ROBERT D. M. ACCOLA
KODAI MATH. J.
29 (2006), 299–317

ON THE CASTELNUOVO-SEVERI INEQUALITY FOR RIEMANN SURFACES

ROBERT D. M. ACCOLA

Abstract

Some consequences of equality in the Castelnuovo-Severi inequality are discussed. In particular, it is shown that if a Riemann surface of genus ten, W_{10}, admits four coverings of tori, each in three sheets, then W_{10} admits an elementary abelian group of order 27. By previous work this last result is then characterized by the vanishing of certain thetanulls. An elementary discussion of the direct product of monodromy groups is an essential part of the proofs.

1. Introduction

The inequality of Castelnuovo-Severi is as follows [2, 4]. Let W_g, a compact Riemann surface of genus g, cover two Riemann surfaces, W_h, genus h, in m sheets, and W_k, genus k, in n sheets, so that the two coverings admit no common non-trivial factorization. (If m and n are primes this will always be the case.) Then the Castelnuovo-Severi inequality (CSI) states:

$$g \leq mh + nk + (m-1)(n-1)$$

A natural (and venerable) question is whether there is a Riemann surface W_ℓ, covered by W_h in n sheets, and covered by W_k in m sheets, so that the resulting square diagram commutes. In the case of equality in the CSI a theorem of Castelnuovo gives much more information, so that in this case the question should be easier to answer. For $h = k = 0$ the answer is: sometimes yes and sometimes no (Section 8). So perhaps the question should be that of finding necessary and sufficient conditions for the existence of such a W_ℓ, which must necessarily be the Riemann sphere $\mathbf{P}^1$. We shall give a characterization when $h = k = 1$, $m = 3$ and n an odd prime (Theorems 4.5 and 4.6). A more general situation is covered in Theorem 4.2 where h and k are large compared to m and n.

This investigation had its origin in a W_{10} covering several tori each in three sheets, where this situation is characterized by the existence on W_{10} of certain half-canonical linear series [3]. The goal was to show that if W_{10} covered four

AMS Mathematical Subject Classification. Primary 14H30, Secondary 30F10.
Received June 13, 2005.

such tori then W_{10} admitted an automorphism group isomorphic to an elementary abelian group of order 27 (Section 7). It turns out that a substantial part of the proof can be carried out when 3 is replaced by any odd prime p, and perhaps this clarifies the proof. Section 5 carries out that part of the proof with 3 replaced by p. Section 6 gives examples.

By Riemann's solution to the Jacobi inversion problem the existence of half-canonical linear series on W_{10} is equivalent to the vanishing of the theta function for the Jacobian of W_{10} at certain half-periods (vanishing thetanulls) [5]. Thus the existence on a W_{10} of such a group of order 27 is characterized by the vanishing of certain thetanulls.

This paper deals with a subject and methods available to mathematicians in the early part of the last century, if not earlier. Considering the extensive literature already existing at that time, the author really has no idea whether some if not all of the material in this paper is already part of the literature. If a reader knows of any reference in the literature that pertains to the work here presented, the author would be very grateful to know of it.

2. Preliminary results

First some notation. W_g will always stand for a compact Riemann surface with the lower case subscript denoting its genus, in this case g. W_F will always denote a compact Riemann surface with a finite set of points, Pt_F, deleted with the upper case subscript denoting its fundamental group (in this case F) after a suitable base point has been chosen. W_F^* will denote the compact Riemann surface obtained by adding the points Pt_F to W_F; that is $W_F^* - Pt_F = W_F$. W_F and W_F^* have the same genus. Occasionally we will use both kinds of notation, W_g and W_F^*, for the same compact Riemann surface.

If A is a subgroup of F of finite index m, then after suitable base points have been chosen the topological mapping $\pi_{AF} : W_A \to W_F$ is defined. (We will always assume, without further comment, such suitable base points chosen so that coverings of Riemann surfaces with subgroup subscripts $F, A, B, A \cap B, \ldots,$ will correspond to the inclusion relations of the subgroups indicated; e.g., $W_{A\cap B} \to W_A$.) The covering map can be extended to $\pi_{AF} : W_A^* \to W_F^*$ where Pt_A maps onto Pt_F. (If W_A^* has genus h and W_F^* has genus ℓ we may denote π_{AF} by $\pi_{h\ell}$.

Let $Ax_1, Ax_2, \ldots, Ax_m$ be a coset decomposition of F. These cosets correspond to the points on W_A above the base point for F. The monodromy map $\mu : F \to S_m$ (the symmetric group on the first m digits) is as follows: $\mu(f)(i) = j$ if $Ax_i f = Ax_j$. (Here we will make no notational distinction between a path f and its homotopy equivalence class.) Let A_0 denote the kernel of μ. Then F/A_0 is isomorphic to $\mathrm{Im}(\mu)$, the monodromy group of the covering π_{AF} denoted $\mathrm{Mono}(W_A/W_F)$. By definition it is the monodromy group of the ramified covering $\pi_{AF} : W_A^* \to W_F^*$, and may be denoted $\mathrm{Mono}(W_A^*/W_F^*)$ or $\mathrm{Mono}(W_h/W_l)$. If $f(\in F)$ "circles" a point $t \in Pt_F$ let $\alpha_1, \alpha_2, \ldots, \Sigma\alpha_i = m$, be the multiplicities of the points in Pt_A above t. By renumbering the x's we may assume that $x_1 f^{\alpha_1} x_1^{-1}, x_2 f^{\alpha_2} x_2^{-1}, \ldots$ are paths in A which circle the points in Pt_A above

t. If $f \in A_0$ there are m such paths and no ramification occurs above t in Pt_A.

Now assume A and B are two different subgroups of F of indices m and n respectively. Also assume that $F = AB$ and that $A \cap B$ has index mn in F. If m and n are prime this will be the case. Let $By_1, By_2, \ldots$ be another coset decomposition of F.

Now suppose that f $(\in F)$ circles $t \in Pt_F$. Suppose that $xf^{\alpha}x^{-1}$ (now dropping subscripts) circles a point a in Pt_A above t, and suppose that $yf^{\beta}y^{-1}$ circles a point b in Pt_B above t, where α and β are minimum positive integers with these properties. Since $(Ax \cap By)f^{\gamma} = Axf^{\gamma} \cap Byf^{\gamma}$ we see that if $\gamma =$ l.c.m.$[\alpha, \beta]$ then $(Ax \cap By)f^{\gamma} = (Ax \cap By)$ where γ is the minimum positive integer with this property. There is at least one point c in $Pt_{A\cap B}$ where c is above a, b, and t, and the ramification at c for the covering $\pi_{A\cap B, A}$ is $[\alpha, \beta]/\alpha$. The number of such c's is g.c.d.(α, β).

LEMMA 2.1. *Assume the above discussion. If $f \in A_0$ then the multiplicities of the points above any such a in the covering $\pi_{A\cap B, A}$ are the same as those above t in the covering $\pi_{B,F}$. The covering $\pi_{A\cap B, B}$ is unramified over any point in Pt_B above t.* □

LEMMA 2.2. *Assume the above discussion. Suppose above $a \in Pt_A$ the covering $\pi_{A\cap B, A}$ is unramified and that $xf^{\alpha}x^{-1}$ $(\in A)$ circles a. Then $f^{\alpha} \in B_0$.*

Proof (Exponentiation by an element in F denotes conjugation.).

$$xf^{\alpha}x^{-1} \in \bigcap\{(A \cap B)^g \mid g \in A\} = A \cap \bigcap\{B^g \mid g \in A\}$$

But $F = AB$ so $\bigcap\{B^g \mid g \in A\} = \bigcap\{B^g \mid g \in F\} = B_0$. Since B_0 is normal in F the result follows. □

In this paper we have assumed (and will continue to assume) many well-known facts from elementary group theory. However, for future reference we wish to draw attention to the following.

LEMMA 2.3. *Let G be a finite group, the direct product of two normal subgroups, M and N. Let A be a normal subgroup of G so that $A \cap M = A \cap N = \langle e \rangle$. Then A is central in G, and A is isomorphic to a subgroup of M which is central in M.* □

LEMMA 2.4. *With the same hypotheses as in Lemma* 2.3, *assume further that A and M are isomorphic to subgroups of S_p (p an odd prime) whose orders are divisible by p. Then A and M are isomorphic to Z_p (Cyclic group of order p).* □

(since the centralizer of Z_p in S_p is Z_p itself.)

