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AN OBSTRUCTION FOR CHERN CLASS FORMS TO BE HARMONIC*

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Abstract

This is a publication of an old preprint which extended Futaki character to an obstruction for Chern class forms to be harmonic. Although there were considerable developments on the subject, this paper is presented "as was".

In his paper [1] Futaki gave a new obstruction to the existence of Einstein Kähler metrics on compact complex manifolds of positive first Chern class. Following Futaki's construction, we show that the straightforward generalization gives an obstruction of Chern forms to be harmonic.

Let M be an m-dimensional compact Kähler manifold, $ds^2 = 2\sum g_{ij}\, dz^i \wedge d\overline{z^j}$ be a Kähler metric on M. Then its Kähler form ω, connection form θ, and curvature form Ω are given as follows.

$$\omega = \sqrt{-1}\sum g_{i\bar{j}}\, dz^i \wedge d\overline{z^j},$$
$$\theta^i_j = \sum g^{i\bar{k}} \partial g_{j\bar{k}},$$
$$\Omega^i_j = d\theta^i_j + \sum \theta^i_k \wedge \theta^k_j = d\theta + 1/2[\theta, \theta].$$

Chern class forms c_k with respect to the metric are defined by

$$\det\left(1 - \frac{t}{2\pi\sqrt{-1}}\Omega\right) = \sum t^k c_k(\Omega),$$

where t is a parameter.

Let $J(M)$ be the Lie algebra of the group of the holomorphic transformations. Then each element of $J(M)$ can be regarded as a vector field on M. Although Futaki considered $J(M)$ as a set of complex holomorphic vector fields, we consider $J(M)$ as a set of real vector fields. For d-closed real (k,k)-form ϕ,

* Editor's note: This paper, written in 1983 and hitherto unpublished, is being published here at the request of A. Futaki, who needs results of this paper in his own paper which appears elsewhere in this volume.

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we denote its harmonic part by $H\phi$, then there exists a real $(k-1,k-1)$-form ψ such that

$$\phi - H\phi = \sqrt{-1}\partial\bar{\partial}\psi.$$

ln particular, there exists a real $(k-1,k-1)$-form F_k such that

$$c_k - Hc_k = \sqrt{-1}\partial\bar{\partial}F_k.$$

We define a linear function f_k on $J(M)$ by

$$f_k(X) = \int L_X F_k \wedge \omega^{m-k+1} \quad \text{for } X \in J(M),$$

where L_X is the Lie differentiation with respect to X.

Theorem 1. *The function f_k does not depend on the choice of Kähler metric, provided the cohomology class $[\omega]$ of the Kähler form ω is fixed. If M admits a metric with harmonic k-th Chern class form c_k, then f_k vanishes.*

If $k = 1$ and $[\omega]$ is equal to the first Chern class, this is nothing but Futaki's main theorem. Calabi obtained the above theorem in the case of $k = 1$. He state it in terms of constancy of scalar curvature, which is equivalent to harmonicity of first Chern class form (or Ricci form), but his proof is different from ours. The last statement of the theorem is trivial, we only have to prove the independence. First we show that f_k is well-defined, namely independent of the choice of F_k.

$$\begin{aligned} f_k(X) &= \int L_X F_k \wedge \omega^{m-k+1} \\ &= -\int F_k \wedge L_X \omega^{m-k+1} \\ &= -(m-k+1)\int F_k \wedge \omega^{m-k} \wedge L_X\omega \end{aligned}$$

There exists a real valued function v so that $L_X\omega = \sqrt{-1}\partial\bar{\partial}v$, and

$$\begin{aligned} f_k(X) &= -(m-k+1)\int F_k \wedge \omega^{m-k} \wedge \sqrt{-1}\partial\bar{\partial}v \\ &= -(m-k+1)\int \sqrt{-1}\partial\bar{\partial}F_k \wedge \omega^{m-k} \wedge v \\ &= -(m-k+1)\int (c_k - Hc_k) \wedge \omega^{m-k} \wedge v \end{aligned}$$

Thus f_k is well-defined. To prove the independence from metrics, it is convenient to write f_k in a different way.

