Y. TADOKORO
KODAI MATH. J.
**29** (2006), 370–382

# THE POINTED HARMONIC VOLUMES OF HYPERELLIPTIC CURVES WITH WEIERSTRASS BASE POINTS

Yuuki Tadokoro

**Abstract**

We give an explicit computation of the pointed harmonic volumes of hyperelliptic curves with Weierstrass base points, which are paraphrased into a combinatorial formula.

## 1. Introduction

Let $X$ be a compact Riemann surface of genus $g \geq 2$ and $p$ a point on $X$. By Pulte [5], the pointed harmonic volume of $(X, p)$ was defined to be the homomorphism $I_p : K \otimes H \to \mathbf{R}/\mathbf{Z}$, using Harris' method for the harmonic volume of $X$ [4]. Here, we denote by $H = H^1(X; \mathbf{Z})$ the first cohomology group of $X$ and $K$ the kernel of the intersection pairing $H \otimes H \to \mathbf{Z}$. In this paper, we compute the pointed harmonic volume of any hyperelliptic curve $C$ with any Weierstrass point $p$. In theorem 5.6, we compute that of some special hyperelliptic curve $C_0$ with Weierstrass points in an analytic way, by the explicit computation of Chen's iterated integrals [2]. Using Proposition 4.1, we can compute the pointed harmonic volumes of all the hyperelliptic curves with Weierstrass base points from those of $C_0$. These results are paraphrased from a combinatorial viewpoint as follows. Let $\{P_j\}_{j=0,1,\dots,2g+1}$ denote the set of Weierstrass points on $C$, and fix a Weierstrass point $P_\nu$, $0 \leq \nu \leq 2g+1$. A certain homomorphism $\kappa_\nu : K \otimes H \to \frac{1}{2}\mathbf{Z}/\mathbf{Z} = \{0, 1/2\}$ is defined in §6, which depends on the choice of $P_\nu$.

Theorem 6.2. *For any hyperelliptic curve $C$ and $A \in K \otimes H$, we have*

$$I_{P_\nu}(A) \equiv \kappa_\nu(A) \mod \mathbf{Z}.$$

The author [6] computed the harmonic volumes of hyperelliptic curves. But the computation of the pointed ones of $(X, p)$ is more complicated than that of $X$. For any hyperelliptic curve $C$, it is tedious to compute $I_p$ in the case $p \in C \setminus \{P_j\}_{j=0,1,\dots,2g+1}$. But we have $I_p \equiv 0$ or $1/2 \mod \mathbf{Z}$ in the case $p \in \{P_j\}_{j=0,1,\dots,2g+1}$. It has been still unknown which elements of $K \otimes H$ and Weier-

Received November 16, 2005.

strass points $p$ have nontrivial $I_p$ or not. In this paper, we compute them completely.

As an application of the pointed harmonic volume of $(X, p)$, Pulte proved the pointed Torelli theorem [5]. We denote by $\pi_1(X, p)$ the fundamental group of $X$ at the base point $p \in X$ and $J_p$ the augmentation ideal of the group ring $\mathbf{Z}\pi_1(X, p)$.

THEOREM 1.1 (The pointed Torelli theorem [5]). *Suppose that $X$ and $Y$ are compact Riemann surfaces and that $p \in X$ and $q \in Y$. With the exception of two points $p$ in $X$, if there is a ring isomorphism*

$$\mathbf{Z}\pi_1(X, p)/J_p^3 \to \mathbf{Z}\pi_1(Y, q)/J_q^3$$

*which preserves the mixed Hodge structure, then there is a biholomorphism $\varphi : X \to Y$ such that $\varphi(p) = q$.*

If $X$ is generic (e.g. $X$ is hyperelliptic), then there are no exceptional points. The pointed harmonic volumes determine the choice of the base points. In the proof of this theorem, the classical Torelli theorem follows from the preservation of the mixed Hodge structure and we obtain the biholomorphism $X \cong Y$. When we choose the base points, the pointed harmonic volume plays an important role. Theorem 6.2 also tells the choice of Weierstrass base points on $C$.

Now we describe the contents of this paper briefly. In §2, we define the pointed harmonic volume of $(X, p)$, using Chen's iterated integrals [2]. In §3, we give a basis of the first homology group $H_1(C; \mathbf{Z}/2\mathbf{Z})$ of the hyperelliptic curve $C$. In §4, we prove $I_{P_v} \in H^0(\Delta_g^1; \mathrm{Hom}(K \otimes H, \mathbf{Z}/2\mathbf{Z}))$. In §5, the pointed harmonic volume of some special hyperelliptic curve $C_0$ with Weierstrass base points is computed in an analytic way. This result can be extended to all the hyperelliptic curves with Weierstrass base points and interpreted from a combinatorial viewpoint. In §6, we obtain a simple combinatorial formula of the pointed harmonic volume of $(C, P_v)$.

*Acknowledgments*. The author is grateful to Nariya Kawazumi for valuable advice and reading the manuscript. He also thanks Masahiko Yoshinaga for reading the manuscript. This work is partially supported by 21st Century COE program (University of Tokyo) by the Ministry of Education, Culture, Sports, Science and Technology.

## 2. The pointed harmonic volume

We recall the definition of the pointed harmonic volume of a pointed Riemann surface $(X, p)$. Here $X$ is a compact Riemann surface of genus $g \geq 2$ and $p$ a point on $X$. We identify the first integral homology group $H_1(X; \mathbf{Z})$ of $X$ with the first integral cohomology group by the Poincaré duality, and denote it by $H$. For closed 1-forms $\omega_{1,i}$ and $\omega_{2,i}$, $i = 1, 2, \dots, m$, on $X$ such that $\int_X \sum_{i=1}^m \omega_{1,i} \wedge \omega_{2,i} = 0$, we obtain the 1-form $\eta$ such that $d\eta = \sum_{i=1}^m \omega_{1,i} \wedge \omega_{2,i}$

and $\int_X \eta \wedge *\alpha = 0$ for any closed 1-form $\alpha$ on $X$. Here, $*$ is the Hodge star operator which depends only on the complex structure and not the choice of Hermitian metric. We identify $H$ with the space of all the real harmonic 1-forms on $X$ with integral periods by the Hodge theorem. We denote by $K$ the kernel of the intersection pairing $(\,,\,) : H \otimes H \to \mathbf{Z}$.

