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STUDY OF SOME SUBCLASSES OF UNIVALENT FUNCTIONS AND THEIR RADIUS PROPERTIES

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Abstract

An analytic function $f(z) = z + a_2 z^2 + \cdots$ in the unit disk $\Delta = \{z : |z| < 1\}$ is said to be in $\mathcal{U}(\lambda, \mu)$ if

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1} - 1 \right| \le \lambda \quad (|z| < 1)$$

for some $\lambda \ge 0$ and $\mu > -1$. For $-1 \le \alpha \le 1$, we introduce a geometrically motivated $\mathcal{S}_p(\alpha)$-class defined by

$$\mathcal{S}_p(\alpha) = \left\{ f \in \mathcal{S} : \left| \frac{zf'(z)}{f(z)} - 1 \right| \le \operatorname{Re} \frac{zf'(z)}{f(z)} - \alpha, z \in \Delta \right\},$$

where $\mathcal{S}$ represents the class of all normalized univalent functions in Δ. In this paper, the authors determine necessary and sufficient coefficient conditions for certain class of functions to be in $\mathcal{S}_p(\alpha)$. Also, radius properties are considered for $\mathcal{S}_p(\alpha)$-class in the class $\mathcal{S}$. In addition, we also find disks $|z| < r := r(\lambda, \mu)$ for which $\frac{1}{r} f(rz) \in \mathcal{U}(\lambda, \mu)$ whenever $f \in \mathcal{S}$. In addition to a number of new results, we also present several new sufficient conditions for f to be in the class $\mathcal{U}(\lambda, \mu)$.

1. Introduction and preliminaries

Denote by $\mathcal{A}$ the class of all functions f, normalized by $f(0) = 0 = f'(0) - 1$, that are analytic in the unit disk $\Delta = \{z \in \mathbf{C} : |z| < 1\}$, and by $\mathcal{S}$ the class of univalent functions $f \in \mathcal{A}$. Denote by $\mathcal{S}^*$ the subclass consisting of functions f in $\mathcal{S}$ that are starlike (with respect to origin), i.e. $tw \in f(\Delta)$ whenever $t \in [0,1]$ and $w \in f(\Delta)$. Analytically, $f \in \mathcal{S}^*$ if and only if $\operatorname{Re}(zf'(z)/f(z)) \ge 0$ in Δ. A simple generalization of $\mathcal{S}^*$ is the so-called class of all starlike functions of order α, $0 \le \alpha \le 1$, denoted by $\mathcal{S}^*(\alpha)$. Indeed, $f \in \mathcal{S}^*(\alpha)$ if and only

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if $\mathrm{Re}(zf'(z)/f(z)) \geq \alpha$ in Δ. We set $\mathcal{S}^*(0) = \mathcal{S}^*$. A function $f \in \mathcal{A}$ is said to be in $\mathcal{U}(\lambda, \mu)$ if

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} - 1 \right| \leq \lambda \quad (|z| < 1)$$

for some $\lambda \geq 0$ and $\mu > -1$. We set $\mathcal{U}(\lambda, 1) = \mathcal{U}(\lambda)$, and $\mathcal{U}(1) = \mathcal{U}$. In [10], the authors studied a subclass $\mathcal{P}(2\lambda)$ of $\mathcal{U}(\lambda)$, consisting of functions f for which

$$\left| \left(\frac{z}{f(z)}\right)'' \right| \leq 2\lambda \quad (|z| < 1).$$

We have the strict inclusion $\mathcal{P}(2) \subsetneq \mathcal{U} \subsetneq \mathcal{S}$, see [1, 7, 13]. Moreover, a close connection between the classes $\mathcal{P}(2\lambda)$ and $\mathcal{U}(\lambda)$ is given by $\mathcal{P}(2\lambda) \subset \mathcal{U}(\lambda)$, see [9, 10]. In [8, 14, 15, 16], the authors considered the problem of finding conditions on λ and μ so that each function in $\mathcal{U}(\lambda, \mu)$ is starlike or in some subsets of $\mathcal{S}$. For example, Ponnusamy and Singh [15] have shown that

$$\mathcal{U}(\lambda, \mu) \subseteq \mathcal{S}^* \quad \text{if } \mu < 0 \text{ and } 0 \leq \lambda \leq \frac{1-\mu}{\sqrt{(1-\mu)^2 + \mu^2}} =: \lambda^*(\mu)$$