LEMMA 2.5 [2, p 23–25]. *Let $\pi_{gh} : W_g \to W_h$ be an m-sheeted covering, and let $\pi_{gk} : W_g \to W_k$ be an n-sheeted covering where the two coverings admit no common non-trivial factorization. Then the number of pairs of points, counted with multiplicity, common to fibers of the two maps is finite.* (*This is the statement that the algebraic curve W_g lying on the surface $W_h \times W_k$ has a finite number of singularities.*) $\square$

LEMMA 2.6 (Castelnuovo [2, p 26]). *Suppose for W_g, W_h, and W_k in Lemma* 2.5 *we have equality in the Castelnuovo-Severi inequality. Then for x_1 and x_2 in W_h $\pi_{gk}(\pi_{gh}^{-1}(x_1))$ is linearly equivalent to $\pi_{gk}(\pi_{gh}^{-1}(x_2))$.* $\square$

LEMMA 2.7 (Castelnuovo Riemann Roch theorem) [2, p 27]. *Let $\pi : W_g \to W_h$ be an m sheeted covering. Let g_N^R be a linear series on W_g where $N - R < g - mh$. Then any fiber of the map π imposes at most $m - 1$ linear conditions on g_N^R.* $\square$

LEMMA 2.8 [1]. *Let W_g admit a linear series g_n^1, without fixed points, and a half-canonical g_{g-1}^r, where $r \geq [n/2]$. Then g_n^1 imposes at most $[n/2]$ conditions on g_{g-1}^r.* $\square$

3. A generalization of the Castelnuovo-Severi inequality

Suppose that W_g admits two coverings

$$\pi_{gh} : W_g \to W_h \text{ in } m \text{ sheets}$$
$$\pi_{gk} : W_g \to W_k \text{ in } n \text{ sheets}$$

where the two coverings admit no non-trivial common factorization. This array of coverings will be denoted $\mathrm{Iv}(g; h, m; k, n)$ where Iv stands for the letter "v" inverted to resemble the arrows in the diagram of these two coverings (with W_g on top).

Suppose further that W_h covers another Riemann surface, W_ℓ in n sheets, and W_k covers the same W_ℓ in m sheets and the square diagram of covers commutes. (W_g on top and W_ℓ on the bottom.) We shall denote this array of coverings by $\mathrm{Sq}(g; h, m; k, n; \ell)$. We will say that W_ℓ *completes* the Iv. (W_ℓ is the unique completion since it corresponds to the intersection of two subfields of the full field of meromorphic functions on W_g.)

Let $W_F = W_\ell - Pt_F$ where, as before, Pt_F is the set of points above which $\pi_{g\ell}$ is ramified. Let A be the fundamental group of $W_h - \pi_{h\ell}^{-1}(Pt_F)$ $(= W_A)$, of index n in F. Let B be the fundamental group of $W_k - \pi_{k\ell}^{-1}(Pt_F)$ $(= W_B)$ of index m in F. Then $A \cap B$ is the fundamental group of $W_g - \pi_{g\ell}^{-1}(Pt_F)$ $(= W_{A\cap B})$ of index mn in F. Also $F = AB$.

Now we wish to find an upper bound on g given m, n, h, k, and ℓ. This will give a weak generalization of the Castelnuovo-Severi inequality since we are assuming the Iv is completed.

Since $F = AB = BA$ let

$$Ab_1, Ab_2, \dots Ab_n \quad b_i \in B$$
$$Ba_1, Ba_2, \dots Ba_m \quad a_j \in A$$

be coset decompositions of F. Let f $(\in F)$ circle a point $t \in Pt_F$. If $d_1, d_2, \dots$ are the multiplicities of the points in W_h above t $(\Sigma d_i = n)$ then the total ramification in W_h above t is $\Sigma(d_i - 1)$. If $e_1, e_2, \dots$ are the multiplicities of the points in W_k above t $(\Sigma e_j = m)$ then the total ramification in W_k above t is $\Sigma(e_j - 1)$. The total ramification of all points in W_g above t is $\Sigma\Sigma(d_i, e_j)\{[d_i, e_j] - 1\}$.

Remark. If d and e are positive integers then $de - d - e + (d, e) \geq 0$. Equality occurs if and only if $(d-1)(e-1) = 0$.

LEMMA 3.1

$$(*) \qquad \Sigma\Sigma(d_i, e_j)\{[d_i, e_j] - 1\} \leq m\Sigma(d_i - 1) + n\Sigma(e_j - 1).$$

Equality occurs if and only if one of the two terms on the right hand side of (*) *is zero.*

Proof

$$m\Sigma(d_i - 1) = (\Sigma e_j)\Sigma(d_i - 1) = \Sigma\Sigma(e_j d_i - e_j)$$
$$n\Sigma(e_j - 1) = (\Sigma d_i)\Sigma(e_j - 1) = \Sigma\Sigma(d_i e_j - d_i)$$

By the remark

$$(*_{ij}) \qquad (e_j d_i - e_j) + (d_i e_j - d_i) \geq (d_i, e_j)\{[d_i, e_j] - 1\}$$

Summing on i and j proves the inequality. If we have equality in (*) we have equality in all the inequalities $(*_{ij})$. If a term $(e_j - 1) \neq 0$ then $(d_i - 1) = 0$ for all i. □

DEFINITION. Suppose for a point t in W_ℓ for the square $\mathrm{Sq}(g; h, m; k, n; \ell)$ at most only one of the two coverings of W_ℓ is ramified. The ramification at such a point will be said to be *pure*. If the ramification at every point of W_ℓ is pure then the covering $\pi_{g\ell}$ will be said to have *separated* ramification.

THEOREM 3.2. *Suppose we have a square* $\mathrm{Sq}(g; h, m; k, n; \ell)$. *Then*

$$(3.1) \qquad 2g - 2 + mn(2\ell - 2) \leq m(2h - 2) + n(2k - 2)$$
$$(or\ \ g + mn\ell \leq mh + nk + (m-1)(n-1))$$

Equality in (3.1) *implies that the ramification for* $\pi_{g\ell}$ *is separated.*

Proof. By the Riemann-Hurwitz formula (with obvious notation)

$$2g-2 = m(2h-2) + \mathrm{ram}(g,h)$$
$$2g-2 = n(2k-2) + \mathrm{ram}(g,k)$$
$$2g-2 = mn(2\ell-2) + \mathrm{ram}(g,\ell)$$

$$2g-2+mn(2\ell-2)+\mathrm{ram}(g,\ell) = m(2h-2)+n(2k-2) + \mathrm{ram}(g,h)+\mathrm{ram}(g,k) \tag{3.2}$$

But

$$\mathrm{ram}(g,\ell) = \mathrm{ram}(g,h) + m\,\mathrm{ram}(h,\ell) = \mathrm{ram}(g,k) + n\,\mathrm{ram}(k,\ell)$$

By Lemma 3.1

$$\mathrm{ram}(g,\ell) \le m\,\mathrm{ram}(h,\ell) + n\,\mathrm{ram}(k,\ell) \tag{3.3}$$

Therefore

$$\mathrm{ram}(g,k) \le m\,\mathrm{ram}(h,\ell)$$

And so: $\mathrm{ram}(g,\ell) \ge \mathrm{ram}(g,h) + \mathrm{ram}(g,k)$. Together with (3.2) this proves the inequality in the statement of the theorem.

If we have equality in (3.1) then we have equality in all the above inequalities, especially (3.3). Thus we have equality in (*) of Lemma 2.1 for each point of Pt_F; that is, the ramification at each point of Pt_F is pure. □

Corollary 3.3. *Equality in* (3.1) *of Theorem* 3.2 *is equivalent to the following.* (*The K's refer to canonical series.*)

1) $\mathrm{ram}(g,\ell) = \mathrm{ram}(g,h) + \mathrm{ram}(g,k)$
2) $\mathrm{ram}(g,h) = n\,\mathrm{ram}(k,\ell)$
3) $\mathrm{ram}(g,\ell) = n\,\mathrm{ram}(k,\ell) + m\,\mathrm{ram}(h,\ell)$
4) *The ramification of* $\pi_{g\ell}$ *is separated.*

(3.4) 5) $K_g + \pi_{g\ell}^{-1}(K_\ell) \equiv \pi_{gh}^{-1}(K_h) + \pi_{gk}^{-1}(K_k)$

Proof. That 4) implies 3) together with the proof of Theorem 3.2 shows this corollary is true for the first four statements in the conclusion. We need only show that equality in (3.1) implies (3.4) since the converse is immediate. Let B_{gh} denote the ramification divisor on W_g for the covering π_{gh} (with similar notation for other coverings.)