Definition 2.1 (The pointed harmonic volume [5]). For $\sum_{i=1}^m a_i \otimes b_i \in K$ and $c \in H$, the pointed harmonic volume is defined to be

$$I_p\left(\left(\sum_{i=1}^m a_i \otimes b_i\right) \otimes c\right) := \sum_{i=1}^m \int_\gamma a_i b_i - \int_\gamma \eta \mod \mathbf{Z}.$$

Here $\eta$ is the 1-form on $X$ which is associated to $\sum_{i=1}^m a_i \otimes b_i$ in the way stated above and $\gamma : [0,1] \to X$ is a loop in $X$ at the base point $p$ whose homology class is equal to $c$. The integral $\int_\gamma a_i b_i$ is Chen's iterated integral [2], that is, $\int_\gamma a_i b_i = \int_{0 \le t_1 \le t_2 \le 1} f_i(t_1) g_i(t_2)\, dt_1 dt_2$ for $\gamma^* a_i = f_i(t)\, dt$ and $\gamma^* b_i = g_i(t)\, dt$. Here $t$ is the coordinate in the unit interval $[0,1]$. See Chen [2] for iterated integrals and Harris [4], Pulte [5] for the (pointed) harmonic volume.

*Remark* 2.2. By the definition of $I_p$, we have $I_p((\sum_{i=1}^m a_i \otimes b_i) \otimes c) \equiv -I_p((\sum_{i=1}^m b_i \otimes a_i) \otimes c) \mod \mathbf{Z}$.

## 3. Hyperelliptic curves

Let $C$ be a hyperelliptic curve and $\mathbf{Z}_2$ the field $\mathbf{Z}/2\mathbf{Z}$. In this section, we explain the first homology group of $C$ with $\mathbf{Z}_2$-coefficients.

We define the hyperelliptic curve $C$ as follows. It is the compactification of the plane curve in the $(z,w)$ plane $\mathbf{C}^2$

$$w^2 = \prod_{i=0}^{2g+1} (z - p_i),$$

where $p_0, p_1, \dots, p_{2g+1}$ are some distinct points on $\mathbf{C}$. It admits the hyperelliptic involution given by $\iota : (z,w) \mapsto (z,-w)$. Let $\pi$ be the 2-sheeted covering $C \to \mathbf{C}P^1$, $(z,w) \mapsto z$, branched over $2g+2$ branch points $\{p_i\}_{i=0,1,\dots,2g+1}$ and $P_i \in C$ a ramification point such that $\pi(P_i) = p_i$. It is known that $\{P_i\}_{i=0,1,\dots,2g+1}$ is just the set of all the Weierstrass points on any hyperelliptic curve $C$.

For points $p_i$ and $p_j$, we denote by $p_i p_j$ a simple path joining $p_i$ and $p_j$. We draw simple paths $p_0p_1, p_1p_2, \dots, p_{2g}p_{2g+1}$ and $p_{2g+1}p_0$ such that all the $2g+2$ arcs do not intersect except for endpoints of them. We take a disk $D \subset \mathbf{C}P^1$ whose boundary is $(\bigcup_{j=0}^{2g} p_j p_{j+1}) \cup p_{2g+1}p_0$ (Figure 1, $g = 2$). We picture two copies of $\mathbf{C}P^1$ as above and call them $\Omega_0$ and $\Omega_1$. We make crosscuts along $p_{2k}p_{2k+1}$, $k = 0,1,\dots,g$, and construct the hyperelliptic curve $C$ by joining every $p_{2k}p_{2k+1}$ on $\Omega_0$ to the corresponding one on $\Omega_1$ for $k = 0,1,\dots,g$. See 102–103 in [3] for example. We may consider $\Omega_i \subset C$ for $i = 0,1$.

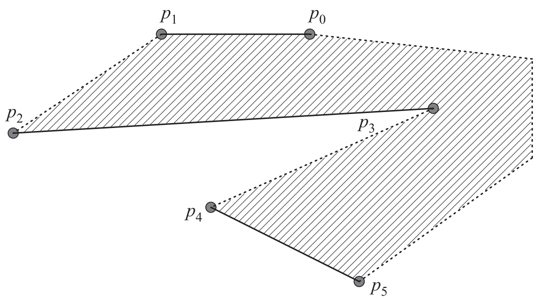


FIGURE 1. $D \subset \mathbf{C}P^1$

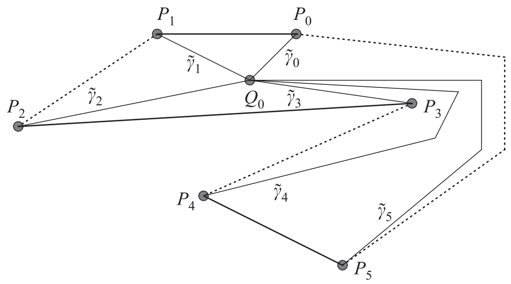


FIGURE 2. $\Omega_0 \subset C$

The hyperelliptic involution $\iota$ interchanges a point on $\Omega_0$ and the corresponding one on $\Omega_1$, and fixes $P_i$, $i = 0, 1, \ldots, 2g+1$. We choose a base point $Q_0 \in \Omega_0$ and denote $Q_1 = \iota(Q_0) \in \Omega_1$. Let $\gamma_j$, $j = 0, 1, \ldots, 2g+1$, be a simple path in $D$ joining $\pi(Q_0)$ and $p_j$. We denote by $\tilde{\gamma}_j$ the lift of $\gamma_j$ in $\Omega_0$ from $Q_0$ to $P_j$ (Figure 2, $g = 2$). Set $e_j = \tilde{\gamma}_j \cdot \iota(\tilde{\gamma}_j)^{-1}$, where the product $\tilde{\gamma}_j \cdot \iota(\tilde{\gamma}_j)^{-1}$ indicates that we traverse $\tilde{\gamma}_j$ first, then $\iota(\tilde{\gamma}_j)^{-1}$. It is a path in $C$ which is to be followed from $Q_0$ to $P_j$ and go to $Q_1$ in Figure 3. It is clear that $e_{j_1} \cdot \iota(e_{j_2})$ is a loop in $C$ at the base point $Q_0$. Moreover we have homotopy equivalences relative to the base point $Q_0$