and in [8], Obradović proved that the above inclusion continues to hold for $0 < \mu \leq 1$ and with the same bound for λ. The sharpness part of these results may be obtained as a consequence of results from [21]. However, it is not known whether each function f in $\mathcal{U}(1, \mu)$ (or more generally, $\mathcal{U}(\lambda, \mu)$ with $\lambda^*(\mu) < \lambda \leq 1$) is univalent in Δ for certain values of μ in the open interval $(0, 1)$. On the other hand, according to a result due to Aksentiev [1] (see also Ozaki and Nunokawa [13] for a reformulated version as given by $\mathcal{U}$), we have the inclusion $\mathcal{U}(\lambda) \subset \mathcal{S}$ for $0 \leq \lambda \leq 1$. We see that the Koebe function $z/(1-z)^2$ belongs to $\mathcal{U}$ but does not belong to $\mathcal{S}^*(\alpha)$ for any $\alpha > 0$. In fact, the bounded function $z + z^2/2$ belongs to $\mathcal{U}$ but not in $\mathcal{S}^*(\alpha)$ for any $\alpha > 0$. That is, $\mathcal{U} \not\subset \mathcal{S}^*(\alpha)$ for any $\alpha > 0$. Thus, $\mathcal{U} \subsetneq \mathcal{S}$ and the inclusion is strict as functions in $\mathcal{S}$ are not necessarily in $\mathcal{U}$. Further work on these classes, including some interesting generalizations of these classes, may be found in [9, 12, 17].

A function $f \in \mathcal{S}^*(\alpha)$ is said to be in $\mathcal{T}^*(\alpha)$ if it can be expressed as

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k.$$

Functions of this form are discussed in detail by Silverman [23] and others [24].

In this paper we shall be mainly concerned with functions $f \in \mathcal{A}$ of the form

$$\left(\frac{z}{f(z)}\right)^{\mu} = 1 + \sum_{n=1}^{\infty} b_n z^n, \quad z \in \Delta, \tag{1.1}$$

where $(z/f(z))^{\mu}$ represents principal powers. The class of functions f of this form for which $b_n \geq 0$ is especially interesting and deserves separate attention. We remark that if $f \in \mathcal{S}$ then $z/f(z)$ is nonvanishing and hence, $f \in \mathcal{S}$ may be expressed as

$$f(z) = \frac{z}{g(z)}, \quad \text{where } g(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \; z \in \Delta.$$

These two representations are convenient for our investigation. Finally, we introduce a subclass $\mathcal{S}_p(\alpha)$, $-1 \leq \alpha \leq 1$, of starlike functions in the following way [19]:

$$\mathcal{S}_p(\alpha) = \left\{ f \in \mathcal{S} : \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \operatorname{Re} \frac{zf'(z)}{f(z)} - \alpha, z \in \Delta \right\}.$$

Geometrically, $f \in \mathcal{S}_p(\alpha)$ if and only if the domain values of $zf'(z)/f(z)$, $z \in \Delta$, is the parabolic region $(\operatorname{Im} w)^2 \leq (1-\alpha)[2 \operatorname{Re} w - (1+\alpha)]$. In [19], Rønning has shown that the class $\mathcal{S}_p(\alpha)$ must contain non-univalent functions if $\alpha < -1$, and $\mathcal{S}_p(\alpha) \subset \mathcal{S}^*$ if $-1 \leq \alpha \leq 1$. We set $\mathcal{S}_p(0) = \mathcal{S}_p$. The class of uniformly convex functions was introduced by Goodman in [4] (see also [5] where Goodman studied the class of uniformly starlike functions). Later Rønning [20] studied these classes along with the class $\mathcal{S}_p$. Moreover, from the work of Rønning [20], it follows easily that $f(z) = z + a_n z^n$ is in $\mathcal{S}_p(\alpha)$ if and only if $(2n-1-\alpha)|a_n| \leq 1-\alpha$.

Let $\mathcal{F}$ and $\mathcal{G}$ be two subclasses of $\mathcal{A}$. If for every $f \in \mathcal{F}$, $r^{-1}f(rz) \in \mathcal{G}$ for $r \leq r_0$, and r_0 is the largest number for which this holds, then we say that r_0 is the $\mathcal{G}$ radius (or the radius of the property connected to $\mathcal{G}$) in $\mathcal{F}$. There are many results of this type in the theory of univalent functions. For example, the $\mathcal{S}_p$ radius in $\mathcal{S}^*$ was found by Rønning in [20] to be $1/3$. Also, $\mathcal{P}(2)$ radius in $\mathcal{U}$ has been obtained by Obradović and Ponnusamy in [11] and is given by $2/3$. At this place, it is appropriate to recall here the following result:

Theorem A [20, Theorem 4]. *If $f \in \mathcal{S}$, then $\frac{1}{r}f(rz) \in \mathcal{S}_p$ if and only if $0 < r \leq 0.33217\ldots$.*

2. Lemmas

For the proof of our results, we need the following result (see [3, Theorem 11 in p. 193 of Vol-2]) which reveals the importance of the area theorem in the theory of univalent functions.

