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HERMITIAN MANIFOLDS WITH FLAT ASSOCIATED CONNECTION

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Abstract

A local classification of the Hermitian manifolds with flat associated connection is given. Hermitian manifolds admitting locally a conformal metric with flat associated connection are characterized by a curvature identity. Locally conformal Kähler manifolds as well as Hermitian surfaces with vanishing associated conformal curvature tensor are characterized as locally conformal to a Kähler manifold of constant holomorphic sectional curvatures.

1. Introduction

Let (M, g, J) be a Hermitian manifold with metric g and complex structure J.

One of the characterizations of the Chern connection D on a Hermitian manifold is that the Chern connection is the unique natural connection ($Dg = DJ = 0$) with torsion T so that the symmetric connection $D - \frac{1}{2}T$ is complex.

In this paper we prove that any symmetric connection ∇ on a complex (Hermitian) manifold admits a unique symmetric complex connection $\tilde{\nabla}$ so that for any two holomorphic vector fields X, Y the vector field $\tilde{\nabla}_X Y$ coincides with the $(1,0)$-part of $\nabla_X Y$. We call the connection $\tilde{\nabla}$ associated with ∇.

If ∇ is the Levi-Civita connection on a Hermitian manifold, then it follows that the associated connection $\tilde{\nabla}$ is the symmetric complex connection $D - \frac{1}{2}T$.

In Theorem 3.1 we precise the conditions under which a given complex symmetric connection is holomorphically projectively flat in dimension 4.

Using the fact that a Hermitian manifold with flat associated connection $\tilde{\nabla}$ admits locally special holomorphic coordinates in which the local components of $\tilde{\nabla}$ are zero, we obtain a local classification of these manifolds. Namely, the local components of the metric with respect to special holomorphic coordinates are found in Theorem 5.2.

In Section 6 we show that any conformal transformation of the metric g induces locally a holomorphically projective transformation (with closed 1-form)

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of the symmetric complex connection $\tilde{\nabla}$ and vice versa. Then the holomorphically projective tensor P_H of the connection $\tilde{\nabla}$ gives rise to the associated conformal curvature tensor $\tilde{W}$ which is a conformal invariant.

In Theorem 6.4 we give the following geometric interpretation of the condition $\tilde{W} = 0$ in the class of locally conformal Kähler manifolds:

Let (M, g, J) $(\dim M = 2n \geq 6)$ *be a locally conformal Kähler manifold with associated conformal curvature tensor* $\tilde{W}$. *Then the following conditions are equivalent*:

(i) $\tilde{W} = 0$;

(ii) $\tilde{W}^{\lambda}_{\alpha\bar{\beta}\gamma} = 0$;

(iii) *there exists locally a conformal metric with flat associated connection*;

(iv) *the metric g is locally conformally equivalent to a Kähler metric of constant holomorphic sectional curvatures.*

In Theorem 6.7 we characterize the Hermitian surfaces with vanishing associated conformal curvature tensor:

Let (M, g, J) *be a Hermitian surface with associated conformal curvature tensor* $\tilde{W}$. *Then the following conditions are equivalent*:

(i) $\tilde{W} = 0$;

(ii) *there exists locally a conformal metric with flat associated connection*;

(iii) *the metric g is locally conformally equivalent to a Kähler metric of constant holomorphic sectional curvatures.*

In Example 1 (Example 2) we give conformal Kähler metrics with flat associated connection.

2. Symmetric connections on a complex manifold

Let (M, J) be a complex manifold with complex structure J. The tangent space to M at a point $p \in M$ and its complexification are denoted by T_pM and $T_p^{\mathbf{C}}M$, respectively. By $\mathfrak{X}M$ and $\mathfrak{X}^{\mathbf{C}}M$ we denote the algebras of real and complex differentiable vector fields on M, respectively. The complex structure J generates the standard splittings

$$T_p^{\mathbf{C}}M = T_p^{1,0}M \oplus T_p^{0,1}M, \quad \mathfrak{X}^{\mathbf{C}}M = \mathfrak{X}^{1,0}M \oplus \mathfrak{X}^{0,1}M.$$

If $\dim_{\mathbf{C}} M = n$ and $z^1, \ldots, z^n$ are local holomorphic coordinates in a neighborhood U, then the vector fields $\partial_\alpha = \partial/\partial z^\alpha$, $\alpha = 1, \ldots, n$ (resp. $\partial_{\bar{\alpha}} = \partial/\partial z^{\bar{\alpha}}$, $\bar{\alpha} = \bar{1}, \ldots, \bar{n}$) form a basis for $T_p^{1,0}M$ (resp. $T_p^{0,1}M$) at any point $p \in U$.