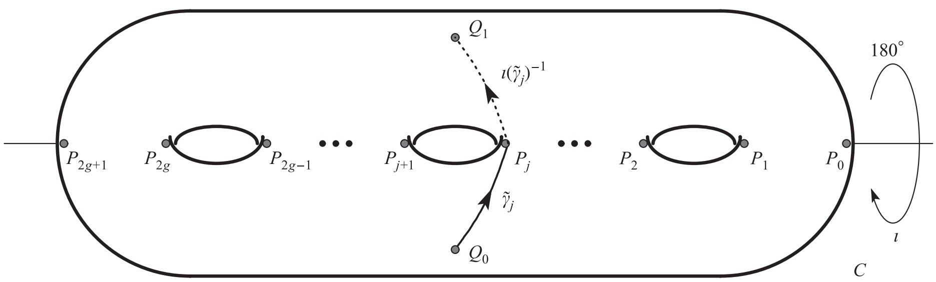


FIGURE 3. $e_j = \tilde{\gamma}_j \cdot \iota(\tilde{\gamma}_j)^{-1}$

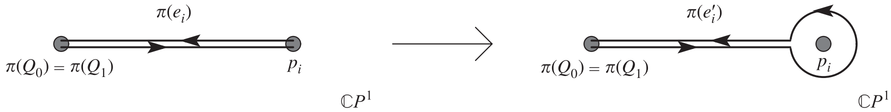


FIGURE 4. A deformation of $e_i$

$$e_j \cdot \iota(e_j) \sim 1 \quad \text{and} \quad e_0 \cdot \iota(e_1) \cdot \cdots \cdot e_{2g} \cdot \iota(e_{2g+1}) \sim 1.$$

We set $a_i = e_{2i-1} \cdot \iota(e_{2i})$ and $b_i = e_{2i-1} \cdot \iota(e_{2i-2}) \cdot \cdots \cdot e_1 \cdot \iota(e_0)$, and denote by $x_i$ and $y_i$ the homology classes of $a_i$ and $b_i$ respectively. Then we have $\{x_i, y_i\}_{i=1,2,\dots,g}$ is a symplectic basis of $H_1(C;\mathbf{Z})$ in Figure 1 in [6].

Let $H_{\mathbf{Z}_2}$ denote $H_1(C;\mathbf{Z}_2)$ and $B$ branch locus $\{p_i\}_{i=0,1,\dots,2g+1}$. We deform the path $e_i$ in $C$ and denote it by $e'_i$ in $C\backslash\pi^{-1}(B)$ as follows. The path $e'_i$ avoids $P_i$ in a sufficiently small neighborhood at $P_i$ so that $\pi(e'_i)$ goes around $p_i$ which does not any $p_k$ with $k \neq i$ (Figure 4) and the set of homology classes $\{\pi(e'_i)\}_{i=0,1,\dots,2g}$ is a basis of $H_1(\mathbf{C}P^1\backslash B;\mathbf{Z}_2)$. Moreover we have $\pi(e'_0) + \pi(e'_1) + \cdots + \pi(e'_{2g+1}) = 0 \in H_1(\mathbf{C}P^1\backslash B;\mathbf{Z}_2)$. Since the coefficients are in $\mathbf{Z}_2$, the homology class of $e'_i$ is independent of the choice of it. From the 2-sheeted covering $\pi$, we have the well-defined homomorphism $v_0 : H \to H_1(\mathbf{C}P^1\backslash B;\mathbf{Z}_2)$ which factors through $H_1(C\backslash\pi^{-1}(B);\mathbf{Z})$ (Arnol'd [1]). We obtain the linear map $v : H_{\mathbf{Z}_2} \to H_1(\mathbf{C}P^1\backslash B;\mathbf{Z}_2)$ induced naturally by $v_0$. It immediately follows that $v(x_i \bmod 2) = \pi(e'_{2i-1}) + \pi(e'_{2i})$, $v(y_i \bmod 2) = \pi(e'_0) + \pi(e'_1) + \cdots + \pi(e'_{2i-1})$, and $v$ is injective. The map $v$ gives the short exact sequence

$$0 \to H_{\mathbf{Z}_2} \xrightarrow{v} H_1(\mathbf{C}P^1\backslash B;\mathbf{Z}_2) \to \mathbf{Z}_2 \to 0.$$

Here the map $H_1(\mathbf{C}P^1\backslash B;\mathbf{Z}_2) \to \mathbf{Z}_2$ is the augmentation map $\pi(e'_i) \mapsto 1$. Fix a Weierstrass point $P_v$. Let $f_i$ denote $\pi(e'_v) + \pi(e'_i)$ for $i = 0, 1, \dots, 2g+1$. We remark that $f_i$ may be considered as an element of $H_{\mathbf{Z}_2}$ and $f_v = 0$. For $i = 1, 2, \dots, g$, we have the identification

$$\begin{cases} x_i = f_{2i-1} + f_{2i}, \\ y_i = f_0 + f_1 + \cdots + f_{2i-1}, \end{cases} \tag{3.1}$$

in $H_{\mathbf{Z}_2}$. It is clear that $f_0 + f_1 + \cdots + f_{2g+1} = 0$.

For any hyperelliptic curve $C$ and Weierstrass point $P_j \in C$, the hyperelliptic involution $\iota$ fixes $I_{P_j}$ and acts on $H_{\mathbf{Z}_2}$ as $(-1)$-times. Then we have the value of $I_{P_j}$ is 0 or $1/2 \bmod \mathbf{Z}$ from the equation $I_{P_j} \equiv (-1)^3 I_{P_j} \bmod \mathbf{Z}$. We may consider $I_{P_j} \in \mathrm{Hom}((K \otimes H)_{\mathbf{Z}_2}, \mathbf{Z}_2)$, where $(K \otimes H)_{\mathbf{Z}_2}$ denotes $(K \otimes H) \otimes \mathbf{Z}_2$.