$$\int L_X F_k \wedge \omega^{m-k+1} = (m-k+1)\int L_X(F_k \wedge \omega^{m-k}) \wedge \omega.$$

Note that

$$c_k \wedge \omega^{m-k} - Hc_k \wedge \omega^{m-k} = \sqrt{-1}\partial\bar{\partial}F_k \wedge \omega^{m-k}$$
$$Hc_k \wedge \omega^{m-k} = H(c_k \wedge \omega^{m-k}) = \text{const}\,\omega^m.$$

and

$$\det\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right) = \sum t^k c_k \wedge \omega^{m-k}.$$

Thus denoting the invariant polynomial corresponding to the determinant by P, we conclude that the theorem is equivalent to the following statement.

If we define $f(t,X)$ for $X \in J(M)$ in the following way, then $f(t,X)$ is independent of the choice of the metric provided the Kähler class $[\omega]$ is fixed.

Let F_t be a $(m-1,m-1)$-form so that

$$P\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^m\right) - HP\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^m\right) = \sqrt{-1}\partial\bar{\partial}F_t,$$

for instance,

$$F_t = \sum t^k F_k \wedge \omega^{m-k}.$$

We define

$$f(t,X) = \int L_X F_t \wedge \omega.$$

Here again the choice of F_t does not matter because of the same reason as before.

Since the space of metrics with the fixed cohomology class is connected, to prove the statement it is sufficient to show the derivative of f as a function of metrics is zero. It is easily seen that any infinitismal variation of the Kähler form with fixed cohomology class is given by $\sqrt{-1}\partial\bar{\partial}u$ with some real valued function u on M. We denote the infinitismal variation of a tensor ϕ which depends on metrics by $\delta\phi(u)$. In a geodesic coordinate at each point,

$$\delta\theta^i_j(u) = \partial u^i_j = D'u^i_j,$$
$$\delta\Omega^i_j(u) = d(\delta\theta) + [\delta\theta,\theta] = D\delta\theta = DD'u^i_j = D''D'u^i_j,$$

where $u^i_j = \sum u_{j\bar{k}}g^{\bar{k}i}$ is given by raising indices of the complex Hessian of u, D is covariant exterior differentiation, and D', D'' are its holomorphic and anti-holomorphic parts.

$$\delta f(t,X)(u) = \int L_X(\delta F_t)(u) \wedge \omega + \int L_X F_t \wedge \delta\omega(u)$$
$$= -\int \delta F_t(u) \wedge L_X\omega + \int L_X F_t \wedge \sqrt{-1}\partial\bar{\partial}u$$

$$\begin{aligned}
&= -\int \delta F_t(u) \wedge \sqrt{-1}\partial\bar{\partial}v + \int L_X F_t \wedge \sqrt{-1}\partial\bar{\partial}u \\
&= -\int \sqrt{-1}\partial\bar{\partial}\delta F_t(u) \wedge v + \int \sqrt{-1}\partial\bar{\partial}L_X F_t \wedge u \\
&= -\int \delta(\sqrt{-1}\partial\bar{\partial}F_t)(u) \wedge v + \int L_X(\sqrt{-1}\partial\bar{\partial}F_t) \wedge u \\
&= -\int \delta P\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^m\right)(u) \wedge v \\
&\quad + \int \delta HP\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^m\right)(u) \wedge v \\
&\quad + \int L_X P\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^m\right) \wedge u \\
&\quad - \int L_X HP\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^m\right) \wedge u
\end{aligned}$$

Here we used the fact that δ and L_X commute with ∂ and $\bar{\partial}$; for L_X it is because X is an infinitismal holomorphic transformation. By the definition of L_X, it is easy to see that if a tensor ϕ is defined by the metric in a canonical way, $L_X\phi$ is nothing but $\delta\phi(v)$, where v is a real valued function satisfying $L_X\omega = \sqrt{-1}\partial\bar{\partial}v$. Hence,

$$\begin{aligned}
\delta f(t,X)(u) = &-\int \delta P\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^m\right)(u) \wedge v \\
&+ \int \delta HP\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^m\right)(u) \wedge v \\
&+ \int \delta P\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^m\right)(v) \wedge u \\
&- \int \delta HP\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^m\right)(v) \wedge u
\end{aligned}$$

