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MIRROR SYMMETRY, KOBAYASHI'S DUALITY, AND SAITO'S DUALITY

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Abstract

M. Kobayashi introduced a notion of duality of weight systems. We tone this notion slightly down to a notion called coupling. We show that coupling induces a relation between the reduced zeta functions of the monodromy operators of the corresponding singularities generalizing an observation of K. Saito concerning Arnold's strange duality. We show that the weight systems of the mirror symmetric pairs of M. Reid's list of 95 families of Gorenstein K3 surfaces in weighted projective 3-spaces are strongly coupled. This includes Arnold's strange duality where the corresponding weight systems are strongly dual in Kobayashi's original sense. We show that the same is true for the extension of Arnold's strange duality found by the author and C. T. C. Wall.

Introduction

The mirror symmetry of Calabi-Yau threefolds has attracted the attention of many physicists and mathematicians. One- and two-dimensional Calabi-Yau varieties are elliptic curves and K3 surfaces respectively. It is well-known that there exist 3 families of weighted projective elliptic plane curves. The cones over these curves are the simple-elliptic singularities of type $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ (see below). They are self-dual with respect to mirror symmetry.

M. Reid classified and listed all families of K3 weighted projective hypersurfaces with Gorenstein singularities (unpublished). It turned out that there are 95 such families. The cones over these surfaces are called simple K3 hypersurface singularities. These singularities were classified and thus Reid's list was rediscovered by T. Yonemura [Yo]. These surfaces include compactifications of the 14 exceptional unimodal hypersurface singularities of V. I. Arnold. It is well-known that the mirror symmetry between the corresponding families of

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K3 weighted projective hypersurfaces corresponds to Arnold's strange duality (see e.g. [D3]). S.-M. Belcastro [Be] determined for which of the 95 families the mirror symmetric family is again in Reid's list.

V. V. Batyrev [Ba] showed that the mirror symmetry of Calabi-Yau hypersurfaces in toric varieties is related to the polar duality between their Newton polytopes. M. Kobayashi [Ko] discovered that Arnold's strange duality corresponds to a duality of weight systems and this is related to Batyrev's result. K. Saito [S1, S2] observed that Arnold's strange duality corresponds to a duality between the characteristic polynomials of the monodromy operators of the corresponding dual singularities. In [Yu, Lect. 3, Problem 8.5] it is asked whether there are any possible relations among all these dualities and mirror symmetry. Here we give a partial answer to this question extending [E3] where it was shown that Saito's duality can be derived from polar duality.

We consider weight systems $(a_1, \ldots, a_n; h)$ with

$$0 < a_0 := h - \sum_{i=0}^{n} a_i, \quad a_0 | h.$$

Let $f(x_1, \ldots, x_n)$ be polynomial of weighted degree h. Then the hypersurface $\tilde{X}$ in the weighted projective space given by

$$x_0^{h/a_0} + f(x_1, \ldots, x_n) = 0$$

is a Calabi-Yau hypersurface (if it is quasismooth). We introduce a notion of coupling of such weight systems which tones down Kobayashi's notion of duality. We relate this to polar duality in the same way as in [E3]. The basic notion is the notion of a weighted magic square C. The partner weight system corresponds to the transpose of this matrix. The natural $\mathbf{C}^*$-action on $\mathbf{C}^n$ induces a monodromy transformation on the homology of the fibre

$$F = \{(x_1, \ldots, x_n) \in \mathbf{C}^n \mid f(x_1, \ldots, x_n) = 1\}.$$

We consider the reduced zeta function $\tilde{\zeta}_C(t)$ of this monodromy operator. We indicate how this rational function can be computed from the matrix C. We show that the function $\tilde{\zeta}_{C^t}(t)$ associated to the transpose matrix C^t is in a sense dual to $\tilde{\zeta}_C(t)$ which generalizes Saito's duality and coincides with it in the case when $n = 3$ and $a_0 = b_0 = 1$.

Then we investigate the coupling of weight systems for the weight systems of Belcastro's list of mirror symmetric pairs inside Yonemura's list of 95 weight systems. It turns out that for any mirror symmetric pair the corresponding weight systems are (strongly) coupled. The cases of Arnold's strange duality are exactly those with $a_0 = b_0 = 1$ where the Saito duality holds in its strong form. Here the corresponding weight systems are strongly dual in Kobayashi's original sense. The 31 cases with $a_0 = 1$ and $f(x_1, x_2, x_3) = 0$ having an isolated singularity at the origin are compactifications of the 31 Fuchsian singularities classified by I. Dolgachev [D1], I. G. Sherbak [Sh], and Ph. Wagreich [Wag]. Many, but not all, of them have mirror symmetric partners inside Yonemura's

list. In [E4] we asked whether the mirror symmetric families to the Fuchsian singularities not involved in Arnold's strange duality and its extension by the author and C. T. C. Wall are realized by singularities. Here we find for 7 of these Fuchsian singularities singularities which are related to these singularities in a way explained in Section 3.

Finally we consider the extension of Arnold's strange duality found by the author and C. T. C. Wall. This again corresponds to mirror symmetry. Here also weighted complete intersections in weighted projective 4-spaces are involved. We associate weight systems to these varieties and we show that the mirror symmetric pairs have (strongly) dual weight systems.

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1. Duality of weight systems

An $(n+1)$-tuple of positive integers $W_{\mathbf{a}} = (a_1, \ldots, a_n; h)$ is called a *weight system*. The integers a_i are called the *weights* of $W_{\mathbf{a}}$ and h is called the *degree* of $W_{\mathbf{a}}$.

Two weight systems $W = (a_1, \ldots, a_n; h)$ and $W' = (a_1', \ldots, a_n'; h')$ are *equivalent* if there exists a permutation $\sigma \in \mathfrak{S}_n$ and a rational number λ such that $\lambda a_{\sigma(i)} = a_i'$ for $i = 1, \ldots, n$ and $\lambda h = h'$. The weight system is called *reduced* if

$$\gcd(a_1, \ldots, a_n) = 1.$$

Each equivalence class contains a unique reduced weight system satisfying

$$a_1 \leq \cdots \leq a_n.$$

Let

$$a_0 := h - \sum_{i=1}^{n} a_i.$$

In the sequel we shall assume that our weight system is reduced, satisfies $a_1 \leq \cdots \leq a_n$, and that $a_0 \neq 0$.