In what follows Greek indices $\alpha, \beta, \gamma, \ldots$ run from 1 to n, while Latin indices $i, j, k, \ldots$ run from 1 to $2n$.

Throughout the whole paper the standard summation convention is assumed.

Let ∇ be an arbitrary symmetric connection on M. Then the condition that (M, J) is a complex manifold is equivalent to the identity

$$(\nabla_X J)JY + (\nabla_{JX} J)Y = (\nabla_Y J)JX + (\nabla_{JY} J)X, \quad X, Y \in \mathfrak{X}M. \tag{2.1}$$

The aim of this section is to prove the following statement.

PROPOSITION 2.1. *Let ∇ be a symmetric connection on a complex manifold (M, J). Then there exists a unique symmetric complex connection $\tilde{\nabla}$ on M, satisfying the condition*:

$$\tilde{\nabla}_X Y = (\nabla_X Y)^{1,0}, \quad X, Y \in \mathfrak{X}^{1,0}M.$$

Proof. Any symmetric connection $\tilde{\nabla}$ on M is determined by the equality

$$\tilde{\nabla}_X Y = \nabla_X Y + Q(X, Y), \quad X, Y \in \mathfrak{X}M,$$

where Q is a symmetric tensor. Since $\tilde{\nabla}$ is complex, then we have

$$JQ(X, Y) - Q(X, JY) = (\nabla_X J)Y$$

or in local holomorphic coordinates

$$Q^{\gamma}_{\alpha\bar{\beta}} = -\frac{i}{2}\nabla_\alpha J^{\gamma}_{\bar{\beta}}, \quad Q^{\bar{\gamma}}_{\alpha\beta} = \frac{i}{2}\nabla_\alpha J^{\bar{\gamma}}_{\beta}. \tag{2.2}$$

The first equality of (2.2) is equivalent to the next one

$$Q(X, Y) + Q(JX, JY) = (\nabla_X J)JY - (\nabla_{JY}J)X, \tag{2.3}$$

while the second equality of (2.2) is equivalent to the following one

$$Q(X, Y) - Q(JX, JY) + JQ(JX, Y) + JQ(X, JY) = (\nabla_X J)JY + (\nabla_{JX}J)Y. \tag{2.4}$$

Let $X, Y \in \mathfrak{X}^{1,0}M$. Then the condition $\tilde{\nabla}_X Y = (\nabla_X Y)^{1,0}$ is equivalent to the identity

$$Q(X, Y) - Q(JX, JY) - JQ(JX, Y) - JQ(X, JY) = 0. \tag{2.5}$$

Now (2.3), (2.4) and (2.5) imply that

$$Q(X, Y) = \frac{1}{4}\{2(\nabla_X J)JY + (\nabla_Y J)JX - (\nabla_{JY}J)X\}.$$

Because of (2.1) the tensor Q is symmetric.

Hence, the connection $\tilde{\nabla}$ is determined by

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{4}\{(2(\nabla_X J)JY + (\nabla_Y J)JX - (\nabla_{JY}J)X\}. \tag{2.6}$$

An immediate verification shows that the connection (2.6) satisfies the conditions of the proposition. QED

We call the connection $\tilde{\nabla}$ from Proposition 2.1 *associated with* ∇.

The local components of the symmetric connection ∇ satisfy the equalities

$$\Gamma^{\bar{\gamma}}_{\alpha\beta} = -\frac{i}{2}\nabla_\alpha J^{\bar{\gamma}}_{\beta}, \quad \Gamma^{\gamma}_{\alpha\bar{\beta}} = \frac{i}{2}\nabla_\alpha J^{\gamma}_{\bar{\beta}}.$$

Hence the local components of the associated connection satisfy the conditions

$$\tilde{\Gamma}^{\bar{\gamma}}_{\alpha\beta} = 0, \quad \tilde{\Gamma}^{\gamma}_{\alpha\bar{\beta}} = 0.$$

Then the essential components of the associated connection $\tilde{\nabla}$ (i.e. those that may not be zero) are:

$$\tilde{\Gamma}^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta}.$$

An application to the Riemannian connection on a Hermitian manifold.