We show first that $B_{gh} = \pi_{gk}^{-1}(B_{k\ell})$. $\pi_{k\ell}$ is ramified only at points t in Pt_F where the f's that circle the t's are in A_0, since the ramification of $\pi_{g\ell}$ is separated. These are the only points over which π_{gh} is ramified. Above these points $\pi_{h\ell}$ and π_{gk} are unramified. Thus $B_{gh} = \pi_{gk}^{-1}(B_{k\ell})$.

Now $K_g \equiv B_{gh} + \pi_{gh}^{-1}(K_h)$. Since $\pi_{g\ell}^{-1}(K_\ell) = \pi_{gk}^{-1}(\pi_{k\ell}^{-1}(K_\ell))$ and $B_{gh} = \pi_{gk}^{-1}(B_{k\ell})$ we have

$$K_g + \pi_{g\ell}^{-1}(K_\ell) \equiv \pi_{gk}^{-1}(B_{k\ell} + \pi_{k\ell}^{-1}(K_\ell)) + \pi_{gh}^{-1}(K_h) \equiv \pi_{gk}^{-1}(K_k) + \pi_{gh}^{-1}(K_h)$$ □

Corollary 3.4. *Suppose for* $\mathrm{Iv}(g; h, m; k, n)$ *we have*

$$g + mn\ell_0 \geq mh + nk + (m-1)(n-1)$$

If the Iv *is completed by* W_ℓ *then* $\ell \leq \ell_0$. □

Corollary 3.5. *Assume we have equality in Theorem* 3.2 *and* $l = 0$. *Then*

$$\mathrm{Mono}(W_g/W_\ell) \cong \mathrm{Mono}(W_h/W_\ell) \times \mathrm{Mono}(W_k/W_\ell)$$

Proof. If f circles a point in Pt_F then $f \in A_0$ or $f \in B_0$ but not both. Such f's generate F so $F = A_0B_0$. Now

$$\mathrm{Mono}(W_g/W_\ell) \cong F/(A_0 \cap B_0) \cong A_0/(A_0 \cap B_0) \times B_0/(A_0 \cap B_0)$$

But $\mathrm{Mono}(W_h/W_\ell) \cong F/A_0 \cong B_0/(A_0 \cap B_0) \ldots$ etc. □

The following corollaries apply to Section 5.

Definition. For a prime p, $p^\alpha \| n$, will mean that p^α divides n but $p^{\alpha+1}$ does not. We say that p^α *strictly* divides n.

Corollary 3.6. *Suppose we have a* $\mathrm{Sq}(g; h, p; k, p; \ell)$ *where* p *is an odd prime. Then* $F = A_0B$, $p \,\|\, [F : A_0]$ *and* $p^2 \,\|\, [F : A_0 \cap B_0]$.

Proof. To show $F = A_0B$ it suffices to show that $A_0 \not\subset B$ since $[F : B] = p$. If $A_0 \subset B$ then $A_0 \subset B_0$ and so $A_0 = A_0 \cap B_0$. But F/A_0 is isomorphic to a subgroup of S_p, and so $p \,\|\, [F : A_0]$. But $A_0 \cap B_0 \subset A \cap B$ and $p^2 = [F : A \cap B]$. This contradiction shows that $F = A_0B$.

Now $F/A_0 \cong B/A_0 \cap B$. It follows that $[A_0 : A_0 \cap B] = p$. Also $\bigcap\{(A_0 \cap B)^g \mid g \in A_0\} = A_0 \cap B_0$ since $F = A_0B$. Consequently $p \,\|\, [A_0 : A_0 \cap B_0]$ and so $p^2 \,\|\, [F : A_0 \cap B_0]$. □

Corollary 3.7. *Suppose we have equality in Theorem* 3.2 *and* $m = n = p$, *an odd prime. Suppose there exists another subgroup* $C \subset F$ *of index* p *containing* $A \cap B$. *Then the covering* $W_g \to W_C^*$ *is unramified.*

Proof. $C \cap (A \cap B) = A \cap B = A \cap C = C \cap B$. Since A, B, and C have index p $F = AC_0 = A_0C = BC_0 = B_0C$. We apply Lemma 2.1. Suppose $f \in F$ circles $t \in Pt_F$. If $f \in A_0$ the covering $(W_g =)\ W_{A\cap C}^* \to W_C^*$ is unramified over f. If $f \in B_0$ the covering $(W_g =)\ W_{B\cap C}^* \to W_C^*$ is unramified over f. All points of Pt_F have been accounted for. □

Corollary 3.8. *Suppose we have the hypotheses of Corollary* 3.7 *and* $\ell = 0$. *Then* $F/(A_0 \cap B_0) \cong Z_p \times Z_p$.

Proof. Consider the finite group $F/(A_0 \cap B_0)$; that is, we assume $A_0 \cap B_0 = A_0 \cap C_0 = C_0 \cap B_0 = \langle e \rangle$. C_0 is a normal subgroup of F, the direct product of A_0 and B_0, and $A_0 \cap C_0 = C_0 \cap B_0 = \langle e \rangle$. The Corollary now follows from Lemma 2.4 since A_0, B_0, and C_0, whose orders are divisible by p, are subgroups of S_p. $\square$

4. Equality in the Castelnuovo-Severi inequality

Definition. For a covering $\pi_{gh} : W_g \to W_h$ and a divisor D on W_g define

$$\sigma_{gh}(D) := \pi_{gh}^{-1}(\pi_{gh}(D)) \quad \text{a divisor of degree } (\deg \pi_{gh})(\deg D)$$

Suppose W_ℓ completes $\mathrm{Iv}(g; h, m; k, n)$. If $x \in W_g$ then $\sigma_{g\ell}(x)$ can be described in two ways: $\sigma_{gk}(\sigma_{gh}(x))$ or $\sigma_{gh}(\sigma_{gk}(x))$. For almost all x in W_g $\sigma_{g\ell}(x)$ is a divisor of mn distinct points. If x is not in $\sigma_{g\ell}(y)$ then $\sigma_{g\ell}(x) \cap \sigma_{g\ell}(y) = \emptyset$. If $\ell = 0$ then the $\sigma_{gl}(x)$'s form a g^1_{mn}.

Lemma 4.1. *Suppose we have an* $\mathrm{Iv}(g; h, m; k, n)$, *and for all but a finite number of* x *on* W_g $\sigma_{gk}(\sigma_{gh}(x)) = \sigma_{gh}(\sigma_{gk}(x))$. *Then there exists a* W_ℓ $(= \{\sigma_{gk}(\sigma_{gh}(x)) \mid x \in W_g\}$ *which completes the* Iv *and* $\sigma_{g\ell}(x) = \sigma_{gk}(\sigma_{gh}(x))$.

Proof. $W_g^{\sim} := \{x \in W_g \mid \deg \sigma_{gk}(\sigma_{gh}(x)) = mn \text{ and } \sigma_{gk}(\sigma_{gh}(x)) = \sigma_{gh}(\sigma_{gk}(x))\}$. Then $W_g^{\sim} = W_g - \{\text{a finite number of points}\}$. We first show that if x_1, $y \in W_g^{\sim}$ and $y \in \sigma_{gk}(\sigma_{gh}(x_1))$ then $\sigma_{gk}(\sigma_{gh}(x_1)) = \sigma_{gk}(\sigma_{gh}(y))$. Now let $\sigma_{gh}(x_1) = x_1 + \cdots + x_m$. There is an i such that $y \in \sigma_{gk}(x_i)$. Thus $\sigma_{gk}(\sigma_{gh}(y)) = \sigma_{gh}(\sigma_{gk}(y)) = \sigma_{gh}(\sigma_{gk}(x_i)) = \sigma_{gk}(\sigma_{gh}(x_i)) = \sigma_{gk}(\sigma_{gh}(x_1))$.