So what we have to prove is that the linearlizations of

$$P\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^m\right) \quad \text{and} \quad HP\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^m\right)$$

are symmetric. Symmetry of $HP\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^m\right)$ is easily seen, because $HP\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^m\right) = p(t)\omega^m$ with a polynomial p which is independent of ω, thus $\delta HP\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^m\right)(u) = p(t)m\omega^{m-1} \wedge \sqrt{-1}\partial\bar{\partial}u = p(t)\omega^m \triangle u$, and $\int p(t)\omega^m \triangle u \wedge v$ is clearly symmetric.

$$
\begin{aligned}
&\delta P\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^m\right)(u) \\
&= mP\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^{m-1} \wedge \left(\delta\omega - \frac{t}{2\pi\sqrt{-1}}\delta\Omega\right)(u)\right) \\
&= mP\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^{m-1} \wedge \left(\sqrt{-1}\partial\bar{\partial}u - \frac{t}{2\pi\sqrt{-1}}D''D'u^i_j\right)\right) \\
&= -mP\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^{m-1} \wedge \left(\sqrt{-1}D''D'\left(u - \frac{t}{2\pi}u^i_j\right)\right)\right) \\
&= m\sqrt{-1}\partial\bar{\partial}P\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^{m-1} \wedge \left(u - \frac{t}{2\pi}u^i_j\right)\right),
\end{aligned}
$$

where we used $D\omega = D\Omega = 0$.

$$
\begin{aligned}
&\int \delta P\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^m\right)(u) \wedge v \\
&= \int mP\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^{m-1} \wedge \left(u - \frac{t}{2\pi}u^i_j\right)\right) \wedge \sqrt{-1}\partial\bar{\partial}v \\
&= \int mP\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^{m-1} \wedge \left(u\sqrt{-1}\partial\bar{\partial}v - \frac{t}{2\pi}u^i_j\sqrt{-1}\partial\bar{\partial}v\right)\right) \\
&= \int mP\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^{m-1} \wedge (u\sqrt{-1}\partial\bar{\partial}v)\right) \\
&\quad - \int mP\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^{m-1}\left(\frac{t}{2\pi}u^i_j\sqrt{-1}\partial\bar{\partial}v\right)\right)
\end{aligned}
$$

Clearly the first term is symmetric in u and v, we want to prove that the second term is symmetric, in fact we will show the integrand is symmetric. We introduce some algebraic notations. Let E be a k-dimensional complex vector space. For the tensor product of the exterior algebra over E, we define a wedge product in the following way.

$$(a \otimes b) \wedge (c \otimes d) = (a \wedge c) \otimes (b \wedge d) \quad \text{for } a, b, c, d \in \bigwedge E,$$

and extend it linearly. We define the transpose ${}^t\phi$ of a element ϕ of $\bigwedge E \otimes \bigwedge E$ as follows.

$${}^t(a \otimes b) = b \otimes a \quad \text{for } a, b \in \bigwedge E,$$

and extend it linearly. Then we get that

$${}^t(\phi \wedge \psi) = {}^t\phi \wedge {}^t\psi, \quad \text{for } \phi, \psi \in \bigwedge E \otimes \bigwedge E.$$

We say that $\phi \in \bigwedge^k E \otimes \bigwedge^k E$ is symmetric, if ${}^t\phi = \phi$. Using a non-zero element $v \in \bigwedge^k E$, we can identify $\bigwedge^k E$ and $\bigwedge^k E \otimes \bigwedge^k E$ with $\mathbf{C}$.