If $a_0 > 0$ and $a_0 | h$ then we shall call the weight system a *Calabi-Yau weight system*. The reason for this is the following: Let $\mathbf{P}(a_0, \mathbf{a}) = \mathbf{P}(a_0, \ldots, a_n)$ be the weighted complex projective space of weight $(a_0, \ldots, a_n)$, i.e. the projective variety Proj $\mathbf{C}[x_0, \ldots, x_n]$ where the degree of x_i is a_i. Denote by $(x_0 : \cdots : x_n)$ the natural homogeneous coordinates of $\mathbf{P}(a_0, \mathbf{a})$. Let $f(x_1, \ldots, x_n)$ be an equation of weighted degree h and define

$$\tilde{f}(x_0, x_1, \ldots, x_n) := x_0^{h/a_0} + f(x_1, \ldots, x_n).$$

Consider the hypersurface $\tilde{X} := \tilde{f}^{-1}(0)$ in $\mathbf{P}(a_0, \mathbf{a})$. Let $\mathbf{C}^{n+1}$ be the affine $(n+1)$-space with coordinates $(x_0, \ldots, x_n)$. Assume that the hypersurface $\tilde{X}$ is

quasismooth, i.e. the cone $C_{\tilde{X}} = \{\tilde{f} = 0\}$ over $\tilde{X}$ in $\mathbf{C}^{n+1}$ is smooth outside of the origin. By [D2, Theorem 3.3.4] the dualizing sheaf $\omega_{\tilde{X}}$ satisfies $\omega_{\tilde{X}} = \mathcal{O}_{\tilde{X}}$. Therefore $\tilde{X}$ is a (possibly singular) Calabi-Yau variety.

We recall some definitions of [Ko]. Let $W_{\mathbf{a}} = (a_1, \ldots, a_n; h)$ and $W_{\mathbf{b}} = (b_1, \ldots, b_n; k)$ be two weight systems.

DEFINITION. Let C be an $n \times n$ matrix with entries in the non-negative integers. The matrix C is called a *weighted magic square* of weight $(W_{\mathbf{a}}, W_{\mathbf{b}})$ if

$$C(a_1, \ldots, a_n)^t = (h, \ldots, h)^t \quad \text{and}$$
$$(b_1, \ldots, b_n)C = (k, \ldots, k).$$

Let $C = (c_{ij})$ be a weighted magic square of weight $(W_{\mathbf{a}}, W_{\mathbf{b}})$. Let B be the $n \times n$ matrix $(c_{ij} - 1)$. Let A be the inverse matrix of B. By [Ko, Lemma 2.3.5(1)], $(\det C)/h = (\det B)/a_0$ and $(\det C)/k = (\det B)/b_0$ and both numbers are integers.

LEMMA 1. *We have*

$$A(1, \ldots, 1)^t = \left(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}\right)^t,$$
$$(1, \ldots, 1)A = \left(\frac{b_1}{b_0}, \ldots, \frac{b_n}{b_0}\right).$$

Proof. By definition, $BA(1, \ldots, 1)^t = (1, \ldots, 1)^t$. We have

$$B\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} c_{11} - 1 & \cdots & c_{1n} - 1 \\ \vdots & \ddots & \vdots \\ c_{n1} - 1 & \cdots & c_{nn} - 1 \end{pmatrix}\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$
$$= \begin{pmatrix} h - \sum_{i=1}^n a_i \\ \vdots \\ h - \sum_{i=1}^n a_i \end{pmatrix} = \begin{pmatrix} a_0 \\ \vdots \\ a_0 \end{pmatrix}.$$

This implies the first claim. The second claim follows in the same way. □

It follows from Lemma 1 that the weight systems $W_{\mathbf{a}}$ and $W_{\mathbf{b}}$ can be retrieved from the matrix C.

DEFINITION. A weighted magic square C of weight $(W_{\mathbf{a}}, W_{\mathbf{b}})$ is called *primitive* if $|\det C| = h = k$.

The weight systems $W_{\mathbf{a}}$ and $W_{\mathbf{b}}$ are called *dual* if there exists a primitive weighted magic square of weight $(W_{\mathbf{a}}, W_{\mathbf{b}})$.

Two dual weight systems are called *strongly dual* if any row and any column of C contains at least one zero.

If two weight systems $W_{\mathbf{a}}$ and $W_{\mathbf{b}}$ are dual, then it follows that $k = h$ and $b_0 = a_0$. We tone down this definition to include the case when $a_0 \neq b_0$.

DEFINITION. A weighted magic square C of weight $(W_{\mathbf{a}}, W_{\mathbf{b}})$ is called *almost primitive* if $|\det C| = hb_0 = ka_0$.

The weight systems $W_{\mathbf{a}}$ and $W_{\mathbf{b}}$ are called *coupled* if there exists an almost primitive weighted magic square of weight $(W_{\mathbf{a}}, W_{\mathbf{b}})$.

Two coupled weight systems are called *strongly coupled* if any row and any column of C contains at least one zero.

Let $C = (c_{ij})$ be a weighted magic square of weight $(W_{\mathbf{a}}, W_{\mathbf{b}})$. We now assume that $a_0 > 0$. We show that the coupling of weight systems is related to the polar duality of associated Newton polytopes (cf. [Ko, E3]).

DEFINITION. The $(n-1)$-simplex $\Gamma(\mathbf{a})$ which is the convex hull of the row vectors of the matrix C in $\mathbf{R}^n$ is called a *Newton diagram* of the weight system $W_{\mathbf{a}}$.

The $(n-1)$-simplex $\Delta(\mathbf{a})$ which is the convex hull of the vectors $(h/a_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, h/a_n)$ in $\mathbf{R}^n$ is called the *full Newton diagram* of the weight system $W_{\mathbf{a}}$.

Let $\bar{\Delta}(\mathbf{a})$ be the n-simplex which is obtained from $\Delta(\mathbf{a})$ by taking the convex hull with the origin in $\mathbf{R}^n$ and translating it by the vector $(-1, \ldots, -1)$, i.e. $\bar{\Delta}(\mathbf{a})$ is the convex hull of the vectors $(-1 + h/a_1, -1, \ldots, -1), \ldots, (-1, \ldots, -1, -1 + h/a_n), (-1, \ldots, -1)$.