Define $W_\ell^{\sim} = \{\sigma_{gk}(\sigma_{gh}(x)) \mid x \in W_g^{\sim}\}$. The maps from W_g to W_h and W_k, restricted to $W_g^{\sim}$ define topological maps from the latter to suitably punctured Riemann surfaces $W_h^{\sim}$ and $W_k^{\sim}$. Then $W_\ell^{\sim}$ completes the Iv formed by $W_g^{\sim}$, $W_h^{\sim}$, and $W_k^{\sim}$. Giving $W_\ell^{\sim}$ an analytic structure in the obvious manner and adding points to all the punctured Riemann surfaces completes the proof. $\square$

Definition. A Castelnuovo-Severi Iv, denoted $\mathrm{CSIv}(g; h, m; k, n)$, will be an Iv where

$$g = mh + nk + (n-1)(m-1) \qquad \square$$

It is difficult to believe that the following result is not somewhere in the literature of the last century.

Theorem 4.2. *Let* q *be a prime in a* $\mathrm{CSIv}(g; h, m; k, q)$. *Assume* $h > (q-1)(q-2)/2$. *Then the* CSIv *admits a completion.*

Proof. For each point $x \in W_k$ $\pi_{gh}(\pi_{gk}^{-1}(x))$ is a divisor of degree q belonging to a single g^r_q on W_h by Lemma 2.6. Since q is prime, $r > 1$ implies that W_h admits a plane model of degree q and so $h \le (q-1)(q-2)/2$. Thus $r = 1$.

Let w_1 be a point on W_g so that the mq points $\sigma_{gk}(\sigma_{gh}(w_1))$ are unramified for π_{gh} and π_{gk}. $\sigma_{gh}(w_1) = w_1 + \cdots + w_m$, and $\sigma_{gk}(\sigma_{gh}(w_1)) = \sigma_{gk}(w_1) + \cdots +$

$\sigma_{gk}(w_m)$. Each of the divisors $\pi_{gh}(\sigma_{gk}(w_1)), \ldots, \pi_{gh}(\sigma_{gk}(w_m))$ belong to g_q^1 and all have the same point $\pi_{gh}(w_1)$ in common. Therefore, they are all the same divisor on W_h. It follows that $\pi_{gh}^{-1}(\pi_{gh}(\sigma_{gk}(w_1))) = \sigma_{gk}(\sigma_{gh}(w_1))$. The result follows from Lemma 4.1. □

The following theorems are rather small steps in characterizing those CSIv's which have completions where the hypotheses of Theorem 4.2 do not hold. We shall be considering a CSIv$(pq+1; 1, p; 1, q)$ where p and q are odd primes. We believe that a completion for such an Iv is characterized by the existence on W_{pq+1} of a complete half-canonical linear series g_{pq}^{p+q-3} which is unique. We shall prove this only when p or q is 3.

Let $g = pq+1$, let $\pi : W_g \to W_1$ be the p-sheeted covering, and let $\theta : W_g \to T_1$ be the q-sheeted covering. Assume that W_g admits a complete half-canonical linear series g_{pq}^{p+q-3}. By the Castelnuovo Riemann-Roch theorem for x on W_1 $\pi^{-1}(x)$ imposes at most $p-1$ conditions on g_{pq}^{p+q-3}. The problem is to show that the words "at most" can be replaced by the word "precisely."

We first prove two lemmas from which our results follow.

Lemma 4.3. *Suppose for a* CSIv$(pq+1; 1, p; 1, q)$, W_g *admits a half-canonical* $g_{pq}^{p+q-3+\varepsilon}$, $\varepsilon \geq 0$. *Suppose there is a* g_q^{q-1} *on* W_1 *so that* $|\pi^{-1}(g_q^{q-1})| = g_{pq}^{p+q-3+\varepsilon}$. *Then if* $y_0 \in T_1$ *and* $\theta^{-1}(y_0)$ *consists of* q *distinct points no two of which lie in a fiber of* π (*there are a finite number of fibers omitted, by Lemma* 2.5) *then* $\theta^{-1}(y_0)$ *imposes precisely* $q-1$ *conditions on* $g_{pq}^{p+q-3+\varepsilon}$.

Proof. Let $\theta^{-1}(y_0) = z_1, \ldots, z_q$. Suppose $z_1, \ldots, z_t$ impose t independent conditions on $g_{pq}^{p+q-3+\varepsilon}$, so that if $D \in g_{pq}^{p+q-3+\varepsilon}$ and D contains $z_1, \ldots, z_t$, then D contains $z_{t+1}, \ldots, z_q$. We want to show that $t \geq q-1$. Assume otherwise. $\pi(z_1 + \cdots + z_{q-1})$ are $q-1$ points on W_1 which determine a divisor in g_q^{q-1}. Let that divisor be $\pi(z_1 + \cdots + z_{q-1}) + x$. Then $|\pi(z_{q-1}) + x| = g_2^1$ on W_1. Choose a divisor C in g_2^1 so that $\pi^{-1}(C) \cap \{z_{q-1}, z_q\}$ is empty. Then $\pi^{-1}(\pi(z_1 + \cdots + z_{q-2}) + C)$ is a divisor in $g_{pq}^{p+q-3+\varepsilon}$ containing $z_1 + \cdots + z_{q-2}$ but not $z_{q-1} + z_q$. This contradiction proves the lemma. □

Lemma 4.4. *Suppose for a* CSIv$(pq+1; 1, p; 1, q)$, W_g *admits a half-canonical* $g_{pq}^{p+q-3+\varepsilon}$, $\varepsilon \geq 0$. *Suppose there exists in* W_1 *an* x_0 *so that* $\pi^{-1}(x_0)$ *imposes precisely* $p-1$ *conditions on* $g_{pq}^{p+q-3+\varepsilon}$. *Then* $\varepsilon = 0$. *If* $D \in g_{pq}^{p+q-3}$ *and* D *contains* $\pi^{-1}(x_0)$ *then* $D = \pi^{-1}(D_q)$ *where* D_q *is a divisor of degree* q *on* W_1.

Proof. If $x_1 \in W_1$, $x_1 \neq x_0$ then $|x_1 + x_0|$ is a g_2^1 on W_1. On W_g $\pi^{-1}(x_0) + \pi^{-1}(x_1)$, being a g_{2p}^1, imposes p conditions on $g_{pq}^{p+q-3+\varepsilon}$ (Lemma 2.8). Thus $\pi^{-1}(x_1)$ imposes one condition on $g_{pq}^{p+q-3+\varepsilon} - \pi^{-1}(x_0)$ $(= g_{pq-p}^{q-2+\varepsilon})$; that is, $g_{pq-p}^{q-2+\varepsilon}$ is composite being the lift of a g_{q-1}^{q-2} on W_1, and so $\varepsilon = 0$. g_{pq}^{p+q-3} is now seen to be the completion of the lift of a g_q^{q-1} on W_1. Thus the divisor in g_{pq}^{p+q-3} containing $\pi^{-1}(x_0)$ is the lift of a divisor in g_q^{q-1}. □

Theorem 4.5. *Consider the* CSIv$(pq+1;1,p;1,q)$, *where p and q are odd primes and the* Iv *admits a completion. Then W_{pq+1} admits a complete, simple half-canonical g_{pq}^{p+q-3}.*

Proof. Since the completion W_ℓ has genus zero formula 3.4 in Corollary 3.3 insures that the g_1^1 on W_ℓ lifts to be half-canonical on W_g. g_1^1 lifted to W_1 completes to be g_q^{q-1}, and lifted to T_1 completes to a g_p^{p-1}. Lifted to W_g g_1^1 therefore has dimension at least $(p-1)+(q-1)-1$. Lemma 4.3 insures that there is a y_0 on T_1 so that $\theta^{-1}(y_0)$ imposes precisely $q-1$ conditions on the half-canonical $g_{pq}^{p+q-3+\varepsilon}$, and Lemma 4.4 implies $\varepsilon=0$. Since p and q are prime it follows that g_{pq}^{p+q-3} is simple. □

Theorem 4.6. *For the* CSIv$(3p+1;1,p;1,3)$, *suppose that W_{3p+1} admits a half-canonical g_{3p}^p. Then the* Iv *admits a completion.*

Proof. First we show that g_{3p}^p is simple. Suppose that it is composite. Since g_{3p}^p is not trigonal (by CSI), W_{3p+1} must cover a W_q in two sheets. The non-fixed points of g_{3p}^p is the lift of a complete $g_{(3p-f)/2}^p$ on W_q where f is odd. Since the Clifford index is negative $q=(p-f)/2$ and again this violates CSI.