Lemma 1. *Let $\mu > 0$ and $f \in \mathcal{S}$ be in the form (1.1). Then we have*

$$\sum_{n=1}^{\infty} (n-\mu)|b_n|^2 \leq \mu.$$

Next we recall well-known coefficient condition that is sufficient for functions to be in $\mathcal{U}(\lambda)$ or $\mathcal{P}(2\lambda)$ or $\mathcal{S}^*(\alpha)$, respectively.

LEMMA 2 [12]. *Let* $\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ *be a non-vanishing analytic function in* Δ *and* $f(z) = z/\phi(z)$. *Then if* $\sum_{n=2}^{\infty}(n-1)|b_n| \le \lambda$, *we have*

(a) $f \in \mathcal{U}(\lambda)$

(b) $f \in \mathcal{U}(\lambda) \cap \mathcal{S}^*$ *for* $0 < \lambda \le \dfrac{\sqrt{2-|b_1|^2} - |b_1|}{2} = \lambda_*(f)$;

(c) *Further, if* $\sum_{n=2}^{\infty} n(n-1)|b_n| \le 2\lambda$, *then we have* $f \in \mathcal{P}(2\lambda)$.

In [18], it was shown that if $\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ is a non-vanishing analytic function in Δ and $f(z) = z/\phi(z)$, then $f \in \mathcal{S}^*(\alpha)$, $0 \le \alpha \le 1$, whenever

$$\sum_{k=2}^{\infty}(k-1+\alpha)|b_k| \le \begin{cases} 1-\alpha-(1-\alpha)|b_1| & \text{if } 0 \le \alpha \le 1/2 \\ 1-\alpha-\alpha|b_1| & \text{if } 1/2 \le \alpha \le 1. \end{cases}$$

3. Coefficient conditions for functions in $\mathcal{S}_p(\alpha)$

THEOREM 1. *If a function* f *of the form* (1.1) *with* $b_n \ge 0$ *and* $\mu > 0$ *is in* $\mathcal{S}_p(\alpha)$, *we then have*

$$\sum_{n=1}^{\infty}(2n - \mu(1-\alpha))b_n \le \mu(1-\alpha). \tag{3.1}$$

Proof. Let $f \in \mathcal{S}_p(\alpha)$. Now, it is easy to see that

$$z\frac{d}{dz}\left(\frac{z}{f(z)}\right)^{\mu} = \mu\left[\left(\frac{z}{f(z)}\right)^{\mu} - \left(\frac{z}{f(z)}\right)^{\mu+1} f'(z)\right]. \tag{3.2}$$

Using the identity (3.2), we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) - \alpha \Leftrightarrow \left|\frac{-\dfrac{z}{\mu}\dfrac{d}{dz}\left(\dfrac{z}{f(z)}\right)^{\mu}}{\left(\dfrac{z}{f(z)}\right)^{\mu}}\right|$$

$$\le \mathrm{Re}\,\frac{\left(\dfrac{z}{f(z)}\right)^{\mu} - \dfrac{z}{\mu}\dfrac{d}{dz}\left(\dfrac{z}{f(z)}\right)^{\mu}}{\left(\dfrac{z}{f(z)}\right)^{\mu}} - \alpha.$$

Since f is in the form (1.1), the last inequality may be equivalently written as

$$\frac{1}{\mu}\left|\frac{-\sum_{n=1}^{\infty} nb_n z^n}{1+\sum_{n=1}^{\infty} b_n z^n}\right| \le \mathrm{Re}\left(1-\frac{1}{\mu}\frac{\sum_{n=1}^{\infty} nb_n z^n}{1+\sum_{n=1}^{\infty} b_n z^n}\right)-\alpha.$$

If $z \in \Delta$ is real and tends to 1^- through reals, then from the last inequality we have

$$\frac{1}{\mu}\left(\frac{\sum_{n=1}^{\infty} nb_n}{1+\sum_{n=1}^{\infty} b_n}\right) \le 1-\alpha-\frac{1}{\mu}\left(\frac{\sum_{n=1}^{\infty} nb_n}{1+\sum_{n=1}^{\infty} b_n}\right),$$

from which we obtain the desired inequality (3.1). □

The case $\mu = 1$ leads to

Corollary 1. *Let $f \in \mathcal{S}_p(\alpha)$ be such that $z/f(z) = 1+\sum_{n=1}^{\infty} b_n z^n$ with $b_n \ge 0$. Then we have*

$$\sum_{n=1}^{\infty}(2n-1+\alpha)b_n \le 1-\alpha.$$

Theorem 2. *Let $z/f(z)$ be a nonvanishing analytic function of the form* (1.1) *with $\mu > 0$. Then the condition*

$$\sum_{n=1}^{\infty}(2n+\mu(1-\alpha))|b_n| \le \mu(1-\alpha) \tag{3.3}$$

is sufficient for f to be in the class $\mathcal{S}_p(\alpha)$.