Let (M, g, J) be a Hermitian manifold. It is well known that Hermitian manifolds are characterized in terms of the Levi-Civita connection ∇ of the metric g by the following identity (e.g. [4])

$$(\nabla_{JX}J)Y = J(\nabla_X J)Y, \quad X, Y \in \mathfrak{X}M,$$

which is equivalent to the identity

$$\nabla_\alpha J^{\bar{\gamma}}_{\beta} = 0$$

in local holomorphic coordinates. Then the equality (2.6) reduces to

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\{(\nabla_X J)JY + (\nabla_Y J)JX\}. \tag{2.7}$$

Another approach to the connection $\tilde{\nabla}$ has been used in [1, 2]. Let D be the Hermitian (Chern) connection of the manifold (M, g, J) and T be its torsion tensor. By using the relation between ∇ and D, it follows that

$$\tilde{\nabla} = D - \frac{1}{2}T,$$

i.e. $\tilde{\nabla}$ is the unique symmetric connection having the same geodesics with the same affine parameter as the Hermitian connection D.

The associated connection has been studied from the point of view of its affine group of transformations in [1]. Hermitian manifolds with flat associated connection have been treated in [3] with respect to their complex holomorphic sectional curvatures. Hermitian manifolds with flat associated connection satisfying the Einstein condition with respect to the Hermitian curvature have been investigated in [2].

3. A note on the holomorphically projective curvature tensor

Let $(M, J, \tilde{\nabla})$ ($\dim M = 2n \geq 4$) be a complex manifold with complex structure J and complex symmetric connection $\tilde{\nabla}$. The curvature tensor of $\tilde{\nabla}$ and its Ricci tensor are denoted by $\tilde{R}$ and $\tilde{\rho}$, respectively. If $\tilde{\nabla}'$ is a complex symmetric connection on M with the same holomorphic planar curves as $\tilde{\nabla}$, then [5]

$$\tilde{\nabla}'_X Y = \tilde{\nabla}_X Y + \omega(X)Y + \omega(Y)X - \omega(JX)JY - \omega(JY)JX, \quad X, Y \in \mathfrak{X}M \tag{3.1}$$

for some 1-form ω on M.

A transformation of $\tilde{\nabla}$ onto $\tilde{\nabla}'$ given by (3.1) is called a holomorphically projective transformation of $\tilde{\nabla}$.

The tensor

$$
\begin{aligned}
(3.2)\qquad P_H(X,Y)Z = {} & \tilde{R}(X,Y)Z - P(Y,Z)X + P(X,Z)Y \\
& + \{P(X,Y) - P(Y,X)\}Z + P(Y,JZ)JX \\
& - P(X,JZ)JY - \{P(X,JY) - P(Y,JX)\}JZ,
\end{aligned}
$$

where

$$
(3.3)\qquad P(X,Y) = \frac{1}{4(n^2-1)}\{(2n-1)\tilde{\rho}(X,Y) + \tilde{\rho}(Y,X) - \tilde{\rho}(JX,JY) - \tilde{\rho}(JY,JX)\}
$$

is called the holomorphically projective curvature tensor.

If P'_H is the corresponding holomorphically projective tensor associated with $\tilde{\nabla}'$, then $P'_H = P_H$ [6].

The complex manifold $(M, J, \tilde{\nabla})$ is said to be holomorphically projectively flat if there exists a 1-form ω, so that the connection $\tilde{\nabla}'$ is flat, i.e. its curvature tensor $\tilde{R}' = 0$. Therefore, if $(M, J, \tilde{\nabla})$ is holomorphically projectively flat, then its holomorphically projective tensor $P_H = 0$.

The inverse problem is treated in [6] and [7] as follows:

Let the holomorphically projective tensor, given by (3.2) and (3.3) vanish identically. To find a 1-form ω, so that the corresponding connection $\tilde{\nabla}'$ (3.1) is flat, is equivalent to solving the equation

$$
(3.4)\qquad (\nabla_X \omega)(Y) - \omega(X)\omega(Y) + \omega(JX)\omega(JY) = P(X,Y),
$$

with respect to ω, P being given by (3.3).