## 4. Pointed harmonic volumes of hyperelliptic curves and the moduli space of Riemann surfaces

We recall some results about the moduli space of Riemann surfaces. Let $\Sigma_g$ be a closed oriented surface of genus $g$. Its mapping class group, denoted here by $\Gamma_g^s$, is the group of isotopy classes of orientation preserving diffeomorphisms of $\Sigma_g$ which fix $s$ points on $\Sigma_g$ for $s = 0, 1$. We denote $\Gamma_g = \Gamma_g^0$. The group $\Gamma_g^1$ acts on the Teichmüller space $\mathscr{T}_g^1$ of $\Sigma_g$ with a marked point and the quotient space $\mathscr{M}_g^1$ is the moduli space of Riemann surfaces of genus $g$ with a marked point. The group $\Gamma_g^1$ acts naturally on the first homology group $H_1(\Sigma_g; \mathbf{Z})$ of $\Sigma_g$.

Let $\mathscr{H}_g^1 \subset \mathscr{M}_g^1$ be the moduli space of hyperelliptic curves of genus $g$ with a marked Weierstrass point $P_v$. For the rest of this paper, we suppose that a marked point is a Weierstrass point. The hyperelliptic mapping class group $\Delta_g^1$ is the subgroup of $\Gamma_g$ defined by

$$\{\varphi \in \Gamma_g; \varphi\iota = \iota\varphi, \varphi(P_v) = P_v\},$$

where $\iota$ is the hyperelliptic involution of $\Sigma_g$. We have $\Delta_g^1 \subset \Gamma_g^1$. The moduli space $\mathscr{H}_g^1$ is known to be connected and has a natural structure of a quasi-projective orbifold. The group $\Delta_g^1$ can be considered as its orbifold fundamental group. For any $\mathbf{Z}\Delta_g^1$-module $M$, we may consider the dual $M^* = \mathrm{Hom}(M, \mathbf{Z}_2)$ as a $\mathbf{Z}_2\Delta_g^1$-module in a natural way. We denote $I_v = I_{P_v}$.

Proposition 4.1. *We have*

$$I_v \in H^0(\Delta_g^1; (K \otimes H)^*),$$

*i.e.* $I_v$ *is a* $\Delta_g^1$*-invariant in the dual* $(K \otimes H)^*$.

*Proof.* Let $\mathscr{L}$ be a locally constant sheaf with a stalk $\mathrm{Hom}_{\mathbf{Z}}(K \otimes H, \mathbf{Z}_2)$. In a similar way to Harris' method [4], $I_v$ varies in $\mathscr{H}_g^1$ continuously. For any hyperelliptic curves, $I_v \equiv 0$ or $1/2 \bmod \mathbf{Z}$. We remark that the pointed harmonic volume is uniquely determined for any point on $\mathscr{H}_g^1$. The locally constant sheaf $\mathscr{L}$ has a global section $\tilde{I}_v$ associated to $I_v$. Moreover $\mathscr{H}_g^1$ is arcwise connected. Therefore $\tilde{I}_v$ is a constant section of $\mathscr{L}$ and $I_v$ is invariant under the action of the orbifold fundamental group $\Delta_g^1$ of $\mathscr{H}_g^1$. □

## 5. Pointed harmonic volumes of a hyperelliptic curve $C_0$

We compute the pointed harmonic volume of a pointed hyperelliptic curve $(C_0, P_v)$. See §3 and 4 in [6] for details. We define the hyperelliptic curve $C_0$ by the equation $w^2 = z^{2g+2} - 1$. We take $Q_i = (0, (-1)^i\sqrt{-1})$, $i = 0, 1$, and $P_j = (\zeta^j, 0)$, $j = 0, 1, \dots, 2g+1$, where $\zeta = \exp(2\pi\sqrt{-1}/(2g+2))$. We define a path $e_j : [0, 1] \to C_0$, $j = 0, 1, \dots, 2g+1$, by

$$\begin{cases} \left(2t\zeta^j, \sqrt{-1}\sqrt{1-(2t)^{2g+2}}\right) & \text{for } 0 \le t \le 1/2, \\ \left((2-2t)\zeta^j, -\sqrt{-1}\sqrt{1-(2-2t)^{2g+2}}\right) & \text{for } 1/2 \le t \le 1. \end{cases}$$

For $i = 1, 2, \dots, g$, we denote by $\omega_i$ a holomorphic 1-form $z^{i-1}\,dz/w$ on $C_0$. It is known that $\{\omega_i\}_{i=1,2,\dots,g}$ is a basis of the space of holomorphic 1-forms on $C_0$. Let $B(u,v)$ denote the beta function $\int_0^1 x^{u-1}(1-x)^{v-1}\,dx$ for $u,v>0$. Set $\omega_i' = \dfrac{(2g+2)\sqrt{-1}}{2B(i/(2g+2),1/2)}\omega_i$. Then we have

$$\int_{a_j}\omega_i' = \zeta^{i(2j-1)}(1-\zeta^i) \quad \text{and} \quad \int_{b_j}\omega_i' = \frac{\zeta^{2ij}-1}{\zeta^i+1},$$

where $i,j \in \{1,2,\dots,g\}$. The integral $\int_\gamma \omega_i'$ depends only on the homology class of $\gamma$, since $\omega_i'$ is a closed 1-form.