Proof. As in the proof of Theorem 1, we notice that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)-\alpha$$

is equivalent to

$$\left|-\frac{\sum_{n=1}^{\infty} nb_n z^n}{1+\sum_{n=1}^{\infty} b_n z^n}\right| \le \mu(1-\alpha)-\mathrm{Re}\left(\frac{\sum_{n=1}^{\infty} nb_n z^n}{1+\sum_{n=1}^{\infty} b_n z^n}\right).$$

Thus, to show that f is in $\mathcal{S}_p(\alpha)$, it suffices to show that the quotient

$$-\frac{\sum_{n=1}^{\infty} nb_n z^n}{1+\sum_{n=1}^{\infty} b_n z^n}$$

lies in the parabolic region

$$(\mathrm{Im}\, w)^2 \le \mu(1-\alpha)[\mu(1-\alpha)+2\,\mathrm{Re}\, w].$$

Geometrically, this condition holds if we can show that

$$\left|\frac{\sum_{n=1}^{\infty} nb_n z^n}{1+\sum_{n=1}^{\infty} b_n z^n}\right| \le \frac{\mu(1-\alpha)}{2}, \quad z \in \Delta. \tag{3.4}$$

From the condition (3.3), we obtain that

$$\sum_{n=1}^{\infty}(2n+\mu(1-\alpha))|b_n|\,|z|^n \le \mu(1-\alpha)$$

and so

$$\sum_{n=1}^{\infty} n|b_n|\,|z|^n \le \frac{\mu(1-\alpha)}{2}\left(1-\sum_{n=1}^{\infty}|b_n|\,|z|^n\right).$$

In view of this inequality, we deduce that

$$\left|\frac{\sum_{n=1}^{\infty} nb_n z^n}{1+\sum_{n=1}^{\infty} b_n z^n}\right| \le \frac{\mu(1-\alpha)}{2}\left(\frac{1-\sum_{n=1}^{\infty}|b_n|\,|z|^n}{1-\sum_{n=1}^{\infty}|b_n|\,|z|^n}\right) = \frac{\mu(1-\alpha)}{2}$$

which is exactly the inequality (3.4) and therefore, $f \in \mathcal{S}_p(\alpha)$. □

Corollary 2. *Let $z/f(z)$ be a nonvanishing analytic function in Δ of the form $z/f(z) = 1+\sum_{n=1}^{\infty} b_n z^n$. Then the condition*

$$\sum_{n=1}^{\infty}(2n+1-\alpha)|b_n| \le 1-\alpha$$

is sufficient for f to be in the class $\mathcal{S}_p(\alpha)$.

The case $\alpha = 0$ of Corollaries 1 and 2 has been obtained recently by Obradović and Ponnusamy [11].

4. Radius problems

Theorem 3. *If $f \in \mathcal{S}$ is given by (1.1) with $0 < \mu < 1$, then $\frac{1}{r}f(rz) \in \mathcal{S}_p(\alpha)$ for $0 < r \le r_0$, where r_0 is the root of the integral equation*

$$\frac{4r^2(1+\mu(2-\alpha)(1-r^2))}{(1-r^2)^2} + \frac{r^2\mu^2(3-\alpha)^2}{1-\mu}\int_0^1 \frac{dt}{1-r^2 t^{1/(1-\mu)}} = \mu(1-\alpha)^2. \tag{4.1}$$

Proof. Let $f \in \mathcal{S}$ be given by (1.1) with $0 < \mu < 1$. Then $z/f(z)$ is nonvanishing in Δ and for $0 < r \le 1$, we have

$$\left(\frac{z}{\frac{1}{r}f(rz)}\right)^{\mu} = 1 + (b_1 r)z + (b_2 r^2)z^2 + \cdots.$$

If

$$S := \sum_{n=1}^{\infty} (2n + \mu(1-\alpha))|b_n| r^n \le \mu(1-\alpha) \tag{4.2}$$

for some r, then $\frac{1}{r} f(rz) \in \mathcal{S}_p(\alpha)$, by Theorem 2. Now, using the Cauchy-Schwarz inequality and Lemma 1, we see that