DEFINITION. Let $M \subset \mathbf{R}^n$. Let $\langle , \rangle$ denote the Euclidean scalar product of $\mathbf{R}^n$. The *polar dual* of M is the following subset of $\mathbf{R}^n$:

$$M^* := \{ y \in \mathbf{R}^n \mid \langle x, y \rangle \geq -1 \text{ for all } x \in M \}.$$

LEMMA 2. *The polar dual* $\bar{\Delta}(\mathbf{a})^*$ *is the n-simplex with vertices* $v_1 := (1, 0, \ldots, 0), \ldots, v_n := (0, \ldots, 0, 1)$, $v_0 := (-a_1/a_0, \ldots, -a_n/a_0)$.

Proof [Ko, Lemma 3.2]. □

PROPOSITION 1. *Let* ∇ *be the convex hull of the vectors* $v_1 - v_0, \ldots, v_n - v_0$ *in* $\mathbf{R}^n$. *Then, in the coordinate system given by taking the rows of A as basis vectors,* ∇ *is the convex hull of the columns of C, hence a Newton diagram of the partner weight system* $W_{\mathbf{b}}$.

Proof. By Lemma 2, the claim is equivalent to the following statement:

$$AC = \begin{pmatrix} 1+\frac{a_1}{a_0} & \frac{a_1}{a_0} & \cdots & \frac{a_1}{a_0} \\ \frac{a_2}{a_0} & 1+\frac{a_2}{a_0} & \cdots & \frac{a_2}{a_0} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_n}{a_0} & \frac{a_n}{a_0} & \cdots & 1+\frac{a_n}{a_0} \end{pmatrix}.$$

If E denotes the $n \times n$ identity matrix and $\mathbf{1}$ the matrix with all entries equal to 1, then we have

$$AC = A(B+\mathbf{1}) = AB + A\mathbf{1} = E + A\mathbf{1}.$$

Hence the claim follows from Lemma 1. □

2. Saito's duality

Let $C = (c_{ij})$ be a weighted magic square of weight $(W_{\mathbf{a}}, W_{\mathbf{b}})$. We shall associate a rational function $\tilde{\zeta}_C(t)$ to the matrix C.

We consider the hypersurface X in $\mathbf{C}^n$ defined by the equation $f(x_1, \ldots, x_n) = 0$, where

$$f(x_1, \ldots, x_n) = x_1^{c_{11}} x_2^{c_{12}} \cdots x_n^{c_{1n}} + x_1^{c_{21}} x_2^{c_{22}} \cdots x_n^{c_{2n}} + \cdots + x_1^{c_{n1}} x_2^{c_{n2}} \cdots x_n^{c_{nn}}.$$

Let

$$F := \{(x_1, \ldots, x_n) \in \mathbf{C}^n \mid f(x_1, \ldots, x_n) = 1\}$$

be the Milnor fibre of $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$.

If $W_{\mathbf{a}}$ is a Calabi-Yau weight system, then there is the following relation with the hypersurface $\tilde{X}$ in $\mathbf{P}(a_0, \mathbf{a})$ defined by the equation

$$x_0^{h/a_0} + f(x_1, \ldots, x_n) = 0.$$

Let V be the hypersurface in $\mathbf{P}(\mathbf{a}) := \mathbf{P}(a_1, \ldots, a_n)$ given by the equation $f(x_1, \ldots, x_n) = 0$. The mapping

$$\pi_{a_0} : \tilde{X} \to \mathbf{P}(\mathbf{a}), \quad (x_0, x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n),$$

is a covering of degree h/a_0 which is branched along the hypersurface V and possibly along the singularities of $\mathbf{P}(\mathbf{a})$. Let $\tilde{X}_0 := \tilde{X} \backslash \pi_{a_0}^{-1}(V)$. Let $\tilde{Y}$ be the hypersurface in $\mathbf{P}(1, \mathbf{a})$ given by the equation $x_0^h + f(x_1, \ldots, x_n) = 0$. Then the mapping $\pi_1 : \tilde{Y} \to \mathbf{P}(\mathbf{a})$ is a covering of degree h branched along the hypersurface V and possibly along the singularities of $\mathbf{P}(\mathbf{a})$. Then $\tilde{Y}_0 := \tilde{Y} \backslash \pi_1^{-1}(V)$ can be identified with the Milnor fibre F (cf. [DD]). Therefore the induced mapping $F = \tilde{Y}_0 \to \tilde{X}_0$ is a (possibly branched) covering of degree a_0.

We have a $\mathbf{C}^*$-action on $\mathbf{C}^n$ defined by

$$\lambda * (x_1, \ldots, x_n) = (\lambda^{a_1} x_1, \ldots, \lambda^{a_n} x_n), \quad \lambda \in \mathbf{C}^*.$$

Then the $\mathbf{C}^*$-action induces a monodromy transformation $\theta : F \to F$ defined by

$$x \mapsto e^{2\pi i/h} * x \quad (x \in F).$$

Let $\theta_* : \tilde{H}_*(F) \to \tilde{H}_*(F)$ be the induced operator on the reduced homology of F. It is the classical monodromy operator of the singularity $f(x_1, \ldots, x_n) = 0$. Let

$$\tilde{\zeta}_C(t) := \prod_{p \geq 0} \{\det(\mathrm{id} - t \cdot \theta_*|_{\tilde{H}_p(F)})\}^{(-1)^p}$$

be the reduced zeta function of θ. If X has an isolated singularity at the origin, the reduced zeta function is related to the characteristic polynomial $\phi_C(t)$ of the monodromy as follows:

$$\phi_C(t) = (\tilde{\zeta}_C(t))^{(-1)^{n-1}}.$$

The reduced zeta function $\tilde{\zeta}_C(t)$ can be computed as follows (cf. [EG2]).

For $J \subset I_0 = \{1, \ldots, n\}$ we denote by $|J|$ the number of elements of J. For $J \neq \emptyset$, let $T_J := \{x \in \mathbf{C}^n \mid x_i = 0 \text{ for } i \notin J, x_i \neq 0 \text{ for } i \in J\}$ be the ("coordinate") complex torus of dimension $|J|$, and let $a_J := \gcd(a_j, j \in J)$. The integer a_J is the order of the isotropy group of the $\mathbf{C}^*$-action on the torus T_J. Let $X_J := X \cap T_J$, $F_J := F \cap T_J$. The operator θ maps F_J to itself; let θ_J be the restriction of θ to F_J. We have

$$\tilde{\zeta}_C(t) = (1-t)^{-1} \prod_{J : |J| \geq 1} \tilde{\zeta}_{C,J}(t),$$

where $\tilde{\zeta}_{C,J}(t)$ is the reduced zeta function of θ_J.