We compute the iterated integrals of real harmonic 1-forms of $C_0$ with integral periods. Let $\Omega_a$ and $\Omega_b$ be the non-singular matrices whose $(i,j)$-entries are

$$\int_{a_j}\omega_i' \quad \text{and} \quad \int_{b_j}\omega_i'$$

respectively. We define real harmonic 1-forms $\alpha_i$ and $\beta_i$, $i=1,2,\dots,g$, by

$$\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_g\end{pmatrix} = \Re\left((\Omega_b)^{-1}\begin{pmatrix}\omega_1'\\ \vdots\\ \omega_g'\end{pmatrix}\right) \quad \text{and} \quad \begin{pmatrix}\beta_1\\ \vdots\\ \beta_g\end{pmatrix} = -\Re\left((\Omega_a)^{-1}\begin{pmatrix}\omega_1'\\ \vdots\\ \omega_g'\end{pmatrix}\right)$$

respectively. It is clear that $\int_{a_j}\alpha_i = \int_{b_j}\beta_i = 0$ and $\int_{b_j}\alpha_i = \delta_{ij} = -\int_{a_j}\beta_i$, where $\delta_{ij}$ is Kronecker's delta. Let $\Theta : H_1(C_0;\mathbf{Z}) \to H^1(C_0;\mathbf{Z})$ denote the Poincaré dual. We have $\Theta(x_i) = \alpha_i$ and $\Theta(y_i) = \beta_i$ for $i=1,2,\dots,g$. Hence, $\{\alpha_i,\beta_i\}_{i=1,2,\dots,g}$ is a symplectic basis of $H^1(C_0;\mathbf{Z})$.

Let $t_u$ be a complex number $\sum_{p=1}^{g}\zeta^{up}$ for any integer $u$. It is obvious that

$$t_u = \begin{cases} g & \text{for } u \in (2g+2)\mathbf{Z},\\ -1 & \text{for } u \in 2\mathbf{Z}\setminus(2g+2)\mathbf{Z},\\ \dfrac{1+\zeta^u}{1-\zeta^u} & \text{for } u \in 2\mathbf{Z}+1.\end{cases}$$

Furthermore, $t_u$ is pure imaginary and $t_{-u} = -t_u$ when $u$ is odd. In addition to the formulas (1), (2), (3), and (4) of Lemma 3.8 in [6], it is to show

LEMMA 5.1. *On the curve* $C_0$, *we have*

$$(5) \qquad \int_{a_k}\alpha_i\beta_j = \frac{-1}{2(g+1)^2}t_{2k-2i}(t_{2k-2j}-t_{2k}),$$

$$(6) \qquad \int_{b_k}\alpha_i\beta_j = \frac{-1}{2(g+1)^2}\sum_{u=1}^{k}\left\{(t_{2u-2i-2}-t_{2u-2i})\sum_{v=1}^{j}t_{2v+2u-2j-2}\right\}.$$

*Here* $i,j,k \in \{1,2,\dots,g\}$.

*Proof.* We compute the case (5) in the following way. Let $A_{j,m}$ and $B_{i,l}$ be $(j,m)$ and $(i,l)$-entries of $(\Omega_a)^{-1}$ and $(\Omega_b)^{-1}$ respectively.

$$\int_{a_k} \alpha_i \beta_j = \int_{a_k} -\Re\left(\sum_{l=1}^{g} B_{i,l}\omega_l'\right)\Re\left(\sum_{m=1}^{g} A_{j,m}\omega_m'\right)$$
$$= -\frac{1}{4}\int_{a_k} \sum_{l,m=1}^{g} (B_{i,l}A_{j,m}\omega_l'\omega_m' + B_{i,l}\bar{A}_{j,m}\omega_l'\overline{\omega}_m' + \bar{B}_{i,l}A_{j,m}\overline{\omega}_l'\omega_m' + \bar{B}_{i,l}\bar{A}_{j,m}\overline{\omega}_l'\overline{\omega}_m')$$
$$= -\frac{1}{2}\Re\left\{\sum_{l,m=1}^{g}\left(B_{i,l}A_{j,m}\int_{a_k}\omega_l'\omega_m' + B_{i,l}\bar{A}_{j,m}\int_{a_k}\omega_l'\overline{\omega}_m'\right)\right\}.$$

We use Lemma 3.5 in [6] and calculate

$$(g+1)^2 \sum_{l,m=1}^{g} B_{i,l}A_{j,m}\int_{a_k}\omega_l'\omega_m'$$
$$= \sum_{l,m=1}^{g} \zeta^{-2il}(1+\zeta^l)\frac{\zeta^m(-1+\zeta^{-2jm})}{1-\zeta^m}\frac{1}{2}\zeta^{(l+m)(2k-1)}(1-2\zeta^m+\zeta^{l+m})$$
$$= \frac{1}{2}\sum_{m=1}^{g}\frac{1-\zeta^{2jm}}{1-\zeta^m}\zeta^{m(2k-2j)}\sum_{l=1}^{g}\zeta^{l(2k-2i-1)}(1+\zeta^l)(1-\zeta^m-\zeta^m(1-\zeta^l))$$
$$= \frac{1}{2}\sum_{m=1}^{g}\sum_{v=2k-2j}^{2k-1}\zeta^{mv}\{(1-\zeta^m)(t_{2k-2i-1}+t_{2k-2i})-\zeta^m(t_{2k-2i-1}-t_{2k-2i+1})\}$$
$$= \frac{1}{2}\sum_{m=1}^{g}\left\{\frac{1-\zeta^{2jm}}{1-\zeta^m}\zeta^{m(2k-2j)}(1-\zeta^m)(t_{2k-2i-1}+t_{2k-2i}) - \sum_{v=2k-2j}^{2k-1}\zeta^{m(v+1)}(t_{2k-2i-1}-t_{2k-2i+1})\right\}$$
$$= \frac{1}{2}\left\{(t_{2k-2i-1}+t_{2k-2i})(t_{2k-2j}-t_{2k}) - (t_{2k-2i-1}-t_{2k-2i+1})\sum_{v=2k-2j}^{2k-1} t_{v+1}\right\}$$
$$= \frac{1}{2}\left\{(t_{2k-2i-1}+t_{2k-2i})(t_{2k-2j}-t_{2k}) - (t_{2k-2i-1}-t_{2k-2i+1})\sum_{v=2k-2j+1}^{2k} t_{v}\right\}.$$