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \sqrt{n-\mu}|b_n| \frac{2n+\mu(1-\alpha)}{\sqrt{n-\mu}} r^n \\ &\le \left(\sum_{n=1}^{\infty} (n-\mu)|b_n|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{(2n+\mu(1-\alpha))^2}{n-\mu} r^{2n}\right)^{1/2} \\ &\le \sqrt{\mu}\left(\sum_{n=1}^{\infty} \frac{(2n+\mu(1-\alpha))^2}{n-\mu} r^{2n}\right)^{1/2} \\ &= \sqrt{\mu}\left(\sum_{n=1}^{\infty} \frac{(2n+\mu(1-\alpha))^2 - \mu^2(3-\alpha)^2}{n-\mu} r^{2n} + \mu^2(3-\alpha)^2 \sum_{n=1}^{\infty} \frac{r^{2n}}{n-\mu}\right)^{1/2} \\ &= \sqrt{\mu}\left(\sum_{n=1}^{\infty} 4(n+\mu(2-\alpha)) r^{2n} + \mu^2(3-\alpha)^2 \sum_{n=1}^{\infty} \frac{r^{2n}}{n-\mu}\right)^{1/2} \\ &= \sqrt{\mu}\left(\frac{4r^2(1+\mu(2-\alpha)(1-r^2))}{(1-r^2)^2} + \frac{r^2\mu^2(3-\alpha)^2}{1-\mu} \int_0^1 \frac{dt}{1-r^2 t^{1/(1-\mu)}}\right)^{1/2}. \end{aligned}$$

In particular, if the last expression is less than or equal to $\mu(1-\alpha)$, then (4.2) holds which gives the condition (4.1). □

In the case $\mu = 1$, Theorem 3 takes the following form which needs a special attention as we see that the radius quantity depends on the second coefficient of the given function f.

Theorem 4. *If $f \in \mathcal{S}$ is of the form $z/f(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, then $\frac{1}{r} f(rz) \in \mathcal{S}_p(\alpha)$ for $0 < r \le r_0$, where r_0, which depends on the second coefficient of f, is the root of the equation*

$$\frac{4r^4(1+(3-\alpha)(1-r^2))}{(1-r^2)^2} - (3-\alpha)^2 r^2 \ln(1-r^2) = (1-\alpha-(3-\alpha)(r/2)|f''(0)|)^2.$$

Proof. Note that, for $f \in \mathcal{S}$ satisfying $z/f(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, we have $b_1 = -f''(0)/2$. Proceeding exactly as in the proof of Theorem 3 (but with

$\mu = 1$) and by considering summation to run from 2 to ∞, we obtain the required conclusion. So we omit the details. $\square$

We remark that, the case $\alpha = 0$ of Theorem 4 is due to Obradović and Ponnusamy in [11].

Now we prove a generalized version of Lemma 2(a) which is useful to prove our next results.

Lemma 3. *Let* $0 \le \alpha < 1$ *and* $\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ *be a non-vanishing analytic function in* Δ *satisfying the coefficient condition*

$$\sum_{n=1}^{\infty} (n - 1 + \alpha)|b_n| \le \lambda(1 - \alpha). \tag{4.3}$$

Then the function f *defined by the equation* $(z/f(z))^{1-\alpha} = \phi(z)$ *is in* $\mathcal{U}(\lambda, 1 - \alpha)$.

Proof. Let f be given by $(z/f(z))^{1-\alpha} = \phi(z)$, where $\phi(z) \ne 0$ in Δ, and we choose here the principal branch so that $(z/f(z))^{1-\alpha}$ at $z = 0$ is 1. Then the power series representation of ϕ and the coefficient condition (4.3), lead to

$$\left| \left(\frac{z}{f(z)} \right)^{2-\alpha} f'(z) - 1 \right| = \left| -\frac{1}{1-\alpha} \sum_{n=1}^{\infty} (n - 1 + \alpha) b_n z^n \right| \le \lambda$$

and therefore, by the definition of the class, f is in $\mathcal{U}(\lambda, 1 - \alpha)$. $\square$

The following result determines the $\mathcal{U}(\lambda, \mu)$ radius in $\mathcal{S}$.

Theorem 5. *Suppose that* $f \in \mathcal{S}$, $0 \le \alpha < 1$, $\lambda > 0$ *and*

$$r_{\alpha,\lambda} = \frac{\lambda\sqrt{2(1-\alpha)}}{\left[\sqrt{(\alpha + 2\lambda^2(1-\alpha))^2 + 4\lambda^2(1-\alpha)^2(1-\lambda^2)} + (\alpha + 2\lambda^2(1-\alpha)) \right]^{1/2}}.$$

Then we have $\frac{1}{r} f(rz) \in \mathcal{U}(\lambda, 1 - \alpha)$ *for*

$$0 < r \le r_{\alpha,\lambda}. \tag{4.4}$$

In particular, $\frac{1}{r} f(rz) \in \mathcal{U}(1, 1 - \alpha)$ *for* $0 < r \le \sqrt{(1-\alpha)/(2-\alpha)}$.

