X)\{g_\theta(S(Z_H, Y_H), W) + g_\theta(S(Z_H, W_H), Y)\} \\
& - \frac{1}{2}\theta(Y)\{g_\theta(S(Z_H, W_H), X) + g_\theta(S(W_H, X_H), Z)\} \\
& + \frac{1}{2}\theta(Z)\{g_\theta(S(W_H, X_H), Y) + g_\theta(S(Y_H, X_H), W)\} \\
& - \frac{1}{2}\theta(W)\{g_\theta(S(Y_H, X_H), Z) + g_\theta(S(Z_H, Y_H), X)\}.
\end{aligned} \tag{102}
$$

By cyclic permutation of the variables Y, Z, W in (102) we obtain another identity of the sort

$$
\begin{aligned}
L(X, Y, Z, W) = {} & L(X, Z, W, Y) - 2\Omega(X, W)A(Z, Y) \\
& + \Omega(Y, W)A(X, Z) + \Omega(X, Z)A(W, Y) \\
& - \Omega(Y, X)A(W, Z) - \Omega(W, Z)A(Y, X) \\
& - \frac{1}{2}\theta(X)\{g_\theta(S(W_H, Z_H), Y) + g_\theta(S(W_H, Y_H), Z)\} \\
& + \frac{1}{2}\theta(Y)\{g_\theta(S(Z_H, X_H), W) + g_\theta(S(W_H, Z_H), X)\} \\
& - \frac{1}{2}\theta(Z)\{g_\theta(S(W_H, Y_H), X) + g_\theta(S(Y_H, X_H), W)\} \\
& - \frac{1}{2}\theta(W)\{g_\theta(S(Y_H, X_H), Z) + g_\theta(S(Z_H, X_H), Y)\}
\end{aligned}
$$

which together with (102) leads to

$$
\begin{aligned}
3L(X, Y, Z, W) = {} & \sum_{YZW} L(X, Y, Z, W) - 2\Omega(W, Z)A(X, Y) \\
& + 3\Omega(X, Z)A(Y, W) - 3\Omega(X, W)A(Y, Z) \\
& + \Omega(Z, Y)A(W, X) + \Omega(Y, W)A(X, Z) \\
& + \frac{3}{2}\theta(X)g_\theta(S(Z_H, W_H), Y) - \frac{1}{2}\theta(Y)g_\theta(S(Z_H, W_H), X) \\
& + \frac{1}{2}\theta(Z)\{2g_\theta(S(Y_H, X_H), W) \\
& \qquad + g_\theta(S(W_H, X_H), Y) + g_\theta(S(W_H, Y_H), X)\}
\end{aligned}
$$

$$-\frac{1}{2}\theta(W)\{2g_\theta(S(Y_H, X_H), Z) + g_\theta(S(Z_H, Y_H), X) + g_\theta(S(Z_H, X_H), Y)\}$$

or

$$\begin{aligned} L(X, Y, Z, W) &= \Omega(Y, W)A(X, Z) - \Omega(Y, Z)A(X, W) \\ &+ \Omega(X, Z)A(Y, W) - \Omega(X, W)A(Y, Z) \\ &+ \frac{1}{2}\{\theta(X)g_\theta(S(Z_H, W_H), Y) - \theta(Y)g_\theta(S(Z_H, W_H), X) \\ &\quad + \theta(Z)g_\theta(S(Y_H, X_H), W) - \theta(W)g_\theta(S(Y_H, X_H, Z))\} \end{aligned}$$

or

$$\begin{aligned} R(X, Y, Z, W) &= 4c\{g_\theta(X, Z)g_\theta(Y, W) - g_\theta(Y, Z)g_\theta(X, W)\} \\ &+ \Omega(Y, W)A(X, Z) - \Omega(Y, Z)A(X, W) \\ &+ \Omega(X, Z)A(Y, W) - \Omega(X, W)A(Y, Z) \\ &+ g_\theta(S(Z_H, W_H), (\theta \wedge I)(X, Y)) \\ &- g_\theta(S(X_H, Y_H), (\theta \wedge I)(Z, W)) \end{aligned} \tag{103}$$

for any $X, Y, Z, W \in T(M)$, where I is the identical transformation and $(\theta \wedge I)(X, Y) = \frac{1}{2}\{\theta(X)Y - \theta(Y)X\}$. Using (103) one may prove Theorem 5 as follows. Let $Y = T$ in (103). As $R(Z, W)T = 0$ and S is $H(M)$-valued we get

$$0 = 4c\{g_\theta(X, Z)\theta(W) - g_\theta(X, W)\theta(Z)\} - \frac{1}{2}g_\theta(S(Z_H, W_H), X), \tag{104}$$

for any $X, Z, W \in T(M)$. In particular for $Z, W \in H(M)$

$$S(Z, W) = 0.$$

Hence $S(Z_H, W_H) = 0$ and (104) becomes

$$c\{g_\theta(X, Z)\theta(W) - g_\theta(X, W)\theta(Z)\} = 0,$$

for any $X, Z, W \in T(M)$. In particular for $Z = X \in H(M)$ and $W = T$ one has $c|X|^2 = 0$ hence $c = 0$ and (103) leads to (31). Then $\tau = 0$ yields $R = 0$. To prove the last statement in Theorem 5 let us assume that M has CR dimension $n \geq 2$ (so that the Levi distribution has rank > 3). Assume that $R = 0$ i.e.

$$\Omega(X, Z)\tau(Y) - \Omega(Y, Z)\tau(X) = A(X, Z)JY - A(Y, Z)JX$$

(by (31)). In particular for $Z = Y$

$$\Omega(X, Y)\tau(Y) = A(X, Y)JY - A(Y, Y)JX. \tag{105}$$

Let $X \in H(M)$ such that $|X| = 1$, $g_\theta(X, Y) = 0$ and $g_\theta(X, JY) = 0$. Taking the inner product of (105) with JX gives $A(Y, Y) = 0$, hence $A = 0$ (as A is symmetric). Q.e.d.

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