Let $Z_J := T_J/\mathbf{C}^*$, $Y_J := X_J/\mathbf{C}^*$. Note that if a_J does not divide h then $Z_J \setminus Y_J$ is empty. In this case, $\tilde{\zeta}_{C,J}(t) = 1$. Suppose $a_J | h$. If we restrict the natural projection $T_J \setminus X_J \to Z_J \setminus Y_J$ to F_J then we get an (h/a_J)-fold covering $F_J \to Z_J \setminus Y_J$. The transformation θ_J is a covering transformation of it and acts as a cyclic permutation of the h/a_J points of a fibre. Therefore

$$\tilde{\zeta}_{C,J}(t) = (1 - t^{h/a_J})^{\chi(Z_J \setminus Y_J)}$$

where $\chi(V)$ denotes the Euler characteristic of the topological space V.

The Euler characteristic $\chi(Z_J \setminus Y_J)$ can be computed as follows. A subset $J \subset I_0$ is called *special* if there exists a subset $I \subset I_0$ with $|I| = |J|$ such that $c_{ij} = 0$ for $i \in I$ and $j \notin J$. Note that in particular $J = \emptyset$ and $J = I_0$ are special. For a special subset $J \neq \emptyset$ denote by C_{IJ} the matrix $(c_{ij})_{j \in J}^{i \in I}$. Define $C_\emptyset := (1)$.

First assume that $|J| = 1$. Then $Z_J = \mathrm{pt}$. The set Y_J is empty if and only if J is special. In this case,

$$\tilde{\zeta}_{C,J}(t) = (1 - t^{h/a_J}).$$

Now suppose that $|J| \geq 2$. Then $\chi(Z_J) = 0$ and $\chi(Z_J \setminus Y_J) = -\chi(Y_J)$. Then $Y_J \neq \emptyset$ if and only if J is special or $J = I_0$. In this case, by [BKKh, Kou] (see also [Va, (7.1) Theorem]) we have

$$\chi(Y_J) = (-1)^{|J|} \frac{a_J |\det C_{IJ}|}{h}.$$

In particular, if $J = I_0$, then J is special, $a_J = 1$ (since $W_{\mathbf{a}}$ is reduced), $C_{IJ} = C$, and

$$\chi(Y_J) = (-1)^n \frac{|\det C|}{h}.$$

For a subset $J \subset I_0$ denote by J' the complementary set $J' := I_0 \backslash J$. Note that if J is special for C then J' is special for C^t. For $J \neq \emptyset$ let $b_J := \gcd(b_j, j \in J)$. Define $a_\emptyset := h$ and $b_\emptyset := k$.

Summarizing we have proved the following theorem.

THEOREM 1. *The reduced zeta functions $\tilde{\zeta}_C(t)$ and $\tilde{\zeta}_{C^t}(t)$ can be computed from the matrix C as follows*:

$$\tilde{\zeta}_C(t) = \prod_{J \text{ special}} (1 - t^{h/a_J})^{(-1)^{|J|+1} a_J |\det C_{IJ}|/h},$$

$$\tilde{\zeta}_{C^t}(t) = \prod_{J \text{ special}} (1 - t^{k/b_{J'}})^{(-1)^{|J'|+1} b_{J'} |\det C_{I'J'}|/k}.$$

Remark 1. Let X have an isolated singularity at the origin. Then its Milnor number $\mu = \operatorname{rank} H_{n-1}(F)$ is equal to

$$\mu = (-1)^{n-1} \sum_{J \text{ special}} (-1)^{|J|+1} |\det C_{IJ}|.$$

The dimension μ_0 of the radical of $H_{n-1}(F)$ is equal to

$$\mu_0 = (-1)^{n-1} \sum_{J \text{ special}} (-1)^{|J|+1} \frac{a_J |\det C_{IJ}|}{h}.$$

In addition, let $n = 3$ and let $(a_1, a_2, a_3; h)$ be a Calabi-Yau weight system. By [D2, Theorem 3.3.4] the hypersurface $\tilde{X}$ in $\mathbf{P}(a_0, \mathbf{a})$ is a simply-connected projective surface with dualizing sheaf $\omega_{\tilde{X}} = \mathcal{O}_{\tilde{X}}$. Resolving its singularities (which are rational double points) we get a non-singular K3 surface with Picard number

$$\rho = 22 - (\mu - \mu_0).$$

If $\mu_0 = 0$, then by [E2, Proposition 1] the discriminant of the Picard lattice, i.e. the determinant of a matrix of the intersection form on the Picard group, is equal to

$$d = (-1)^{\rho-1} \tilde{\zeta}_C(1) = (-1)^{\rho-1} \prod_{J \text{ special}} \left(\frac{h}{a_J}\right)^{(-1)^{|J|+1} a_J |\det C_{IJ}|/h}.$$

Following K. Saito [S1, S2], for a rational function

$$\psi(t) = \prod_{\ell|h}(1 - t^{\ell})^{\alpha_\ell}, \quad \alpha_\ell \in \mathbf{Z},$$

we define the Saito dual (rational) function $\psi^*(t)$ by

$$\psi^*(t) = \prod_{m|h}(1 - t^m)^{-\alpha_{(h/m)}}.$$

In particular, if $\sum_{\ell|h} \alpha_\ell = 0$, then one has

$$\psi^*(1) = \prod_{\ell|h}\left(\frac{h}{\ell}\right)^{-\alpha_\ell} = h^{\sum \alpha_\ell} \prod_{\ell|h} \ell^{\alpha_\ell} = \psi(1).$$

COROLLARY 1. *Let C be primitive, $a_0 = b_0 = 1$, and $n = 3$. Then*

$$\tilde{\zeta}_{C^t}(t) = \tilde{\zeta}^*_C(t).$$

Proof. By the assumptions, we have $h = k$, $a_0 = b_0 = 1$, and $I_0 = \{1, 2, 3\}$. We show that for any special subset $J \subset I_0$ we have

$$a_J = \frac{h}{|\det C_{IJ}|}.$$

This is clear if $|J| = 1$, $J = \emptyset$ or $J = I_0$.