Similarly, we have

$$(g+1)^2 \sum_{l,m=1}^{g} B_{i,l}\bar{A}_{j,m}\int_{a_k}\omega_l'\overline{\omega}_m'$$
$$= \frac{1}{2}\left\{(t_{2k-2i-1}+t_{2k-2i})(t_{2k-2j}-t_{2k}) - (t_{2k-2i-1}-t_{2k-2i+1})\sum_{v=2k-2j+1}^{2k} t_{-v}\right\}.$$

Therefore, we obtain the result

$$\int_{a_k} \alpha_i \beta_j = \frac{-1}{2(g+1)^2} \frac{1}{2} \Re \left\{ 2(t_{2k-2i-1} + t_{2k-2i})(t_{2k-2j} - t_{2k}) \right.$$
$$\left. - 2(t_{2k-2i-1} - t_{2k-2i+1}) \sum_{\substack{v=2k-2j+1 \\ \text{even}}}^{2k} t_v \right\}$$
$$= \frac{-1}{2(g+1)^2} t_{2k-2i}(t_{2k-2j} - t_{2k}).$$

Similarly we compute the case (6). □

Using the symplectic basis $\{x_i, y_i\}_{i=1,2,\dots,g} \subset H_1(C; \mathbf{Z})$ stated in §3, we choose a basis of $K$ as follows:

$$\left\{ \begin{array}{lcc} (1) & z_i \otimes z'_j & (i \neq j) \\ (2) & x_i \otimes y_i - x_1 \otimes y_1 & (i \neq 1) \\ (3) & x_i \otimes y_i + y_i \otimes x_i & (i = 1, 2, \dots, g) \\ (4) & z_i \otimes z_i & (i = 1, 2, \dots, g) \end{array} \right\},$$

where $z_i$ denotes $x_i$ or $y_i$, and so on. By the definition of the pointed harmonic volume $I_v$, we obtain

$$I_v((x_i \otimes y_i + y_i \otimes x_i) \otimes z''_k) \equiv 0 \mod \mathbf{Z} \quad \text{for any } i, k,$$

and

$$I_v(z_i \otimes z_i \otimes z''_k) \equiv \begin{cases} 1/2 \mod \mathbf{Z} & \text{if } z_i \otimes z_i \otimes z''_k = x_i \otimes x_i \otimes y_i \text{ or } y_i \otimes y_i \otimes x_i, \\ 0 \mod \mathbf{Z} & \text{otherwise}, \end{cases}$$

for any hyperelliptic curve $C$. It is enough to consider the case (1) and (2). For the rest of this paper, we omit mod $\mathbf{Z}$, unless otherwise stated.

We compute the pointed harmonic volume of $(C_0, Q_0)$. From Lemma 5.1, Lemma 3.8 in [6], and the equation $\int_{e_j} \eta = 0$ (Lemma 4.2 in [6]), it is to show

PROPOSITION 5.2. *Case* (1). *If* $i \neq k$ *and* $j \neq k$, *then we have*

$$I_{Q_0}(z_i \otimes z'_j \otimes z''_k) \equiv 0.$$

*If* $i = k$ *or* $j = k$, *then we have*

$$I_{Q_0}(x_i \otimes x_j \otimes y_i) \equiv \mu,$$
$$I_{Q_0}(x_i \otimes y_j \otimes y_i) \equiv \begin{cases} (g - j + 1)\mu & \text{if } i < j, \\ (2g - j + 2)\mu & \text{if } i > j, \end{cases}$$

$$I_{Q_0}(y_i \otimes x_j \otimes x_i) \equiv (2g+1)\mu,$$

$$I_{Q_0}(y_i \otimes y_j \otimes x_i) \equiv \begin{cases} (g+j+1)\mu & \text{if } i < j, \\ j\mu & \text{if } i > j, \end{cases}$$

*Case* (2). *If* $i \neq k$ *and* $k \neq 1$, *then we have*

$$I_{Q_0}((x_i \otimes y_i - x_1 \otimes y_1) \otimes z_k'') \equiv 0.$$

*If* $i = k$ *or* $k = 1$, *then we have*

$$I_{Q_0}((x_i \otimes y_i - x_1 \otimes y_1) \otimes x_i) \equiv (g+2)\mu,$$

$$I_{Q_0}((x_i \otimes y_i - x_1 \otimes y_1) \otimes y_i) \equiv (2g-i+2)\mu,$$

$$I_{Q_0}((x_i \otimes y_i - x_1 \otimes y_1) \otimes x_1) \equiv g\mu,$$

$$I_{Q_0}((x_i \otimes y_i - x_1 \otimes y_1) \otimes y_1) \equiv (g+2)\mu.$$

*Here we denote* $\mu = 1/(2g+2)$.

*Remark* 5.3. From Remark 2.2, we do not need to compute $I_{Q_0}(x_j \otimes x_i \otimes y_i)$, $I_{Q_0}((y_i \otimes x_i - y_1 \otimes x_1) \otimes x_i)$, and so on.

We calculate the difference between $I_v$ and $I_{Q_0}$. For $h_1 \otimes h_2 \otimes h_3 \in K \otimes H$, we set $\Lambda_v(h_1 \otimes h_2 \otimes h_3) := I_v(h_1 \otimes h_2 \otimes h_3) - I_{Q_0}(h_1 \otimes h_2 \otimes h_3) \bmod \mathbf{Z}$. Let $\ell_v : [0,1] \to C_0$ be a path $t \mapsto (t\zeta^v, \sqrt{-1}\sqrt{1-t^{2g+2}}) \in C_0$. It is clear that $\ell_v^{-1} \cdot e_j \cdot \ell_v$'s are loops in $C_0$ at the base point $P_v$. From the equation (2.2) in [4], we have

Lemma 5.4.

$$\Lambda_v(h_1 \otimes h_2 \otimes h_3) \equiv (h_1, h_3)\int_{\ell_v} h_2 - (h_2, h_3)\int_{\ell_v} h_1 \mod \mathbf{Z}.$$

It is clear that

$$\int_{\ell_v} \alpha_i = \frac{1}{2(g+1)}\Re(t_{v-2i} + t_{v-2i+1}) \quad \text{and} \quad \int_{\ell_v} \beta_i = \frac{-1}{2(g+1)}\Re\left(\sum_{u=v-2i+1}^{v} t_u\right).$$

These equations and Lemma 5.4 give the following Lemma.