Let $g=3p+1$, let $\pi: W_g \to W_1$ be the p-sheeted covering, and let $\theta: W_g \to T_1$ be the 3-sheeted covering. Choose $y_0 \in T_1$ so that $\theta^{-1}(y_0)$ is a divisor of three distinct points. Then $\theta^{-1}(y_0)$ imposes at most two conditions on g_{3p}^p by the Castelnuovo Riemann-Roch theorem. If $\theta^{-1}(y_0)$ imposed one condition g_{3p}^p would be composite, a contradiction. Thus $\theta^{-1}(y_0)$ imposes precisely two conditions. By Lemma 4.4 any divisor in g_{3p}^p containing $\theta^{-1}(y_0)$ is the lift of a divisor on T_1 of degree p. By Lemma 4.3, for a general choice of x_0 on W_1, $\pi^{-1}(x_0)$ imposes precisely $p-1$ conditions on g_{3p}^p. By Lemma 4.4, again, any divisor in g_{3p}^p containing such a $\pi^{-1}(x_0)$ is the lift of a divisor of degree 3 on W_1. For a general point z_0 on W_g, $\pi^{-1}(\pi(z_0))$ imposes $p-1$ conditions on g_{3p}^p. A divisor in g_{3p}^p requires an one additional condition to contain $\theta^{-1}(\theta(z_0))$. Thus it requires exactly p conditions for a divisor in g_{3p}^p to contain $\pi^{-1}(\pi(z_0))$ and $\theta^{-1}(\theta(z_0))$, and so to be simultaneously lifted from divisors in W_1 and T_1. This gives a g_{3p}^1 in g_{3p}^p which completes the Iv. □

In this case the half-canonical g_{3p}^p is the unique linear series on W_{3p+1} of dimension p and degree $3p$.

5. Several Castelnuovo-Severi coverings

Definition. For p an odd prime, let $g(p)=p^3-2p^2+1$ and let $h(p)=(p-1)(p-2)/2$.

In this section we shall always assume that all the CSIv$(g(p);h(p),p;h(p),p)$'s which occur have completions. This generalizes the case $p=3$, where we have given conditions which insure that completions occur [3].

Consider the fundamental group, F, of a punctured Riemann surface W_F, together with three subgroups A, B, and C, all of index p in F. Then we have three additional subgroups $A \cap B$, $A \cap C$, and $B \cap C$, which we assume all have index p^2 in F, and we have $A \cap B \cap C$ which we assume has index p^3 in F. Then W_F is covered by seven Riemann surfaces W_H with fundamental group H, where H is any one of the seven proper subgroups of F mentioned above. In this array of eight Riemann surfaces W_F will be called the 0^{th} or bottom level; W_H, $H = A, B$, or C will be called the first level; W_H, $H = A \cap B, A \cap C$, or $B \cap C$ will be called the second level, and $W_{A \cap B \cap C}$ will be called the third or top level. The Riemann surfaces on the same level (ith) will always be assumed to have the same genus (h_i). This array of eight Riemann surfaces will be denoted $\mathrm{Cu}(h_3, h_2, h_1, h_0)$. This array can be visualized as a cube standing on one of its vertices, the vertices standing for the Riemann surfaces and the edges pointing downward standing for the covering maps. The cubes of interest will be $\mathrm{Cu}(g(p), h(p), 0, 0)$ which we will abbreviate by $\mathrm{Cu}(p)$.

We shall also consider the possibility of four subgroups of F, A, B, C, D all of index p in F, the corresponding six subgroups, $A \cap B, A \cap C, \ldots, C \cap D$ all of index p^2 in F, the four subgroups $A \cap B \cap C, \ldots, B \cap C \cap D$ all of index p^3 in F and $A \cap B \cap C \cap D$ of index p^4 in F. There are five levels for the corresponding Riemann surfaces, and if h_i is the genus of all Riemann surfaces at the ith level, this array of 16 Riemann surfaces will be denoted $\mathrm{HyCu}(h_4, h_3, h_2, h_1, h_0)$. This can perhaps be visualized as a hypercube standing "vertically" on one of its vertices, with vertices and downward pointing edges standing for Riemann surfaces and covering maps as in a cube. Our purpose will be to show that a $\mathrm{HyCu}(g(p), h(p), 0, 0, 0)$, denoted $\mathrm{HyCu}(p)$, does not exist.

Then we will show that a $W_{g(p)}$ covering four different $W_{h(p)}$'s, each in p sheets with all Iv's admitting completions, must admit a group of automorphisms isomorphic to the elementary abelian group of order p^3, with four subgroups of order p giving rise to the four coverings $W_{g(p)} \to W_{h(p)}$. Except for $p = 3$, we will avoid the problem of completing the $\mathrm{Iv}(g(p); h(p), p; h(p), p)$. If we have such an Iv completed to a square we will denote it $\mathrm{Sq}(g(p), h(p), 0)$ or more simply $\mathrm{Sq}(p)$.

Lemma 5.1. *Suppose W_g admits three coverings $W_g \to W_h$, $W_g \to W_k$, $W_g \to W_\ell$, in p, q, and r sheets respectively* (*all primes*). *Suppose the three* Iv*'s* ($\mathrm{Iv}(g; h, p; k, q)$ *etc.*) *all have completions. Then the array of seven Riemann surfaces with three coverings and three squares can be completed to a cube of eight Riemann surfaces.*

Proof. For almost all $w \in W_g$ we want a divisor of degree pqr containing w, equal to $\sigma_{g\tau(h)}(\sigma_{g\tau(k)}(\sigma_{g\tau(\ell)}(w)))$ for any permutation τ of the letters h, k, ℓ. Now $\sigma_{gh}(\sigma_{gk}(\sigma_{g\ell}(w))) = \sigma_{gh}(\sigma_{g\ell}(\sigma_{gk}(w)))$. But for each point $v \in \sigma_{g\ell}(w)$ $\sigma_{gh}(\sigma_{gk}(v)) = \sigma_{gk}(\sigma_{gh}(v))$. Therefore $\sigma_{gh}(\sigma_{gk}(\sigma_{g\ell}(w))) = \sigma_{gk}(\sigma_{gh}(\sigma_{gl}(w)))$. Therefore, any permutation of the σ-symbols is possible. The proof is now completed as in Lemma 4.1. $\square$

An entirely analogous proof gives:

LEMMA 5.2. *Suppose* W_g *admits* 4 *coverings* $W_g \to W_{g'}$ $g' = h, k, \ell, m$, *all of prime degree. Suppose all* 6 Iv*'s have completions. Then the array of* 15 *Riemann aurfaces* (4 *coverings*, 6 *squares*, *and* 4 *cubes*) *can be completed to a hypercube of* 16 *Riemann surfaces.* □

Let us summarize the results of Theorem 3.2 in the case of a $\mathrm{Sq}(p)$. Let F be the fundamental group of the punctured $\mathbf{P}^1$, W_F. Let A and B be the subgroups of F which are the fundamental groups for the two punctured $W_{h(p)}$. Then $A \cap B$ is the fundamental group of the punctured $W_{g(p)}$. If $t \in Pt_F$ and $f(\in F)$ circles t then $f \in A_0 \cup B_0$ since the ramification of $W_{g(p)} \to W_F^*$ $(= \mathbf{P}^1)$ is separated. Therefore, $F = A_0 B_0$ since A_0 and B_0 are normal in F. $W_F^* = \mathbf{P}^1$. $\mathrm{Mono}(W_{g(p)}/W_F^*) \cong \mathrm{Mono}(W_A^*/W_F^*) \times \mathrm{Mono}(W_B^*/W_F^*)$.

Now we examine the cube $\mathrm{Cu}(p)$. Assume that the corresponding array of subgroups of F arise from subgroups A, B, and C (all of index p in F). In this case W_A^*, W_B^*, and W_C^* are all of genus zero.

Corollary 3.6 says that $F = AB_0$. We also obtain true statements by replacing A and B by any two of the letters A, B, or C. We shall have further results stated in terms of a set of subgroups of F where the result holds for the statement modified by permuting the names of the subgroups. We shall use such statements by referring to the original statement without further comment.

LEMMA 5.3. *In the cube* $\mathrm{Cu}(p)$, *suppose* $t \in Pt_F$ *and* $f(\in F)$ *circles* t. *If* $f \in A_o$, *then* $f \in B_0 \cup C_0$.