Proof. Let $f \in \mathcal{S}$. Then $z/f(z) \ne 0$ in Δ. So, we may consider f in the form

$$\left(\frac{z}{f(z)} \right)^{1-\alpha} = 1 + \sum_{n=1}^{\infty} b_n z^n. \tag{4.5}$$

Now, Lemma 1 gives

$$\sum_{n=1}^{\infty}(n-1+\alpha)|b_n|^2 \le 1-\alpha.$$

On the other side, for $0 < r \le 1$, we obtain from (4.5) that

$$\left(\frac{z}{\frac{1}{r}f(rz)}\right)^{1-\alpha} = 1 + \sum_{n=1}^{\infty}(b_n r^n)z^n.$$

According to Lemma 3, it suffices to verify the inequality

$$\sum_{n=1}^{\infty}(n-1+\alpha)|b_n r^n| \le \lambda(1-\alpha)$$

for $0 < r \le r_{\alpha,\lambda}$. Now, as before, we have

$$\begin{aligned}\sum_{n=1}^{\infty}(n-1+\alpha)|b_n r^n| &\le \left(\sum_{n=1}^{\infty}(n-1+\alpha)|b_n|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}(n-1+\alpha)r^{2n}\right)^{1/2}\\ &\le \sqrt{1-\alpha}\left(\frac{r^4}{(1-r^2)^2}+\alpha\frac{r^2}{1-r^2}\right)^{1/2}\\ &= \sqrt{1-\alpha}\left(\frac{r}{1-r^2}\right)\left(\alpha+(1-\alpha)r^2\right)^{1/2}\\ &\le \lambda(1-\alpha),\end{aligned}$$

if $\frac{r}{1-r^2}\sqrt{\alpha+(1-\alpha)r^2} \le \lambda\sqrt{1-\alpha}$. Note that

$$\frac{r}{1-r^2}\sqrt{\alpha+(1-\alpha)r^2} \le \lambda\sqrt{1-\alpha}$$

is equivalent to (4.4), and so we complete the proof. □

5. Conditions for functions to be in $\mathcal{U}(\lambda,\mu)$

To present our next result, we consider the class of functions of Bazilevič type, see [6, 22]. The result is simple and surprising as it identifies a subclass which lies in $\mathcal{U}(\lambda,\mu)$. This generalizes the result of Obradović and Ponnusamy, see [11, Theorem 5].

THEOREM 6. *Let* $0 < \mu \le 1$. *If* $f \in \mathcal{S}$ *is given by* (1.1) *with* $b_n \ge 0$, *and satisfies the condition that* $\operatorname{Re}\left(f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right) > 0$. *Then* $f \in \mathcal{U}(1,\mu)$.

Proof. Using the equation (3.2), we notice that

$$\mathrm{Re}\left(f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right) > 0 \Leftrightarrow \mathrm{Re}\,\frac{\dfrac{zf'(z)}{f(z)}}{\left(\dfrac{z}{f(z)}\right)^{\mu}} > 0$$

$$\Leftrightarrow \mathrm{Re}\,\frac{\left(\dfrac{z}{f(z)}\right)^{\mu} - \dfrac{z}{\mu}\dfrac{d}{dz}\left(\dfrac{z}{f(z)}\right)^{\mu}}{\left(\dfrac{z}{f(z)}\right)^{2\mu}} > 0$$

$$\Leftrightarrow \mathrm{Re}\,\frac{1+\sum_{n=1}^{\infty}(1-n/\mu)b_n z^n}{(1+\sum_{n=1}^{\infty} b_n z^n)^2} > 0.$$

Since $b_n \geq 0$, allow $z \to 1^-$ along the real axis, we get

$$\mathrm{Re}\,\frac{1-\sum_{n=1}^{\infty}(n/\mu-1)b_n}{(1+\sum_{n=1}^{\infty} b_n)^2} \geq 0,$$

which gives that

$$\sum_{n=1}^{\infty}(n-\mu)b_n \leq \mu$$

and so by Lemma 3, we have $f \in \mathscr{U}(1,\mu)$. □

Theorem 7. *Let* $0 < \mu \leq 1$. *A function* f *of the form* (1.1) *with* $b_n \geq 0$ *and* $z/f(z) \neq 0$, *is in* $\mathscr{U}(1,\mu)$ *if and only if*

$$\sum_{n=1}^{\infty}(n-\mu)b_n \leq \mu. \tag{5.1}$$

Proof. In view of Lemma 3, it suffices to prove the necessary part. To do this, we let $f \in \mathscr{U}(1,\mu)$ and f is of the form (1.1). Then using (3.2), we get

$$\left|\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) - 1\right| = \left|\left(\frac{z}{f(z)}\right)^{\mu} - \frac{z}{\mu}\frac{d}{dz}\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| = \frac{1}{\mu}\left|\sum_{n=1}^{\infty}(n-\mu)b_n z^n\right| \leq 1.$$

Because $b_n \geq 0$, choosing values of z on the real axis and then letting $z \to 1^-$ through real values, we obtain the coefficient condition (5.1). □

The following result gives a sufficient condition for starlike functions of order α to be in the class $\mathscr{U}(\lambda,\mu)$.