Lemma 5.5. *Case* (1). *If* $i \neq k$ *and* $j \neq k$, *then we have*

$$\Lambda_v(z_i \otimes z_j' \otimes z_k'') \equiv 0.$$

*If* $i = k$ *or* $j = k$, *then we have*

$$\Lambda_v(x_i \otimes x_j \otimes y_i) \equiv \begin{cases} g\mu & \text{if } v = 1 \text{ or } 2, \\ (2g+1)\mu & \text{if } v \neq 1 \text{ and } 2, \end{cases}$$

$$\Lambda_v(x_i \otimes y_j \otimes y_i) \equiv \begin{cases} j\mu & \text{if } v > 2i-1, \\ (g+j+1)\mu & \text{if } v \leq 2i-1, \end{cases}$$

$$\Lambda_v(y_i \otimes x_j \otimes x_i) \equiv \begin{cases} (g+2)\mu & \text{if } v = 1 \text{ or } 2, \\ \mu & \text{if } v \neq 1 \text{ and } 2, \end{cases}$$

$$\Lambda_v(y_i \otimes y_j \otimes x_i) \equiv \begin{cases} (2g-j+2)\mu & \text{if } v > 2i-1, \\ (g-j+1)\mu & \text{if } v \leq 2i-1. \end{cases}$$

*Case* (2). *If* $i \neq k$ *and* $k \neq 1$, *then we have*

$$\Lambda_v((x_i \otimes y_i - x_1 \otimes y_1) \otimes z_k'') \equiv 0.$$

*If* $i = k$ *or* $k = 1$, *then we have*

$$\Lambda_v((x_i \otimes y_i - x_1 \otimes y_1) \otimes x_i) \equiv \begin{cases} g\mu & \text{if } v = 2j-1 \text{ or } 2j, \\ (2g+1)\mu & \text{if } v \neq 2j-1 \text{ and } 2j, \end{cases}$$

$$\Lambda_v((x_i \otimes y_i - x_1 \otimes y_1) \otimes y_i) \equiv \begin{cases} i\mu & \text{if } v > 2i-1, \\ (g+i+1)\mu & \text{if } v \leq 2i-1, \end{cases}$$

$$\Lambda_v((x_i \otimes y_i - x_1 \otimes y_1) \otimes x_1) \equiv \begin{cases} (g+2)\mu & \text{if } v = 1 \text{ or } 2, \\ \mu & \text{if } v \neq 1 \text{ and } 2, \end{cases}$$

$$\Lambda_v((x_i \otimes y_i - x_1 \otimes y_1) \otimes y_1) \equiv \begin{cases} (2g+1)\mu & \text{if } v > 1, \\ g\mu & \text{if } v \leq 1. \end{cases}$$

By combining Proposition 5.2 and Lemma 5.5, we have the pointed harmonic volume $I_v$ of $(C_0, P_v)$.

THEOREM 5.6. *Case* (1). *Elements of* $K \otimes H$ *at which the values of the pointed harmonic volume* $I_v$ *are* $1/2 \bmod \mathbf{Z}$ *are given by*

$$\begin{array}{lll} x_i \otimes x_j \otimes y_i, & x_j \otimes x_i \otimes y_i & \text{if } v = 2j-1 \text{ or } 2j, \\ x_i \otimes y_j \otimes y_i, & y_j \otimes x_i \otimes y_i & \text{if } (i < j, v > 2j-1) \text{ or } (i > j, v \leq 2j-1), \\ y_i \otimes x_j \otimes x_i, & x_j \otimes y_i \otimes x_i & \text{if } v = 2j-1 \text{ or } 2j, \\ y_i \otimes y_j \otimes x_i, & y_j \otimes y_i \otimes x_i & \text{if } (i < j, v > 2j-1) \text{ or } (i > j, v \leq 2j-1). \end{array}$$

*The values at the other elements are* $0 \bmod \mathbf{Z}$.

*Case* (2). *Elements of* $K \otimes H$ *at which the values of the pointed harmonic volume* $I_v$ *are* $1/2 \bmod \mathbf{Z}$ *are given by*

$$\begin{array}{lll} (x_i \otimes y_i - x_1 \otimes y_1) \otimes x_i, & (y_i \otimes x_i - y_1 \otimes x_1) \otimes x_i & \text{if } v \neq 2i-1 \text{ and } 2i, \\ (x_i \otimes y_i - x_1 \otimes y_1) \otimes y_i, & (y_i \otimes x_i - y_1 \otimes x_1) \otimes y_i & \text{if } v \leq 2i-1, \\ (x_i \otimes y_i - x_1 \otimes y_1) \otimes x_1, & (y_i \otimes x_i - y_1 \otimes x_1) \otimes x_1 & \text{if } v \neq 1 \text{ and } 2, \\ (x_i \otimes y_i - x_1 \otimes y_1) \otimes y_1, & (y_i \otimes x_i - y_1 \otimes x_1) \otimes y_1 & \text{if } v > 1. \end{array}$$

*The values at the other elements are* $0 \bmod \mathbf{Z}$.

From Proposition 4.1, this theorem can be extended to any hyperelliptic curve $C$ with Weierstrass base points. But this extension is complicated. We reconsider Theorem 5.6 from a combinatorial viewpoint. We apply an element

$A \in K \otimes H$ to the identification (3.1) in the group $(K \otimes H)_{\mathbf{Z}_2}$. Then we have $(A \bmod 2) = \sum_{p,q,r \neq v} A_{p,q,r} f_p \otimes f_q \otimes f_r$, where $A_{p,q,r} \in \mathbf{Z}_2 = \{0, 1\}$. The notation $\#$ means the cardinality of a set. A counting function $\kappa_v : K \otimes H \to \frac{1}{2}\mathbf{Z}/\mathbf{Z} = \{0, 1/2\}$ is well-defined by

$$\kappa_v(A) := \frac{1}{2}(\#\{(p,q,r); A_{p,q,r} = 1, \#\{p,q,r\} = 2\}) \bmod \mathbf{Z}.$$

Here $\#\{p,q,r\} = 2$ means $p = q \neq r$ or $q = r \neq p$ or $r = p \neq q$. By the long but easy computation, we obtain the correspondence.