Proof. Let $Ax_1, \ldots, Ax_p$ be a coset decomposition of F. The points above t in W_A^* are circled by the p curves $x_i f x_i^{-1}$. Since $W_{A\cap B\cap C}^* \to W_A^*$ has separated rami-fication, $x_i f x_i^{-1} \in \bigcap\{(A \cap B)^a \mid a \in A)$ or $x_i f x_i^{-1} \in \bigcap\{(A \cap C)^a \mid a \in A)$. By Lemma 2.2, $f \in B_0$ or $f \in C_0$. □

LEMMA 5.4. *For a* $\mathrm{Cu}(p)$ *let* $f(\in F)$ *circle a point* t *in* Pf_F. *If* $f \notin A_0 \cup B_0 \cup C_0$ *then* $f^p \in A_0 \cap B_0 \cap C_0$.

Proof. Suppose the multiplicities of the covering maps over t are as follows:

With respect to $W_A^* \to W_F^*$ they are $\alpha_1, \alpha_2, \ldots, \Sigma\alpha_i = p$
With respect to $W_B^* \to W_F^*$ they are $\beta_1, \beta_2, \ldots, \Sigma\beta_j = p$
With respect to $W_C^* \to W_F^*$ they are $\gamma_1, \gamma_2, \ldots, \Sigma\gamma_k = p$

We wish to show that $\alpha_1 = \beta_1 = \gamma_1 = p$. Suppose otherwise. Assume α', one of the α's, is the smallest of all the α's, β's, and γ's, $1 \le \alpha' < p$. We claim that there is a β, call it β' so that $\beta' > \alpha'$. If not, all the β's equal α', and so α' divides p. Thus $\alpha' = 1$ and so $f \in B_0$. Contradiction.

Similarly there is a γ' so that $\gamma' > \alpha'$.

Let $Bv_1, \ldots, Bv_p$ be a coset decomposition of F. There is a v, call it v_j, so that $Bv_j f^{\beta'} = Bv_j$ and β' is the smallest positive integer with this property. Therefore $Bv_j f^{\alpha'} \neq Bv_j$, so $f^{\alpha'} \notin B_0$. Similarly, $f^{\alpha'} \notin C_0$.

But there is an x so that $Axf^{\alpha'} = Ax$ or $xf^{\alpha'}x^{-1} \in A$. Since $W^*_{A\cap B\cap C} \to W^*_A$ has separated ramification $xf^{\alpha'}x^{-1} \in (A \cap B_0) \cup (A \cap C_0)$ or $f^{\alpha'} \in B_0 \cup C_0$. This contradiction proves the lemma. □

We summarize the discussion so far in a theorem.

THEOREM 5.5. *Let the cube* Cu(p) *correspond to the fundamental group F of W_F. Let F have subgroups A, B, and C as before. For $t \in Pt_F$ let $f \in F$ circle t. Then either $f \in (A_0 \cap B_0) \cup (A_0 \cap C_0) \cup (B_0 \cap C_0)$ or $f^p \in A_0 \cap B_0 \cap C_0$. That is, for the coverings $W^*_H \to W^*_F$, $H = A, B, C$, above t, either two of the three are unramified or all are ramified with multiplicity p. Above one and only one point $t \in Pt_F$ does the latter hold.*

Proof. Only the last sentence needs verification. Since the ramification is not separated from level two to level zero, there must be at least one such point. If there were two such points all the ramification between Riemann surfaces of level one and level zero would be accounted for since these Riemann surfaces are all Riemann spheres. This is a contradiction. □

COROLLARY 5.6. *Continue the hypotheses of Theorem* 5.5. *Then $F = A_0(B_0 \cap C_0)$.*

$$\mathrm{Mono}(W_{A\cap B\cap C}/W_F) \cong \mathrm{Mono}(W_A/W_F) \times \mathrm{Mono}(W_B/W_F) \times \mathrm{Mono}(W_C/W_F).$$

Proof. That $F = A(B_0 \cap C_0)$ is shown by a proof analogous to that of Corollary 3.6. Theorem 5.5 implies that F is generated by elements in $A_0 \cap B_0$, $A_0 \cap C_0$, and $B_0 \cap C_0$. For notational convenience assume that $A_0 \cap B_0 \cap C_0 = \langle e \rangle$, that is F is now isomorphic to $\mathrm{Mono}(W_{A\cap B\cap C}/W_F)$ and $F = (A_0 \cap B_0) \times (A_0 \cap C_0) \times (B_0 \cap C_0)$. Now it is seen that $F = A_0(B_0 \cap C_0)$. $\mathrm{Mono}(W_A/W_F) \cong F/A_0 \cong (B_0 \cap C_0)$ etc. □

THEOREM 5.7. *A* HyCu(p) *does not exist.*

Proof. Suppose such a hypercube does exist. Let the fundamental groups at the various levels be F, A, B, C, D, $A \cap B, \ldots, A \cap B \cap C, \ldots, A \cap B \cap C \cap D$. For $t \in Pt_F$ let $f \in F$ circle t. We show

1) If $f \in A_0$ then $f \in (B_0 \cap C_0) \cup (B_0 \cap D_0) \cup (C_0 \cap D_0)$.

2) If $f \notin A_0 \cup B_0 \cup C_0 \cup D_0$ then $f^p \in A_0 \cap B_0 \cap C_0 \cap D_0$.

1) As in Lemma 5.3, suppose $f \in A_0$. If $Ax_1, \ldots, Ax_p$ is a coset decomposition for F then $x_i f x_i^{-1} \in A_0$ for all i. The covering $W^*_{A\cap B\cap C\cap D} \to W^*_A$ is a Cu(p), so either $x_i f x_i^{-1} \in (A_0 \cap (B_0 \cap C_0)) \cup (A_0 \cap (B_0 \cap D_0)) \cup (A_0 \cap (C_0 \cap D_0))$ or $x_i f^p x_i^{-1} \in A_0 \cap B_0 \cap C_0 \cap D_0$. But this latter alternative would hold for all paths $x_i f x_i^{-1}$ circling the in Pt_A above t. Since there is at most only one point in Pt_A above t with this property (Theorem 5.5) the second alternative is ruled out.

2) If $f \notin A_0 \cup B_0 \cup C_0 \cup D_0$ then there are four coverings $W_H^* \to W_F^*$, $H = A, B, C, D$, and all are ramified over t. Using the arguments of Lemma 5.4 (multiplicities over t with respect to $W_D^* \to W_F^*$ are $\delta_1, \delta_2, \ldots, \Sigma\delta_\ell = p$) let α', the smallest of all the α's, β's, γ's, δ's. We can then find β', γ', δ', so that $\beta' > \alpha'$, $\gamma' > \alpha'$, $\delta' > \alpha'$. As in Lemma 5.4 we conclude that $f^{\alpha'} \notin B_0 \cup C_0 \cup D_0$. There is a x_1 so that $Ax_1 f^{\alpha'} = Ax_1$ or $x_1 f^{\alpha'} x_1^{-1} \in A$. Let $x_1 f^{\alpha'} x_1^{-1}$ circle $a \in Pt_A$. Since $W_{A\cap B\cap C\cap D}^* \to W_A^*$ is a Cu(p) the ramification of $W_{A\cap H}^* \to W_A^*$ $(H = B, C, D)$ is not pure above a. We conclude that $(x_1 f^{\alpha'} x_1^{-1})^p \in A \cap B_0 \cap C_0 \cap D_0$.

Consider the square with $W_{A\cap B}^*$ at the top and W_F^* at the bottom. a is the point in Pt_A above t where the multiplicity of $\pi_{A,F}$ is α'. Let b be the point in Pt_B above t where the multiplicity of $\pi_{B,F}$ is β'. If c is a point in $Pt_{A\cap B}$ above t that maps onto a and b, the last paragraph shows that the multiplicity of $\pi_{A\cap B,F}$ at c is $\alpha' p$. But by the discussion preceding Lemma 2.1 this multiplicity is also $[\alpha', \beta']$. Since $\alpha' < \beta' \le p$, it follows that $\beta' = p$. Similarly $\gamma' = \delta' = p$.