Therefore let $|J| = 2$. For simplicity we assume that $J = \{1, 2\}$. By Cramer's rule we have

$$a_1 = (c_{22} - c_{12})\frac{h}{\det C_{IJ}}, \quad a_2 = (c_{11} - c_{21})\frac{h}{\det C_{IJ}}.$$

This shows that $h/|\det C_{IJ}|$ divides a_1 and a_2 and hence a_J. Let

$$a_J = e\frac{h}{|\det C_{IJ}|}$$

for some integer $e \geq 1$. Then e divides $c_{22} - c_{12}$ and $c_{11} - c_{21}$. If we subtract the second row of the matrix B from the first row then we obtain the matrix

$$\begin{pmatrix} c_{11} - c_{21} & c_{12} - c_{22} & 0 \\ c_{21} - 1 & c_{22} - 1 & -1 \\ c_{31} - 1 & c_{32} - 1 & c_{33} - 1 \end{pmatrix}.$$

Expanding the determinant of this matrix with respect to the first row we see that e divides the determinant of this matrix which is equal to $\det B = 1$. This implies that $e = 1$ and hence the claim.

Analogously, one can show that for J special

$$b_{J'} = \frac{k}{|\det C_{I'J'}|}.$$

Hence it follows that for any special subset $J \subset I_0$

$$\frac{k}{b_{J'}} = |\det C_{I'J'}| = \frac{h}{|\det C_{IJ}|} = a_J.$$

Moreover,

$$(-1)^{|J'|+1} \frac{b_{J'}|\det C_{I'J'}|}{k} = (-1)^{|J'|+1} = -(-1)^{|J|+1} \frac{a_J|\det C_{IJ}|}{h}.$$

Hence the claim follows from Theorem 1. □

3. Simple K3 hypersurface singularities

First consider the case $n = 2$. Then the (Calabi-Yau) weight systems corresponding to quasismooth plane curves are indicated in Table 1. They are self-dual. The corresponding weighted magic squares are given in that table. They are indicated as follows:

$$x^{c_{11}}y^{c_{12}}, x^{c_{21}}y^{c_{22}}.$$

The corresponding functions $f(x, y) = x^{c_{11}}y^{c_{12}} + x^{c_{21}}y^{c_{22}}$ have isolated singularities at the origin. The characteristic polynomial $\phi_C(t)$ of the monodromy operator satisfies $\phi_C^*(t) = (\phi_C(t))^{-1}$ (cf. [EG1]).

Table 1. Weighted elliptic plane curves

Name	$a_0, a_1, a_2; h$	C	Dual
$\tilde{E}_8$	1, 2, 3; 6	x^3, y^2	$\tilde{E}_8$
$\tilde{E}_7$	1, 1, 2; 4	y^2, x^2y	$\tilde{E}_7$
$\tilde{E}_6$	1, 1, 1; 3	x^2y, xy^2	$\tilde{E}_6$

Now consider the case $n = 3$. Then the (Calabi-Yau) weight systems corresponding to quasismooth surfaces have been classified by Reid (unpublished) and Yonemura [Yo]. The cones over these surfaces are called *simple K3 hypersurface singularities*. Belcastro [Be] determined the mirror symmetric pairs inside that list.

THEOREM 2. *Let $W_{\mathbf{a}}$ and $W_{\mathbf{b}}$ be the weight systems of a mirror symmetric pair of simple K3 hypersurface singularities. Then $W_{\mathbf{a}}$ and $W_{\mathbf{b}}$ are strongly coupled weight systems.*

For the proof of Theorem 2 we indicate in each case an almost primitive weighted magic square C such that each row and each column of C contains at least one zero. This is done in Table 2 for the weight systems with $a_i = b_j = 1$ for some $i, j \in \{0, 1\}$ and in Table 3 for the remaining cases. We use the indexing of [Yo] for the weight systems. We list all the weight systems such that

the mirror family is again in Yonemura's list. In the first column we indicate the index of the weight system. Let

$$f(x, y, z) = x^{c_{11}} y^{c_{12}} z^{c_{13}} + x^{c_{21}} y^{c_{22}} z^{c_{23}} + x^{c_{31}} y^{c_{32}} z^{c_{33}}.$$

If $f(x, y, z) = 0$ defines an isolated hypersurface singularity in Arnold's [Ar] or Wall's [Wal] list of singularities, we give the name of the singularity in the second column. In the case when $a_0 = 1$, $f(x, y, z) = 0$ defines a Fuchsian singularity (for the definition see [E4]). In the cases where there is a name missing we indicate the signature $\{g; \alpha_1, \ldots, \alpha_r\}$ of these singularities (here 'nh' means that the central curve is non-hyperelliptic). In the third column we list the weight system. In the 4th column we indicate the weighted magic square C in the following way:

$$x^{c_{11}} y^{c_{12}} z^{c_{13}}, x^{c_{21}} y^{c_{22}} z^{c_{23}}, x^{c_{31}} y^{c_{32}} z^{c_{33}}.$$

In the last column we indicate the index of the partner weight system.

There are examples of strongly dual weight systems where the corresponding families of K3 surfaces are not mirror symmetric, e.g. (cf. [Ko])

8 24		$1, 2, 3, 6; 12$ $1, 2, 4, 5; 12$	y^2z, x^3y^2, z^2 y^3, x^2y^2, xz^2	24 8

The weight systems correspond to the singularities $W_{1,0}$ and $Q_{2,0}$ respectively, but the equations $f(x, y, z) = 0$ are not equations of these singularities, they even have non-isolated singularities at the origin.

By inspection of the Tables 2 and 3, we see that the cases $a_0 = b_0 = 1$ are exactly the cases of Arnold's strange duality. In these cases the matrices C are primitive and hence the corresponding weight systems are strongly dual. In all other cases the weight systems are not strongly dual but only strongly coupled.

The remaining singularities with $a_0 = 1$ are Fuchsian singularities of signature $\{g; \alpha_1, \ldots, \alpha_r\}$ with $g > 0$. They are coupled to weight systems which again correspond to isolated singularities. We list these singularities together with their partners in Table 4. Here ρ denotes the Picard number of the K3 surface corresponding to the weight system on the left-hand side as it can be found in the table of [Be] and d denotes the discriminant of the Picard lattice. The numbers μ^*, μ_0^*, and d^* are the Milnor number, the dimension of the radical, and the discriminant of the Milnor lattice respectively of the singularity on the right-hand side. For the definition of v^* see below. The singularities Q_{17} and S_{17} are bimodal singularities belonging to the list of Arnold [Ar] of the 14 bimodal exceptional singularities. The weight system of the singularity $V^{\#}NC_{18}^1$ appears in the list of [S1, Appendix 1] of regular weight systems with $\mu = 24$ (see also [E4, Table 3]). The singularity VNC_{13}^1 is a (minimally) elliptic hypersurface singularity and appears in [E1, Table 2] (there we used the name $V^{\#}NC_{(1)}$). The

singularities Z_{25} and W_{25} also appear in the lists of Arnold [Ar]. They have modality 4. The singularities V'_{29} and N_{33} do not occur in the lists of [Ar] and [Wal]. Here we use as names the name of the series (according to [Ar]) to which they belong indexed by the Milnor number.