THEOREM 8. *If* $f \in \mathcal{S}^*(\alpha)$ *is of the form* (1.1) *with* $b_n \geq 0$ *and* $\mu > 0$, *then*

$$\sum_{n=1}^{\infty}(n - \mu(1-\alpha))b_n \leq \mu(1-\alpha). \tag{5.2}$$

In particular, $f \in \mathcal{U}(1-\alpha, \mu)$.

Proof. It is easy to see that

$$f \in \mathcal{S}^*(\alpha) \Leftrightarrow \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \alpha \Leftrightarrow \left|\frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + 1 - 2\alpha}\right| \leq 1.$$

Now, using this relation and the identity (3.2), we have the following

$$\left|\frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + 1 - 2\alpha}\right| = \left|\frac{-z\frac{d}{dz}\left(\frac{z}{f(z)}\right)^{\mu}}{2\mu(1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu} - z\frac{d}{dz}\left(\frac{z}{f(z)}\right)^{\mu}}\right|$$
$$= \left|\frac{-\sum_{n=1}^{\infty} nb_n z^n}{2\mu(1-\alpha)(1+\sum_{n=1}^{\infty} b_n z^n) - \sum_{n=1}^{\infty} nb_n z^n}\right| \leq 1.$$

Since $b_n \geq 0$, if $z \to 1^-$ along the real axis, we see from the last inequality that

$$\frac{\sum_{n=1}^{\infty} nb_n}{2\mu(1-\alpha) - \sum_{n=1}^{\infty}(n - 2\mu(1-\alpha))b_n} \leq 1.$$

This gives the desired inequality (5.2).

Finally, since $n - \mu \leq n - \mu(1-\alpha)$, we have

$$\sum_{n=1}^{\infty}(n-\mu)b_n \leq \sum_{n=1}^{\infty}(n - \mu(1-\alpha))b_n \leq \mu(1-\alpha).$$

From Lemma 3, we conclude that $f \in \mathcal{U}(1-\alpha, \mu)$. □

As a consequence of Theorem 8, we next see that $\mathcal{T}^*(\alpha) \subset \mathcal{U}(1-\alpha)$.

COROLLARY 3. *If* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ *is in* $\mathcal{S}^*(\alpha)$, *then* $f \in \mathcal{U}(1-\alpha)$.

Proof. Let $f \in \mathcal{S}^*(\alpha)$ be of the form $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$. Then $z/f(z)$ is nonvanishing in the unit disk and so it can be expressed as

$$\frac{z}{f(z)} = \frac{1}{1 - |a_2|z - |a_3|z^2 - \cdots} = 1 + b_1 z + b_2 z^2 + \cdots,$$

where $b_n \geq 0$ for all $n \in \mathbf{N}$. Then by Theorem 8, $f \in \mathcal{U}(1-\alpha)$. □

From [12], we collect the following result.

LEMMA 4. *Let* $0 \leq \lambda$, $\gamma \leq 1$ *and* $f \in \mathcal{U}(\lambda)$. *Define*

$$\lambda_\gamma^* = \frac{-|f''(0)|\cos(\pi\gamma/4) + \sin(\pi\gamma/4)\sqrt{16\cos^2(\pi\gamma/4) - |f''(0)|^2}}{2\cos(\pi\gamma/4)}$$

and let $\lambda_\gamma^{\mathcal{R}}$ *be given by the inequality*

$$\sin(\pi\gamma/2)\sqrt{4-\lambda^2} \geq (|f''(0)| + \lambda)\sqrt{4 - (|f''(0)| + \lambda)^2} + \lambda\cos(\pi\gamma/2).$$

Then

(i) $f \in \mathcal{U}(\lambda) \Rightarrow f \in \mathcal{S}_\gamma$ *for* $0 < \lambda \leq \lambda_\gamma^*/2$,

(ii) $f \in \mathcal{U}(\lambda) \Rightarrow f \in \mathcal{R}_\gamma$ *for* $0 < \lambda \leq \lambda_\gamma^{\mathcal{R}}/2$,

where

$$\mathcal{R}_\gamma := \left\{ f \in \mathcal{A} : |\arg f'(z)| \leq \frac{\pi\gamma}{2} \right\} \quad \text{and}$$

$$\mathcal{S}_\gamma := \left\{ f \in \mathcal{A} : |\arg(zf'(z)/f(z))| \leq \frac{\pi\gamma}{2} \right\}.$$