Since $\alpha' < p$, there is another point $e \in Pt_A$ over t. If ε is the multiplicity of $\pi_{A,F}$ at e then $\varepsilon < p$. Since we know that $\beta' = \gamma' = \delta' = p$ we have $\beta' = \gamma' = \delta' = p > \varepsilon$. The same argument as above now shows that there is a x_2 so that $x_2 f^\varepsilon x_2^{-1} \in A$ and $(x_2 f^\varepsilon x_2^{-1})^p \in A \cap B_0 \cap C_0 \cap D_0$. W_A^* has two points, a and e, over which each of the three coverings $W_{A\cap H}^* \to W_A^*$, $H = B, C, D$ has ramification p. Since this contradicts Theorem 5.5, we conclude that $\alpha' = p$, and so $f^p \in A_0 \cap B_0 \cap C_0 \cap D_0$.

We now conclude the proof. As before there is one and only one point in Pt_F where alternative 2) holds. This implies that the ramification for any square between levels one and three is separated. Since this is not the case, this contradiction shows that a HyCu(p) does not exist. □

For Theorem 5.7 a proof similar to the one presented appears necessary for there exists a HyCu$((3p^4 - 5p^3 + 2)/2, g(p), h(p), 0, 0)$ where items 1) and 2) in the proof are satisfied.

Theorem 5.8. *Let $W_{g(p)}$ cover four different $W_{h(p)}$'s, each in p sheets, so that each of the six* Iv*'s admit completions. Then $W_{g(p)}$ admits an automorphism group, G, iso-morphic to $Z_p \times Z_p \times Z_p$, and the four $W_{h(p)}$'s are $W_{g(p)}$ modulo four of the Z_p's in G.*

Proof. Three of the coverings $W_{g(p)} \to W_{h(p)}$ give rise to a Cu(p) since three of these coverings cannot be in a square (Corollary 3.7). The fourth such covering must occur within this cube (Theorem 5.7); that is, there are four cubes within one cube. Let A, B, C, D be the four subgroups of F of index p, as before. Then $A \cap B \cap C = A \cap B \cap D = \cdots = A \cap B \cap C \cap D$. For notational convenience let $A_0 \cap B_0 \cap C_0 \cap D_0 = \langle e \rangle$. By Theorem 5.6 F is equal to a product of three different groups $H_0 \cap L_0$ where $H_0 \neq L_0$ and $\{H, L\}$ is any of the three pairs in a subset of $\{A, B, C, D\}$ of order three. Then $(A_0 \cap B_0) \times (A_0 \cap C_0) \times (B_0 \cap C_0) = (A_0 \cap B_0) \times (A_0 \cap D_0) \times (B_0 \cap D_0)$, and so $(A_0 \cap D_0)$ is a normal subgroup of $(A_0 \cap C_0) \times (B_0 \cap C_0)$ intersecting each factor in the direct product in $\langle e\rangle$. $A_0 \cap D_0$ is isomorphic to $\mathrm{Mono}(W_B/W_F)$ and so is isomorphic to a subgroup of S_p, a subgroup whose order is divisible by p. By Lemma 2.4 $A_0 \cap D_0$ is isomorphic to Z_p. By similar arguments we complete the proof of the theorem. $\square$

6. Examples

Let $W_F = \mathbf{P}^1 - \{t_1, t_2, \dots, t_s\}$, the s-fold punctured Riemann sphere. Let $f_1, f_2, \dots, f_s$, be paths in W_F which circle the t's so that $F = \langle f_1, f_2, \dots, f_s \mid f_1 f_2 \cdots f_s = e\rangle$. Let G be a finite group generated by $a_1, a_2, \dots, a_{s-1}$. Let $\mu : F \to G$ be defined by $\mu(f_i) = a_i$ for $i = 1, 2, \dots, s-1$, and $\mu(f_s) = (a_1 a_2 \cdots a_{s-1})^{-1}$. μ extends to a homomorphism from F onto G. Let H be a subgroup of G of index n such that $H_0 = \langle e\rangle$. Let $A = \mu^{-1}(H)$. Then W_A is an n-sheeted covering of W_F corresponding to H and $F/A_0 \cong G$. Suppose G has order m. $W_{A_0}^*$ is the Galois closure for the covering $W_A^* \to W_F^*$.

Notation. $W_m(f_1, f_2, \dots, f_s)$ will denote W_{A_0}, W_A will be denoted by $W_n(f_1, f_2, \dots, f_s)$.

In the following we will consider only cyclic groups, Z_p, and dihedral groups, D_p, as subgroups of S_p, in order to build our direct products, G. Any other subgroup of S_p containing Z_p, such as A_p or S_p, would do, although the computations would be more complicated. Dihedral groups will be denoted $\langle a, \alpha\rangle$, $\langle b, \beta\rangle$, $\langle c, \gamma\rangle$ where $a^2 = b^2 = c^2 = \alpha^p = \beta^p = \gamma^p = e$, and cyclic groups will be denoted simply by $\langle\alpha\rangle$, $\langle\beta\rangle$, $\langle\gamma\rangle$. Thus a $2p$-sheeted dihedral Galois covering of $\mathbf{P}^1$ is denoted $W_{2p}(a, a\alpha, \alpha^{-1})$, and the p-sheeted covering of $\mathbf{P}^1$, $W_{2p}(a, a\alpha, \alpha^{-1})/\langle a\rangle$, is denoted by $W_p(a, a\alpha, \alpha^{-1})$. There are, of course, many such p-sheeted coverings corresponding to the conjugates of $\langle a\rangle$ in D_p.

An example of a $\mathrm{Sq}((p-1)^2; 0, p; 0, p; 0)$ is given by $W_{p^2}(a, a, a\alpha, a\alpha, b, b, b\beta, b\beta)$ where $G = D_p \times D_p$. Examples of $\mathrm{Cu}(g(p), h(p), 0, 0)$ are:

(i)	$W_{p^3}(a, a\alpha, b, b\beta, c, c\gamma, (\alpha\beta\gamma)^{-1})$	$G = D_p \times D_p \times D_p$
(ii)	$W_{p^3}(a, a\alpha, b, b\beta, \gamma, (\alpha\beta\gamma)^{-1})$	$G = D_p \times D_p \times Z_p$
(iii)	$W_{p^3}(a, a\alpha, \beta, \gamma, (\alpha\beta\gamma)^{-1})$	$G = D_p \times Z_p \times Z_p$
(iv)	$W_{p^3}(\alpha, \beta, \gamma, (\alpha\beta\gamma)^{-1})$	$G = Z_p \times Z_p \times Z_p$

As an example we will work out the case (ii). Let λ be the genus of $W_{4p^3}(a, a\alpha, b, b\beta, \gamma, (\alpha\beta\gamma)^{-1})$. Then

$$2\lambda - 2 = -8p^3 + 4(4p^3/2) + 2(4p^2(p-1))$$

W_λ is a 4-sheeted Galois covering of $W_{p^3}(a, a\alpha, b, b\beta, \gamma, (\alpha\beta\gamma)^{-1})$ of genus g. $W_g = W_\lambda/\langle a, b\rangle$ where $\langle a, b\rangle \cong Z_2 \times Z_2$. Since an involution in D_p has p conjugates, an involution in $\langle a, \alpha\rangle$ will have $2(4p^3/2)/p$ $(= 4p^2)$ fixed points. ab

is an involution without fixed points. Thus the total ramification for $W_\lambda \to W_g$ is $2(4p^2)$ and so

$$2\lambda - 2 = 4(2g-2) + 8p^2, \quad \text{and so } g = g(p).$$

To find the genus of the Riemann surface covered by W_g in p sheets, call it W_h, we could either consider $W_{2p^2}(b, b\beta, \gamma, (\beta\gamma)^{-1})$ and proceed as above, or consider $W_\lambda/\langle a, \alpha, b\rangle$. In the latter case $\langle a, \alpha, b\rangle$ has $p+1$ involutions with ramification $2p^2$ for each. All the other subgroups are unramified or contain one of the ramified involutions. Therefore,

$$2\lambda - 2 = 4p(2h-2) + (p+1)4p^2, \quad \text{or} \quad h = h(p).$$

$W_\lambda/\langle a, b, \gamma\rangle$ also has genus $h(p)$ since the ramification accounted for in $\langle a, b, \gamma\rangle$ is $2(4p^2) + 4p^2(p-1)$.