Table 2. Coupled weight systems with $a_i = b_i = 1$ for $i = 0$ or $i = 1$

No.	Name	$a_0, a_1, a_2, a_3; h$	C	Partner
14	E_{12}	1, 6, 14, 21; 42	x^7, y^3, z^2	14
		6, 1, 14, 21; 42	$x^{21}z, y^3, z^2$	28
		6, 1, 14, 21; 42	$x^{28}y, y^3, z^2$	45
		14, 1, 6, 21; 42	$x^{36}y, y^7, z^2$	51
28		3, 1, 7, 10; 21	x^{21}, y^3, xz^2	14
		3, 1, 7, 10; 21	$x^{11}z, y^3, xz^2$	28
		3, 1, 7, 10; 21	$x^{14}y, y^3, xz^2$	45
		7, 1, 3, 10; 21	$x^{18}y, y^7, xz^2$	51
45		4, 1, 9, 14; 28	x^{28}, xy^3, z^2	14
		4, 1, 9, 14; 28	$x^{14}z, xy^3, z^2$	28
		4, 1, 9, 14; 28	$x^{19}y, xy^3, z^2$	45
		14, 1, 4, 9; 28	$x^{24}y, y^7, xz^3$	51
51		12, 1, 5, 18; 36	x^{36}, xy^7, z^2	14
		12, 1, 5, 18; 36	$x^{18}z, xy^7, z^2$	28
		18, 1, 5, 12; 36	$x^{24}z, xy^7, z^3$	45
		12, 1, 5, 18; 36	$x^{31}y, xy^7, z^2$	51
50	E_{13}	1, 4, 10, 15; 30	x^5y, y^3, z^2	38
		15, 1, 4, 10; 30	$x^{26}y, y^5z, z^3$	77
38	Z_{11}	1, 6, 8, 15; 30	x^5, xy^3, z^2	50
		15, 1, 6, 8; 30	$x^{22}z, y^5, yz^3$	82
77		13, 1, 5, 7; 26	x^{26}, xy^5, yz^3	50
		13, 1, 5, 7; 26	$x^{19}z, xy^5, yz^3$	82
82		11, 1, 3, 7; 22	x^{22}, y^5z, xz^3	38
		11, 1, 3, 7; 22	$x^{19}y, y^5z, xz^3$	77
13	E_{14}	1, 3, 8, 12; 24	x^4z, y^3, z^2	20
		8, 1, 3, 12; 24	$x^{21}y, y^4z, z^2$	59
20	Q_{10}	1, 6, 8, 9; 24	x^4, y^3, xz^2	13
		8, 1, 6, 9; 24	$x^{15}z, y^4, yz^2$	72
59		7, 1, 5, 8; 21	x^{21}, xy^4, yz^2	13
		7, 1, 5, 8; 21	$x^{13}z, xy^4, yz^2$	72
72		5, 1, 2, 7; 15	x^{15}, y^4z, xz^2	20
		5, 1, 2, 7; 15	$x^{13}y, y^4z, xz^2$	59
78	Z_{12}	1, 4, 6, 11; 22	x^4y, xy^3, z^2	78
39	Z_{13}	1, 3, 5, 9; 18	x^3z, xy^3, z^2	60
60	Q_{11}	1, 4, 6, 7; 18	x^3y, y^3, xz^2	39
22	Q_{12}	1, 3, 5, 6; 15	x^3z, y^3, xz^2	22

Table 2. (continued)

No.	Name	$a_0, a_1, a_2, a_3; h$	C	Partner
9	W_{12}	1, 4, 5, 10; 20	x^5, z^2, y^2z	9
		4, 1, 5, 10; 20	$x^{15}y, z^2, y^2z$	71
71		3, 1, 4, 7; 15	x^{15}, xz^2, y^2z	9
		3, 1, 4, 7; 15	$x^{11}y, xz^2, y^2z$	71
37	W_{13}	1, 3, 4, 8; 16	x^4y, z^2, y^2z	58
58	S_{11}	1, 4, 5, 6; 16	x^4, xz^2, y^2z	37
87	S_{12}	1, 3, 4, 5; 13	x^3y, xz^2, y^2z	87
4	U_{12}	1, 3, 4, 4; 12	x^4, y^3, z^3	4

Table 3. Coupled weight systems: remaining cases

No.	Name	$a_0, a_1, a_2, a_3; h$	C	Partner
12		6, 1, 2, 9; 18	x^9z, x^2y^8, z^2	27
		6, 1, 2, 9; 18	$x^{16}y, x^2y^8, z^2$	49
27		8, 2, 3, 11; 24	x^9y^2, y^8, xz^2	12
49		14, 2, 5, 21; 42	$x^{16}y^2, xy^8, z^2$	12
40		7, 1, 2, 4; 14	$x^{10}z, x^2y^6, yz^3$	81
81		13, 2, 3, 8; 26	$x^{10}y^2, y^6z, xz^3$	40
24		4, 1, 2, 5; 12	x^{12}, x^2y^5, yz^2	11
11		10, 2, 3, 15; 30	$x^{12}y^2, y^5z, z^2$	24
6		2, 1, 2, 5; 10	x^5z, x^2y^4, z^2	26
		2, 1, 2, 5; 10	x^8y, x^2y^4, z^2	34
		5, 1, 2, 2; 10	x^8z, x^2y^4, yz^4	76
26		4, 2, 5, 9; 20	x^5y^2, y^4, xz^2	6
34		6, 2, 7, 15; 30	x^8y^2, xy^4, z^2	6
76		13, 2, 5, 6; 26	x^8y^2, y^4z, xz^4	6
10		4, 1, 1, 6; 12	$x^{11}y, y^6z, z^2$	65
		6, 1, 1, 4; 12	$x^{11}y, y^8z, z^3$	80
		4, 1, 1, 6; 12	x^{12}, xy^{11}, z^2	46
65		11, 3, 5, 14; 33	x^{11}, xy^6, yz^2	10
80		22, 4, 5, 13; 44	x^{11}, xy^8, yz^3	10
46		22, 5, 6, 33; 66	$x^{12}y, y^{11}, z^2$	10
42	$Z_{2,0}$	1, 1, 3, 5; 10	x^5z, xy^3, z^2	68
		5, 1, 1, 3; 10	x^9y, y^7z, xz^3	92
		5, 1, 1, 3; 10	x^9y, y^{10}, xz^3	83
68	Q_{17}	3, 4, 10, 13; 30	x^5y, y^3, xz^2	42
92		19, 3, 5, 11; 38	x^9z, xy^7, yz^3	42
83		27, 4, 5, 18; 54	x^9z, xy^{10}, z^3	42