Using the containment results of Lemma 4 and Corollary 3, one can derive a number of interesting results. For instance, we obtain the following:

COROLLARY 4. *If* $0 \leq \gamma \leq 1$ *and* $f(z) = z - \sum_{n=3}^{\infty} |a_n| z^n \in \mathcal{S}^*\left(1 - \sin\frac{\pi\gamma}{6}\right)$, *then* $f \in \mathcal{R}_\gamma$. *In particular, if* $f''(0) = 0$, *then* $f \in \mathcal{S}^*(1/2)$ *implies that* $\operatorname{Re} f'(z) \geq 0$.

COROLLARY 5. *If* $0 \leq \gamma \leq 1$ *and* $f(z) = z - \sum_{n=3}^{\infty} |a_n| z^n \in \mathcal{S}^*\left(1 - \sin\frac{\pi\gamma}{4}\right)$, *then* $f \in \mathcal{S}_\gamma$. *In particular, if* $f''(0) = 0$, *then* $f \in \mathcal{S}^*(1/2)$ *implies that* $|\arg(zf'(z)/f(z))| \leq \pi/3$.

6. Conclusion

To present a meromorphic analog of the class $\mathcal{U}(\lambda)$, we recall, for example, the following result.

LEMMA 5 [17, Theorem 1.2]. *If* $f \in \mathcal{U}(\lambda)$ *and* $a = |f''(0)|/2 \leq 1$, *then* $f \in \mathcal{S}^*(\delta)$ *whenever* $0 \leq \lambda \leq \lambda(\delta)$, *where*

$$(6.1)\qquad \lambda(\delta)=\begin{cases}\dfrac{\sqrt{(1-2\delta)(2-a^2-2\delta)}-a(1-2\delta)}{2(1-\delta)} & \text{if } 0\le\delta<\dfrac{1+a}{3+a},\\[2mm] \dfrac{1-\delta(1+a)}{1+\delta} & \text{if } \dfrac{1+a}{3+a}\le\delta<\dfrac{1}{1+a}.\end{cases}$$

In particular,

$$f\in\mathscr{U}(\lambda),\ f''(0)=0\Rightarrow f\in\mathscr{S}^*\quad \textit{whenever}\ 0\le\lambda\le 1/\sqrt{2}.$$

After the paper was submitted to the journal, Fournier and Ponnusamy [2] settled the question of sharpness of the bound for λ for which $\mathscr{U}(\lambda)\subset\mathscr{S}^*$. As a motivation for our next result, we consider the class, denoted by Σ, of all functions of the form

$$F(\zeta)=\zeta+\sum_{n=0}^{\infty}c_n\zeta^{-n}$$

that are analytic and univalent for $|\zeta|>1$. Thus

$$F\in\Sigma\Leftrightarrow f\in\mathscr{S},\quad f(z)=\frac{1}{F(1/z)}=\frac{z}{1+\sum_{n=1}^{\infty}c_{n-1}z^n}.$$

Also, we note that

$$f'(z)\left(\frac{z}{f(z)}\right)^2=F'(1/z)\quad\text{and}\quad\frac{zf'(z)}{f(z)}=\frac{(1/z)F'(1/z)}{F(1/z)}.$$

Consequently, for $0\le\lambda\le 1$, $f\in\mathscr{U}(\lambda)$ if and only if $|F'(\zeta)-1|\le\lambda$ for $|\zeta|>1$. Similarly, for $0\le\alpha\le 1$, $f\in\mathscr{S}^*(\alpha)$ if and only if

$$\operatorname{Re}\left(\frac{\zeta F'(\zeta)}{F(\zeta)}\right)\ge\alpha\quad\text{for } |\zeta|>1.$$

The class of all such functions satisfying the later condition is denoted by $\Sigma^*(\alpha)$. Thus, Lemma 5 takes the following form:

THEOREM 9. *Let $F(\zeta)=\zeta+\sum_{n=0}^{\infty}c_n\zeta^{-n}$ be analytic and univalent for $|\zeta|>1$. If F satisfies the condition*

$$|F'(\zeta)-1|\le\lambda\quad \textit{for}\ |\zeta|>1$$

and $a=|-c_0|\le 1$, then $F\in\Sigma^*(\delta)$ whenever $0<\lambda\le\lambda(\delta)$, where $\lambda(\delta)$ is given by (6.1). *In particular, for $c_0=0$, $F\in\Sigma^*(\delta)$ whenever $0<\lambda\le 1/\sqrt{2}$.*

In view of the above observations, we can state a number of results for various subclasses of the class of meromorphic univalent functions.

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