Corollary 5.7. *On the curve $C_0$, we have*

$$I_v(A) \equiv \kappa_v(A) \bmod \mathbf{Z}.$$

*Example* 5.8. (1) If $A = x_i \otimes x_j \otimes y_i$ ($i < j$ and $v = 2j - 1$), we have

$$\begin{aligned} \kappa_v(A) &= \kappa_v((f_{2i-1} + f_{2i}) \otimes f_{2j} \otimes (f_0 + f_1 + \cdots + f_{2i-1})) \\ &\equiv \kappa_v(f_{2i-1} \otimes f_{2j} \otimes f_{2i-1}) = 1/2. \end{aligned}$$

(2) If $A = x_i \otimes x_j \otimes y_i$ ($i > j$ and $2i < v$), we have

$$\begin{aligned} \kappa_v(A) &= \kappa_v((f_{2i-1} + f_{2i}) \otimes (f_{2j-1} + f_{2j}) \otimes (f_0 + f_1 + \cdots + f_{2i-1})) \\ &\equiv \kappa_v(f_{2i-1} \otimes f_{2j-1} \otimes f_{2j-1} + f_{2i-1} \otimes f_{2j-1} \otimes f_{2i-1} + f_{2i-1} \otimes f_{2j} \otimes f_{2j} \\ &\quad + f_{2i-1} \otimes f_{2j} \otimes f_{2i-1} + f_{2i} \otimes f_{2j-1} \otimes f_{2j-1} + f_{2i} \otimes f_{2j} \otimes f_{2j}) \\ &= 1/2 + 1/2 + 1/2 + 1/2 + 1/2 + 1/2 \equiv 0. \end{aligned}$$

## 6. A combinatorial formula of $I_v$

In this section, we compute the pointed harmonic volume $I_v = I_{P_v}$ of $(C, P_v)$ by another combinatorial way. Let $S_{2g+1}$ be the $(2g+1)$-th symmetric group. Using the natural projection $\Delta_g^1 \to S_{2g+1}$, the group $H_{\mathbf{Z}_2}$ is naturally considered as a $\mathbf{Z}_2 S_{2g+1}$-module (Arnol'd, V. I. [1]). From the slight modification of Lemma 5.5 and Proposition 5.7 in [6], we have

Lemma 6.1.

$$H^0(\Delta_g^1; (K \otimes H)^*) = H^0(S_{2g+1}; (H^{\otimes 3})^*) = \mathbf{Z}_2.$$

*Moreover the unique nontrivial element $\psi_v \in H^0(S_{2g+1}; (H^{\otimes 3})^*)$ is an $S_{2g+1}$-homomorphism $H^{\otimes 3} \to \mathbf{Z}_2$ defined by*

$$\psi_v(f_i \otimes f_j \otimes f_k) = \begin{cases} 1 & \textit{if } \#\{i,j,k\} = 2, \\ 0 & \textit{otherwise}, \end{cases}$$

*for any $i$, $j$, $k$ except for $v$.*

From Lemma 6.1, we have

THEOREM 6.2. *For $A \in K \otimes H$, we have*

$$I_v(A) \equiv \kappa_v(A) \mod \mathbf{Z}.$$

Using the equation $f_i = \pi(e'_v) + \pi(e'_i)$, we obtain $(A \bmod 2) = \sum_{p,q,r} A'_{p,q,r} \pi(e'_p) \otimes \pi(e'_q) \otimes \pi(e'_r)$. Another counting function $\kappa'_v : K \otimes H \to \frac{1}{2}\mathbf{Z}/\mathbf{Z} = \{0, 1/2\}$ is defined by

$$\kappa'_v(A) := \frac{1}{2}(\#\{(p,q,r); A'_{p,q,r} = 1, \#\{p,q,r\} = 2, p,q,r \neq v\}) \mod \mathbf{Z}.$$

COROLLARY 6.3.

$$I_v(A) \equiv \kappa'_v(A) \mod \mathbf{Z}.$$

*Proof.* We use the notation $e(p,q,r) = \pi(e'_p) \otimes \pi(e'_q) \otimes \pi(e'_r)$ only here. The equation

$$\begin{aligned} f_p \otimes f_q \otimes f_r = {} & e(p,q,r) + e(p,q,v) + e(p,v,r) + e(p,v,v) \\ & + e(v,q,r) + e(p,q,v) + e(v,v,r) + e(v,v,v) \end{aligned}$$

gives $\kappa_v(A) \equiv \kappa'_v(A)$. □

## REFERENCES


[1] V. I. ARNOL'D, A remark on the branching of hyperelliptic integrals as functions of the parameters, Funkcional. Anal. i Priložen. **2** (1968), 1–3.

[2] K. T. CHEN, Algebras of iterated path integrals and fundamental groups, Trans. Amer. Math. Soc. **156** (1971), 359–379.

[3] H. M. FARKAS AND I. KRA, Riemann surfaces, Graduate texts in mathematics **71**, 2nd ed., Springer-Verlag, New York, 1992.

[4] B. HARRIS, Harmonic volumes, Acta Math. **150** (1983), 91–123.

[5] MICHAEL J. PULTE, The fundamental group of a Riemann surface: mixed Hodge structures and algebraic cycles, Duke Math. J. **57** (1988), 721–760.

[6] Y. TADOKORO, The harmonic volumes of hyperelliptic curves, Publ. Res. Inst. Math. Sci. **41** (2005), 799–820.



Yuuki Tadokoro
NATURAL SCIENCE EDUCATION
KISARAZU NATIONAL COLLEGE OF TECHNOLOGY
2-11-1, KIYOMIDAI-HIGASHI
KISARAZU, CHIBA 292-0041
JAPAN
E-mail: tado@nebula.n.kisarazu.ac.jp