Table 3. (continued)

No.	Name	$a_0, a_1, a_2, a_3; h$	C	Partner
25	$Q_{3,0}$	1, 1, 3, 4; 9	$x^6y,\ y^3,\ xz^2$	43
		3, 1, 1, 4; 9	$x^8y,\ y^5z,\ xz^2$	88
		3, 1, 1, 4; 9	$x^8y,\ y^9,\ xz^2$	48
43	Z_{25}	4, 3, 11, 18; 36	x^6z, xy^3, z^2	25
88		9, 2, 5, 11; 27	$x^8z, xy^5,\ yz^2$	25
48		16, 3, 5, 24; 48	x^8z, xy^9, z^2	25
7	$X_{2,0}$	1, 1, 2, 4; 8	$x^6y,\ y^2z, z^2$	64
64	S_{17}	3, 4, 7, 10; 24	$x^6,\ y^2z, xz^2$	7
66	$S^*_{2,0}$	1, 1, 2, 3; 7	$x^7, xz^2,\ y^2z$	35
35	W_{25}	4, 3, 7, 14; 28	$x^7y, z^2,\ y^2z$	66
21	2; 2	1, 1, 1, 2; 5	$x^4y,\ y^3z, xz^2$	86
		1, 1, 1, 2; 5	$x^4y,\ y^5, xz^2$	30
86	$V^{\#}NC^1_{18}$	5, 4, 7, 9; 25	$x^4z, xy^3,\ yz^2$	21
30	N_{33}	8, 5, 7, 20; 40	x^4z, xy^5, z^2	21
5	2;	1, 1, 1, 3; 6	$x^5y,\ y^3z, z^2$	56
		3, 1, 1, 1; 6	$x^6, xy^5,\ yz^5$	73
56	VNC^1_{13}	5, 6, 8, 11; 30	$x^5, xy^3,\ yz^2$	5
73		25, 7, 8, 10; 50	$x^6y,\ y^5z, z^5$	5
1	3; (nh)	1, 1, 1, 1; 4	$x^4, xy^3,\ yz^3$	52
52	V'_{29}	9, 7, 8, 12; 36	$x^4y,\ y^3z, z^3$	1
32		2, 2, 3, 7; 14	xy^4, x^4y^2, z^2	32

Table 4. Fuchsian singularities with $g > 0$ and their partners

No.	Name	μ	μ_0	ρ	d	b_0	d^*	μ_0^*	μ^*	ν^*	Name	No.
42	$Z_{2,0}$	21	2	3	2	3	−6	0	17	0	Q_{17}	68
7	$X_{2,0}$	21	2	3	4	3	−12	0	17	0	S_{17}	64
21	2; 2	24	4	2	−5	5	25	0	24	0	$V^{\#}NC^1_{18}$	86
						8	−10	0	33	6	N_{33}	30
5	2;	25	4	1	2	5	−10	0	19	0	VNC^1_{13}	56
25	$Q_{3,0}$	20	2	4	−3	4	−6	0	25	2	Z_{25}	43
66	$S^*_{2,0}$	20	2	4	−7	4	−14	0	25	2	W_{25}	35
1	3; (nh)	27	6	1	4	9	−12	0	29	6	V'_{29}	52

There is the following relation between these singularities. The K3 surfaces corresponding to the weight systems on the left-hand side are compactifications of the corresponding Fuchsian singularities. Let $g(x, y, z) = 0$ be the equation of a singularity on the right-hand side. Let $W_{\mathbf{b}} = (b_1, b_2, b_3; k)$ be the weight system of this singularity. Let $\tilde{X}^*$ be the hypersurface in $\mathbf{P}(b_0, b_1, b_2, b_3)$ given by the

equation $w^{k/b_0} + g(x, y, z) = 0$. As in Section 2 we consider the natural mapping $\pi_{b_0} : \tilde{X}^* \to \mathbf{P}(b_1, b_2, b_3)$. Let $\tilde{X}_0^* := \tilde{X}^* \backslash \pi_{b_0}^{-1}(V^*)$ where V^* is the hypersurface in $\mathbf{P}(b_1, b_2, b_3)$ defined by $g(x, y, z) = 0$ and let F^* be the Milnor fibre of $g : (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$. Then we have a mapping $F^* \to \tilde{X}_0^*$ which is a (possibly branched) covering of degree b_0. Denote by v^* the total branching order of this covering. One can easily see that this covering is either unbranched or branched along the singularity $(0 : 0 : 1) \notin V^*$ of $\mathbf{P}(b_1, b_2, b_3)$ of branching order v^*. Then we obtain from the Riemann-Hurwitz formula in all cases

$$\mu^* + v^* + 1 = b_0(\rho + 3).$$

4. Extension of Arnold's strange duality

The author and C. T. C. Wall [EW] have found an extension of Arnold's strange duality embracing also isolated complete intersection singularities. Such a singularity is defined by the germ of an analytic mapping $(g, f) : (\mathbf{C}^4, 0) \to (\mathbf{C}^2, 0)$. It is weighted homogeneous of weights q_1, q_2, q_3, q_4 and degrees d_1, d_2 where we assume $d_1 \leq d_2$ and where we have $1 + q_1 + q_2 + q_3 + q_4 = d_1 + d_2$. We consider the compactification of such a singularity in the weighted projective space $\mathbf{P}(1, q_1, q_2, q_3, q_4)$ with coordinates w, x, y, z, t given by the equations

$$g(x, y, z, t) = 0,$$

$$f(x, y, z, t) + w^{d_2} = 0.$$

More precisely, this correspondence embraces the following singularities. We use the notation of [E3].