Since the integrability conditions for the system (3.4) are

$$
(3.5)\qquad (\nabla_X P)(Y,Z) = (\nabla_Y P)(X,Z), \quad X, Y, Z \in \mathfrak{X}M,
$$

the main point is to prove that the condition $P_H = 0$ implies (3.5). Applying the second Bianchi identity to both sides of (3.2) and taking the two possible contractions in the resulting equality, it follows that

$$
\begin{aligned}
(3.6)\qquad & (\nabla_X P)(Y,Z) - (\nabla_Y P)(X,Z) + (\nabla_Y P)(Z,X) \\
& - (\nabla_Z P)(Y,X) + (\nabla_Z P)(X,Y) - (\nabla_X P)(Z,Y) = 0
\end{aligned}
$$

and

$$
\begin{aligned}
(3.7)\qquad & (2n-1)[(\nabla_X P)(Y,Z) - (\nabla_Y P)(X,Z)] \\
& + (\nabla_X P)(JY,JZ) - (\nabla_{JY} P)(X,JZ) \\
& + (\nabla_{JX} P)(Y,JZ) - (\nabla_Y P)(JX,JZ) \\
& - (\nabla_Y P)(JZ,JX) + (\nabla_{JZ} P)(Y,JX) \\
& - (\nabla_{JZ} P)(X,JY) + (\nabla_X P)(JZ,JY) = 0.
\end{aligned}
$$

Here we precise the exact corollaries from the above equalities concerning the components of the tensor $(\tilde{\nabla}_X P)(Y,Z) - (\tilde{\nabla}_Y P)(X,Z)$ with respect to local holomorphic coordinates. From (3.6) and (3.7) we obtain

$$
\begin{aligned}
&(n-2)(\tilde{\nabla}_\alpha P_{\beta\gamma} - \tilde{\nabla}_\beta P_{\alpha\gamma}) = 0,\\
&(n+1)(\tilde{\nabla}_\alpha P_{\beta\bar{\gamma}} - \tilde{\nabla}_\beta P_{\alpha\bar{\gamma}}) = 0,\\
&(n-1)(\tilde{\nabla}_\alpha P_{\bar{\beta}\gamma} - \tilde{\nabla}_{\bar{\beta}} P_{\alpha\gamma}) = 0.
\end{aligned} \tag{3.8}
$$

Hence, the condition $P_H = 0$ implies (3.5) in dim $M = 2n \geq 6$.

Taking into account (3.8), we obtain the following statement in dim $M = 4$.

THEOREM 3.1. *Let $(M, J, \tilde{\nabla})$ be a complex surface with complex symmetric connection $\tilde{\nabla}$. Then $(M, J, \tilde{\nabla})$ is holomorphically projectively flat if and only if:*

$$P_H = 0 \quad \textit{and} \quad \tilde{\nabla}_\alpha P_{\beta\gamma} - \tilde{\nabla}_\beta P_{\alpha\gamma} = 0.$$

Finally we consider the holomorphically projective transformations (3.1) with closed 1-form ω. We call them *special* holomorphically projective transformations. Since the 1-form ω is closed, then the tensor

$$(\nabla_X \omega) Y - \omega(X)\omega(Y) + \omega(JX)\omega(JY)$$

is symmetric and the relation between the the tensors $P(X, Y)$ and $P'(X, Y)$ implies that $P'(X, Y) - P'(Y, X) = P(X, Y) - P(Y, X)$. From here and (3.3) it follows that the tensor

$$\tilde{\rho}(X, Y) - \tilde{\rho}(Y, X), \quad X, Y \in \mathfrak{X}M \tag{3.9}$$

is an invariant of the special holomorphically projective transformations.

Let us denote $\tilde{S}(X, Y) = \frac{1}{2}\{\tilde{\rho}(X, Y) + \tilde{\rho}(Y, X)\}$. Then the tensor

$$
\begin{aligned}
P^0_H(X, Y)Z = {} & \tilde{R}(X, Y)Z - P^0(Y, Z)X + P^0(X, Z)Y\\
& + P^0(Y, JZ)JX - P^0(X, JZ)JY\\
& - \{P^0(X, JY) - P^0(Y, JX)\}JZ,
\end{aligned} \tag{3.10}
$$

where

$$P^0(X, Y) = \frac{1}{2(n^2-1)}\{n\tilde{S}(X, Y) - \tilde{S}(JX, JY)\} \tag{3.11}$$

is an invariant of the special holomorphically projective transformations.