We apply the above notation for $E = T_p^* M$: complexified cotangent space at each point $p \in M$. We identify $\mathrm{End}(T_p M)$ (the space of endmorphisms of holomorphic tangent space) with the space of $(1,1)$-forms in the following way. In a local coordinate system

$$T_j^i \mapsto \sqrt{-1} \sum g_{k\bar{j}} T_i^k \, dz^i \wedge d\overline{z^j}.$$

For example the identity map is identified with ω. We denote the obtained form by the same letter T, then $T^m = \det(T_j^i)\omega^m$. Thus using the identification $\bigwedge^{2m} T_p M = \mathbf{C}$ by the volume form,

$$T^m = m! \det(T_j^i)$$

We apply this notation for ω and Ω, then we get that

$$\int P\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^{m-1} \wedge (u_j^i \sqrt{-1}\partial\bar{\partial}v)\right)$$
$$= 1/m! \int \left(\omega \otimes \omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^{m-1} \wedge (\sqrt{-1}\partial\bar{\partial}u \otimes \sqrt{-1}\partial\bar{\partial}v),$$

where the integrand in the right hand side is identified with a scalar and the integral is taken over the volume form. Then

$$\left(\omega \otimes \omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^{m-1} \wedge (\sqrt{-1}\partial\bar{\partial}u \otimes \sqrt{-1}\partial\bar{\partial}v)$$
$$= {}^t\left[\left(\omega \otimes \omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^{m-1} \wedge (\sqrt{-1}\partial\bar{\partial}u \otimes \sqrt{-1}\partial\bar{\partial}v)\right]$$
$$= \left(\omega \otimes \omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^{m-1} \wedge (\sqrt{-1}\partial\bar{\partial}v \otimes \sqrt{-1}\partial\bar{\partial}u),$$

where we used the fact that any scalar and Ω are symmetric. Thus we get that the integrand is symmetric in u and v.

Next we prove a theorem which corresponds to the theorem 3.1 in [1]. Calabi also obtained it in the case $k = 1$.

THEOREM 2. *Let ϕ be a biholomorphic map which preserves the Kähler class $[\omega]$. If we denote the pull back of a tensor T by $\phi^* T$ then*

$$f(t, \phi^* X) = f(t, X) \quad \textit{for } x \in J(M).$$

Proof. Pulling back the equation which defines F_t, we find that $\phi^{-1^*} F_t$ plays the roll of F_t for the metric $\phi^{-1^*}\omega$. Because $[\phi^{-1^*}\omega] = [\omega]$, and theorem 1,

$$f(t, X) = \int L_X \phi^{-1^*} F_t \wedge \phi^{-1^*}\omega = \int \phi^{-1^*}(L_{\phi^* X} F_t \wedge \omega) = \int L_{\phi^* X} F_t \wedge \omega = f(t, \phi^* X).$$

Corollary 1. *If* $X, Y \in J(M)$, $f(t, [X, Y]) = 0$.

Proof. Let ϕ_s be the one parameter family of biholomorphic maps generated by X, then ϕ_s preserves the Kähler class. Thus

$$f(t, {\phi_s}^* Y) = f(t, Y).$$

Differentiating in s, we get the desired result.

From now on we will deal with more than one manifold, thus when we want to make clear which manifold is considered, we will write explicitly. Define $p(t, M)$ as

$$p(t, M) = \int P\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^m\right) = \int HP\left(\left(\omega - \frac{t}{2\pi\sqrt{-1}}\Omega\right)^m\right),$$

which depends only on $[\omega]$. Let N be another compact Kähler manifold of dimension n. For $M \times N$ we assign a Kähler form $\omega_M + \omega_N$, where ω_M, ω_N are the Kähler forms of M, N, respectively.

Theorem 3.

$$f(t, X, M \times N) = f(t, X_M, M)p(t, N) + p(t, M)f(t, X_N, N),$$

for $X \in J(M \times N) = J(M) \times J(N)$, *where* X_M, X_N *mean* $J(M)$, $J(N)$ *components of* X, *respectively.*

Proof.