In case (iv) we have four coverings $W_{g(p)} \to W_{h(p)}$, one for each of the four punctures in $\mathbf{P}^1$. The kernel of $\mu : F \to Z_p \times Z_p \times Z_p$ is a characteristic subgroup of F, so any fractional linear transformation (FLT) of $\mathbf{P}^1$ that permutes the four points lifts to $W_{g(p)}$. There is always a $Z_2 \times Z_2$ of such FLT's, but by special arrangements of the four points we can have a dihedral group D_4 (order 8) or an alternating group A_4 (order 12) permuting the four points. Thus $W_{g(p)}$ always admits an automorphism group of order $4p^3$, but can also admit groups of order $8p^3$ and $12p^3$.

7. Genus ten

On a Riemann surface W_{10} a *quartet* is a set of four complete half-canonical linear series: g_9^2, h_9^2, k_9^2, ℓ_9^3, whose sum is bicanonical, and where ℓ_9^3 is the unique linear series on W_{10} of dimension 3 and degree 9. By Riemann's solution to the Jacobi inversion problem [5] this is equivalent to the vanishing of the theta function for W_{10} at four half periods, whose sum is zero, to orders 3, 3, 3, 4, where the last half-period is the only point on the Jacobian where the theta function vanishes to order 4 or more.

In [3] it was shown that the existence of a quartet on a W_{10} is equivalent to the existence of a *full* three-sheeted covering $W_{10} \to W_1$ ("full" means that K_{10} is the completion of the lift of a g_6^5 on W_1). Unfortunately, the methods do not appear to distinguish between cyclic and dihedral coverings.

The existence of two quartets, (necessarily with the same ℓ_9^3) gives a CSIv which admits a completion by Theorem 4.6. The existence of three quartets leads to a $\mathrm{Cu}(10, 1, 0, 0)$. The existence of four quartets leads to the existence of an elementary abelian group of order 27 on W_{10}, four subgroups of order three giving rise to the four coverings $W_{10} \to W_1$ (Theorem 5.8).

Conversely, the existence of such a group of automorphisms on W_{10} implies the existence of six $\mathrm{CSIv}(10; 1, 3; 1, 3)$'s all of which have completions. The proof of Theorem 4.5 shows that all the coverings $W_{10} \to W_1$ are full. Thus W_{10} admits four quartets.

THEOREM 7. *For a Riemann surface of genus* 10, W_{10}, *the following two statements are equivalent.*

(1) W_{10} *admits an elementary abelian group of order* 27. *Four cyclic subgroups give rise to quotients of genus one, and the remaining* 9 *cyclic subgroups are fixed point free.*

(2) W_{10} *admits* 4 *quartets.*

Remarks. 1) The W_{10} in Theorem 7 can be simply described as a W_{10} admitting an elementary abelian group, G, of order 27 so that the genus of W_{10}/G is zero.

2) By the remarks at the end of Section 6 we have the following. Any such W_{10} admits an automorphism group of order 4·27. There are such W_{10}'s admitting automorphism groups of order 8·27 and 12·27.

3) The curve $x^6 + y^6 + z^6 = 0$ is not such a Riemann surface since the 3-Sylow subgroup is not abelian.

8. Coverings of the Riemann sphere

In this section we consider CSIv$(g; 0, p; 0, q)$'s where $g = (p-1)(q-1)$, and p and q need not be prime.

We first do some naïve dimension counting. A generic W_g on the top of such an Iv admits a plane model as a plane curve of degree $p+q$ with two ordinary singularities of multiplicities p and q. The dimension of such a family of plane curves is $[(p+q+3)(p+q) - p(p+1) - q(q+1)]/2 - 4 = pq + p + q - 4$. The two singularities arise from picking a divisor from each of the two distinguished linear series on the Riemann surface, so the dimension in moduli space for genus $pq+1$ is $pq + p + q - 6$.

To find the dimension for Sq$(g; 0, p; 0, q; 0)$'s we want the fundamental group, F, for the punctured Riemann sphere to have the maximum number of punctures. Note that Riemann spheres occur at the two middle levels of the square. Let G, as in the examples in Section 6, be $S_p \times S_q$, and let the square arise from

$$W_{(p-1)(q-1)}(a_1, a_2, \ldots, a_{2p-2}; b_1, b_2, \ldots, b_{2q-2})$$

where the a's are transpositions generating S_p, and the b's are transpositions generating S_q. So the dimension in moduli space for genus $(p-1)(q-1)$ for such squares is $(2p-2) + (2q-2) - 3 = 2p + 2q - 7$. It's codimension in the space of the above Iv's is $(p-1)(q-1)$.

We now give a geometric interpretation for a Sq$((p-1)(q-1); 0, p; 0, q; 0)$.

Let C_{p+q} be a plane curve of degree $p+q$ with two ordinary singularities R_p, R_q of multiplicities p and q (genus $= (p-1)(q-1)$). Let x be a point on the curve. A line through x and R_p cuts the curve in q points. q lines connect these q points to R_q. A line through x and R_q cuts the curve in p points. p lines connect these p points to R_p. This set of $p+q$ lines intersect (in general) in pq points. At least $p+q-1$ of these points lie on the curve.

DEFINITION. x will be called *total* of all pq points lie on the curve.

THEOREM 8.1. *If one point of C_{p+q} is total then all points of C_{p+q} are total.*

We will prove this theorem in several steps.

LEMMA 8.2. *Let g_p^1 be the fibers of the p-sheeted covering in the* Iv. *Let g_q^1 be the fibers in the q-sheeted covering in the* Iv. *Then $(q-1)g_p^1$ (and also $(p-1)g_q^1$) is not special.*

Proof. Suppose $(q-1)g_p^1$ is special. Fix a divisor, E, in $K_g-(q-1)g_p^1$. Let y_1 be a general point in W_g. Let $y_1+y_2+\cdots+y_q$ be the divisor in g_q^1 containing y_1. Let D_i, $i=2,3,\ldots,q$, be the divisors in g_p^1 containing y_i. Then ΣD_i is a divisor in $(q-1)g_p^1$ not containing y_1. By Riemann-Roch y_1 is in E since g_q^1 is always special. This contradiction proves the lemma. □

LEMMA 8.3. *The* CSIv$(g;0,p;0,q)$ *admits a completion if and only if pg_q^1 is equivalent to qg_p^1.*

Proof. For notational convenience denote the above CSIv by CSIv$(g;h,p;k,q)$ so that $h=k=0$. If this Iv admits a completion $\mathbf{P}^1$, then the g_1^1 on $\mathbf{P}^1$ lifts to a g_q^q on W_h, which lifts in turn to qg_p^1 on W_g. Lifting g_1^1 thru W_k gives a pg_q^1 on W_g, and so pg_q^1 is equivalent to qg_p^1. For the converse note that $|pg_q^1|=g_{pq}^{p+q-1}$ since pg_q^1 is not special. The incomplete qg_p^1 has dimension q and the incomplete pg_q^1 has dimension p. Thus there is a g_{pq}^1 in g_{pq}^{p+q-1} common to both of these incomplete linear series. This g_{pq}^1 completes the Iv. □

Note that this shows that completing the CSIv is equivalent to all points on C_{p+q} being total.

Proof of Theorem 8.1. If one point is total then pg_q^1 is equivalent to qg_p^1. The result follows from the above lemmas. □

For $g=4$, the completion of the Iv is equivalent to $3g_3^1\equiv 3h_3^1$. Since $g_3^1+h_3^1$ is canonical this is equivalent to $6g_3^1$ being tricanonical. By Riemann's theorem this in turn is equivalent to the theta function vanishing to order 2 at a $1/6^{\text{th}}$-period (which is not a half-period).

Alan Landman has shown that Theorem 8.1 is a special case of a theorem where the hypotheses are quite a bit more general and the conclusion is the same.

REFERENCES

[1] R. D. M. ACCOLA, Plane models for Riemann surfaces admitting certain half-canonical linear series, **II**, Trans of the AMS **263** (1981), 243–259.

[2] R. D. M. Accola, Topics in the theory of Riemann surfaces, Lecture notes in mathematics **1595**, Springer, 1994.

[3] R. D. M. Accola, Vanishing thetanulls for some dihedral and cyclic coverings of Riemann surfaces, Kodai Math. J. **28** (2005), 73–91.

[4] Ernst Kani, On Castelnuovo's equivalence defect, Journal fur die Reine und Angewandte Mathematik **352** (1984), 24–69.

[5] J. Lewittes, Riemann surfaces and the theta function, Acta Mathematica **111** (1964), 37–61.

Department of Mathematics
Brown University
Providence RI, 02912
E-mail: raccola@math.brown.edu