The invariance of the tensor given by (3.10) is equivalent to the invariance of both tensors given by (3.2) and (3.9).

4. Relations between the basic linear connections on a Hermitian manifold

In this section we consider the relations between the connections and their curvatures, which we deal with in the next.

Let (M, g, J) be a Hermitian manifold with fundamental form

$$\Phi(X, Y) = g(JX, Y), \quad X, Y \in \mathfrak{X}M.$$

The tensor

$$F(X, Y, Z) = g((\nabla_X J) Y, Z) = (\nabla_X \Phi)(Y, Z), \quad X, Y, Z \in \mathfrak{X}M$$

gives the deviation of Hermitian geometry from Kähler geometry. The essential components of F with respect to local holomorphic coordinates are

$$F_{\bar{\gamma}\alpha\beta} = \nabla_{\bar{\gamma}} \Phi_{\alpha\beta} = \partial_\alpha \Phi_{\beta\bar{\gamma}} - \partial_\beta \Phi_{\alpha\bar{\gamma}}. \tag{4.1}$$

The Lee form of the Hermitian structure (g, J) at a point $p \in M$ is given by

$$\theta(X) = -F(e_i, JX, e_j) g^{ij}, \quad X \in \mathfrak{X}M,$$

where $\{e_i\}$ $i = 1, \ldots, 2n$ is an arbitrary basis at p and (g^{ij}) is the inverse of the matrix $(g_{ij}) = (g(e_i, e_j))$. In local holomorphic coordinates the defining equality for θ has the form

$$\theta_\alpha = -i g^{\lambda\bar{\mu}} F_{\bar{\mu}\alpha\lambda}, \tag{4.2}$$

where $(g^{\lambda\bar{\mu}})$ is the inverse of the matrix $(g_{\lambda\bar{\mu}})$.

In this paper we deal with the following linear connections on (M, g, J).

1) *The Levi-Civita connection* ∇

The local components Γ^C_{AB} of ∇ satisfy the following relations:

$$\Gamma^{\bar{\gamma}}_{\alpha\beta} = \nabla_\alpha J^{\bar{\gamma}}_{\beta} = 0, \quad \Gamma^{\gamma}_{\alpha\bar{\beta}} = \frac{i}{2} F_{\alpha\bar{\beta}\bar{\sigma}} g^{\gamma\bar{\sigma}}. \tag{4.3}$$

There are no conditions for the components $\Gamma^\gamma_{\alpha\beta}$ of ∇.

The Riemannian curvature tensor R is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X, Y, Z \in \mathfrak{X}M$$

and the corresponding curvature tensor of type $(0, 4)$ is given by

$$R(X, Y, Z, U) = g(R(X, Y)Z, U), \quad X, Y, Z, U \in \mathfrak{X}M.$$

The curvature tensor R has two types of essential local components:

$$R^\lambda_{\alpha\bar{\beta}\gamma} \ (\text{or } R_{\alpha\bar{\beta}\gamma\bar{\delta}} = R^\lambda_{\alpha\bar{\beta}\gamma} g_{\lambda\bar{\delta}}) \quad \text{and} \quad R^\lambda_{\alpha\beta\gamma} \ (\text{or } R_{\alpha\beta\gamma\bar{\delta}} = R^\lambda_{\alpha\beta\gamma} g_{\lambda\bar{\delta}}).$$

The two Ricci tensors with respect to R at any point $p \in M$ are defined by

$$\rho(Y, Z) = g^{ij} R(e_i, Y, Z, e_j), \quad \rho^*(Y, Z) = g^{ij} R(e_i, Y, JZ, Je_j),$$

where $\{e_i\}$ $i = 1, \ldots, 2n$ is an arbitrary basis of the tangent space at p. In local holomorphic coordinates these tensors satisfy the conditions:

$$\rho_{\beta\bar{\gamma}} = \rho_{\bar{\gamma}\beta}, \quad \rho_{\beta\gamma} = \rho_{\gamma\beta};$$
$$\rho^*_{\beta\bar{\gamma}} = \rho^*_{\bar{\gamma}\beta}, \quad \rho^*_{\beta\gamma} = -\rho^*_{\gamma\beta}.$$