$$\det\left(\omega_{M\times N} - \frac{t}{2\pi\sqrt{-1}}\Omega_{M\times N}\right) = \det\left(\omega_M - \frac{t}{2\pi\sqrt{-1}}\Omega_M\right) \times \det\left(\omega_N - \frac{t}{2\pi\sqrt{-1}}\Omega_N\right).$$

$$\begin{aligned}
&\det\left(\omega_{M\times N} - \frac{t}{2\pi\sqrt{-1}}\Omega_{M\times N}\right) - H\det\left(\omega_{M\times N} - \frac{t}{2\pi\sqrt{-1}}\Omega_{M\times N}\right)\\
&\quad = \left[\det\left(\omega_M - \frac{t}{2\pi\sqrt{-1}}\Omega_M\right) - H\det\left(\omega_M - \frac{t}{2\pi\sqrt{-1}}\Omega_M\right)\right]\\
&\qquad \times \det\left(\omega_N - \frac{t}{2\pi\sqrt{-1}}\Omega_N\right) + H\det\left(\omega_M - \frac{t}{2\pi\sqrt{-1}}\Omega_M\right)\\
&\qquad \times \left[\det\left(\omega_N - \frac{t}{2\pi\sqrt{-1}}\Omega_N\right) - H\det\left(\omega_N - \frac{t}{2\pi\sqrt{-1}}\Omega_N\right)\right]\\
&\quad = \sqrt{-1}\partial\bar\partial\left[F_t(M)\det\left(\omega_N - \frac{t}{2\pi\sqrt{-1}}\Omega_N\right) + H\det\left(\omega_M - \frac{t}{2\pi\sqrt{-1}}\Omega_M\right)F_t(N)\right].
\end{aligned}$$

$$F_t(M \times N) = F_t(M)\det\left(\omega_N - \frac{t}{2\pi\sqrt{-1}}\Omega_N\right) + H\det\left(\omega_M - \frac{t}{2\pi\sqrt{-1}}\Omega_M\right)F_t(N).$$

$$
\begin{aligned}
f(t,X,M\times N) &= \int L_X F_t(M\times N)\wedge\omega_{M\times N}\\
&= -\int F_t(M)\,\det\left(\omega_N-\frac{t}{2\pi\sqrt{-1}}\Omega_N\right)\wedge L_X\omega_{M\times N}\\
&\quad -\int H\,\det\left(\omega_M-\frac{t}{2\pi\sqrt{-1}}\Omega_M\right)\wedge F_t(N)\wedge L_X\omega_{M\times N}.
\end{aligned}
$$

For example, if $X=X_M$, $L_X\omega_{M\times N}=L_X(\omega_M+\omega_N)=L_X\omega_M=\sqrt{-1}\partial\bar{\partial}v$ with a real function v on M, the second term vanishes, and using the Fubini's theorem,

$$
f(t,X,M\times N)=\int F_t(M)\wedge L_X\omega_M\int\det\left(\omega_N-\frac{t}{2\pi\sqrt{-1}}\Omega_N\right)=f(t,X,M)p(t,N).
$$

Similarly for $X=X_N$.

Theorem 3 corresponds to the theorem 3.4 in [1]. For a Kähler manifold M with a Kähler form ω_M, let us define λM, $\lambda>0$ as the same manifold with the Kähler form $\lambda\omega_M$.

Theorem 4.

$$
\begin{aligned}
f(t,X,\lambda M) &= \lambda^{m+1}f(t/\lambda,X,M),\\
p(t,\lambda M) &= \lambda^m p(t/\lambda,M),
\end{aligned}
$$

Proof. Because $\Omega_{\lambda M}=\Omega_M$.

Theorem 5. *f_k is non-trivial obstruction, that is there exist a Kähler manifold M and Kähler class such that f_k is not identically zero.*

Proof. Let us redefine f_k as the t^k coefficient of $f(t,X,M)$, which is different from the previous one only by some non-zero constant. As Futaki showed, there exists a Kähler manifold M which has non-trivial f_1. We consider $M\times\lambda P^n$, where P^n is the projective space with the Fubini-Study metric. Then for $X\in J(M)$,

$$
f(t,X,M\times\lambda P^n)=f(t,X,M)\lambda^n p(t/\lambda,P^n),
$$

$$
f_k(X,M\times\lambda P^n)=\sum f_i(X,M)p_{k-i}\lambda^{n-k+i},
$$

where p_i are the coefficients of $p(t,P^n)$ which is non-zero. Since $f_1(X,M)$ is not identically zero, we get that for some λ, $f_k(X,M\times\lambda P^n)$ is not zero.

References

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