(a) Arnold's 14 exceptional unimodal hypersurface singularities.

(b) The six bimodal hypersurface singularities $J_{3,0}$ (12), $Z_{1,0}$ (40), $Q_{2,0}$ (24), $W_{1,0}$ (8), $S_{1,0}$ (63), $U_{1,0}$ (18). The compactifications of these singularities occur in Yonemura's list. The index is indicated in brackets. The first three of these singularities already occurred in Table 3.

The remaining singularities are ICIS defined by the germ of an analytic mapping $(g, f) : (\mathbf{C}^4, 0) \to (\mathbf{C}^2, 0)$ as above. Here we distinguish between three types:

(c) The singularities J_9', J_{10}', J_{11}, K_{10}', K_{11}', $J_{2,0}'$, $K_{1,0}'$ where $g(x, y, z, t) = xt - y^2$, $f(x, y, z, t) = f'(x, y, t) + z^2$ for some $f' : (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$.

(d) The singularities L_{10}, L_{11}, M_{11}, $L_{1,0}$, $M_{1,0}$ where $g(x, y, z, t) = xt - yz$.

(e) The ICIS $I_{1,0}$ given by

$$g(x, y, z, t) = x^3 - yt,$$

$$f(x, y, z, t) = (a + 1)x^3 + yz + z^2 + zt, \quad a \neq 0, 1.$$

The correspondence between these singularities is indicated in Table 5. The compactifications of all these singularities are K3 surfaces and the dual families are mirror symmetric.

We also relate this correspondence to a duality of weight systems. For this purpose, we associate a Calabi-Yau weight system to an ICIS as follows. In the

Table 5. Extension of Arnold's strange duality

No.	Name	$a_0, a_1, a_2, a_3; h$	C	Dual
12	$J_{3,0}$	$1, 2, 6, 9; 18$	x^3y^2, y^3, z^2	J'_9
40	$Z_{1,0}$	$1, 2, 4, 7; 14$	x^3y^2, xy^3, z^2	J'_{10}
24	$Q_{2,0}$	$1, 2, 4, 5; 12$	x^2y^2, y^3, xz^2	J'_{11}
8	$W_{1,0}$	$1, 2, 3, 6; 12$	x^3z, z^2, y^2z	K'_{10}, L_{10}
63	$S_{1,0}$	$1, 2, 3, 4; 10$	x^3z, xz^2, y^2z	K'_{11}, L_{11}
18	$U_{1,0}$	$1, 2, 3, 3; 9$	x^3y, y^2z, yz^2	M_{11}
	J'_9	$1, 6, 2, 9; 18$	x^3, x^2y^3, z^2	12
	J'_{10}	$1, 4, 2, 7; 14$	x^3y, x^2y^3, z^2	40
	J'_{11}	$1, 3, 2, 6; 12$	x^2z, x^2y^3, z^2	24
	K'_{10}, L_{10}	$1, 4, 1, 6; 12$	x^3, z^2, xy^2z	8
	K'_{11}, L_{11}	$1, 3, 1, 5; 10$	x^3y, z^2, xy^2z	63
	M_{11}	$1, 3, 1, 4; 9$	x^3, xy^2z, yz^2	18
	$J'_{2,0}$	$1, 2, 2, 5; 10$	x^3y^2, x^2y^3, z^2	$J'_{2,0}$
	$L_{1,0}, K'_{1,0}$	$1, 2, 1, 4; 8$	x^2z, z^2, xy^2z	$L_{1,0}, K'_{1,0}$
	$M_{1,0}$	$1, 2, 1, 3; 7$	x^3y, xy^2z, yz^2	$M_{1,0}$
	$I_{1,0}$	$1, 2, 3, 0; 6$	x^3, y^2z^2, y^2z	$I_{1,0}$

cases (c) and (d) we associate the weight system $(q_1, q_2 - q_1, q_3; d_2)$ to the singularity $(X, 0)$. Since $d_1 = q_1 + q_4$, this is a Calabi-Yau weight system.

In case (e) we associate the weight system $(q_1, q_2, q_3 - q_2; d_2)$ to the singularity $I_{1,0}$. Since $d_1 = q_2 + q_4$, we have $1 + q_1 + q_2 + q_3 - q_2 = d_2$. However, $a_3 := q_3 - q_2 = 0$.

Then we have the following extension of Theorem 2:

THEOREM 3. *Let $W_{\mathbf{a}}$ and $W_{\mathbf{b}}$ be the weight systems of a mirror symmetric pair of the above singularities. Then $W_{\mathbf{a}}$ and $W_{\mathbf{b}}$ are strongly dual.*

In each case, a primitive weighted magic square C is indicated in Table 5. For a singularity of type (b), the matrix C corresponds to some points of the Newton diagram. The corresponding function $f(x, y, z)$ has a non-isolated singularity at the origin and does not define the given one. These points differ from the points given in [E3]. The points there correspond to non-primitive (and even not almost primitive) matrices C.

For the ICIS of types (c) or (d), one obtains some points of a Newton diagram of the Laurent polynomial associated to the singularity in [E3] by subtracting the second column of C from the first one. In case (e), one has to subtract the third column from the second one.

In all cases, the reduced zeta function $\tilde{\zeta}_C(t)$ differs from the characteristic polynomial $\Delta(t)$ of the monodromy of the corresponding singularity only in the exponents: If

$$\Delta(t) = \prod_{m|h} (1 - t^m)^{\chi_m}$$

then

$$\tilde{\zeta}_C(t) = \prod_{m|h}(1-t^m)^{\varepsilon_m}$$

where $\varepsilon_m = -1, 0, 1$ if $\chi_m < 0$, $\chi_m = 0$, $\chi_m > 0$ respectively. From Corollary 1 we get Saito's duality of the characteristic polynomials of the monodromy up to the absolute value of the exponents.

For a generalization of the construction of the polar dual in §1 for this extension of Arnold's strange duality which precisely yields Saito's duality we refer to [E3].

By inspection of Table 5 we observe the strange fact that the weight systems associated to the ICIS with the exception of $I_{1,0}$ again occur in Yonemura's list. However, comparing [Be, Table 3] and [E2, Table 6], we see that the Picard lattices of the corresponding K3 surfaces are different. If we omit the zero in the weight system of $I_{1,0}$, we obtain the weight system of $\tilde{E}_8$.

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