Further we need the trace $R^\alpha_{\alpha\beta\gamma}$, which is

$$R^\alpha_{\alpha\beta\gamma} = \frac{1}{2}(\rho_{\beta\gamma} + \rho^*_{\beta\gamma}). \tag{4.4}$$

Applying the Ricci identity to the derivations of the fundamental form Φ we find

$$\nabla_\gamma \nabla_{\bar\delta} \Phi_{\alpha\beta} - \nabla_{\bar\delta} \nabla_\gamma \Phi_{\alpha\beta} = 2iR_{\alpha\beta\gamma\bar\delta}.$$

Then, by virtue of (4.3) we obtain

$$2iR_{\alpha\beta\gamma\bar\delta} = \nabla_\gamma F_{\bar\delta\alpha\beta} - \frac{i}{2} F_{\bar\delta\gamma\lambda} F_{\bar\mu\alpha\beta} g^{\lambda\bar\mu}.$$

After a contraction the last equality, (4.2) and (4.4) imply

$$R^\alpha_{\alpha\beta\gamma} = \frac{1}{2}(\rho_{\beta\gamma} + \rho^*_{\beta\gamma}) = -\frac{1}{2}\nabla_\gamma \theta_\beta - \frac{1}{4} F_{\bar\delta\gamma\lambda} F_{\bar\mu\alpha\beta} g^{\lambda\bar\mu}.$$

Hence

$$\begin{aligned} &R^\alpha_{\alpha\beta\gamma} + R^\alpha_{\alpha\gamma\beta} = \rho_{\beta\gamma}, \\ &R^\alpha_{\alpha\beta\gamma} - R^\alpha_{\alpha\gamma\beta} = \rho^*_{\beta\gamma} = \frac{1}{2}(\nabla_\beta \theta_\gamma - \nabla_\gamma \theta_\beta) = \frac{1}{2}\, d\theta_{\beta\gamma}. \end{aligned} \tag{4.5}$$

2) *The associated connection* $\tilde\nabla$

According to (2.7), the only essential components of the associated connection $\tilde\nabla$ are

$$\tilde\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta}. \tag{4.6}$$

We denote the curvature tensor of $\tilde\nabla$ by $\tilde R$. This curvature tensor has two types of essential local components:

$$\begin{aligned} &\tilde R^\lambda_{\alpha\beta\gamma} = \partial_\alpha \Gamma^\lambda_{\beta\gamma} - \partial_\beta \Gamma^\lambda_{\alpha\gamma} + \Gamma^\sigma_{\beta\gamma}\Gamma^\lambda_{\sigma\alpha} - \Gamma^\sigma_{\alpha\gamma}\Gamma^\lambda_{\sigma\beta} = R^\lambda_{\alpha\beta\gamma}, \\ &\tilde R^\lambda_{\alpha\bar\beta\gamma} = -\partial_{\bar\beta}\Gamma^\lambda_{\alpha\gamma}. \end{aligned} \tag{4.7}$$

The Ricci tensor $\tilde\rho(X, Y)$ with respect to the associated curvature tensor $\tilde R$ is introduced in the standard way:

$$\tilde\rho(Y, Z) = g^{ij}\tilde R(e_i, Y, Z, e_j).$$

3) *The Hermitian (Chern) connection* D

This connection is the unique complex metric connection ($DJ = Dg = 0$) with torsion tensor T satisfying the property

$$T(JX, JY) = -T(X, Y), \quad X, Y \in \mathfrak{X}M.$$

The only essential local components $D^\gamma_{\alpha\beta}$ of D satisfy the relation

$$D^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} + \frac{1}{2} T^\gamma_{\alpha\beta} \tag{4.8}$$

and the essential local components of the torsion tensor are given by

$$T^\gamma_{\alpha\beta} = g^{\gamma\bar\sigma}(\partial_\alpha g_{\beta\bar\sigma} - \partial_\beta g_{\alpha\bar\sigma}). \tag{4.9}$$

In view of (4.9) and (4.1) the equality (4.2) becomes

$$\theta_\alpha = T^\sigma_{\alpha\sigma}. \tag{4.10}$$

Denoting by K the curvature tensor of D, we have the following essential local components of K:

$$K^\lambda_{\alpha\bar\beta\gamma} = -\partial_{\bar\beta} D^\lambda_{\alpha\gamma}. \tag{4.11}$$

The Ricci tensor s associated with the Hermitian curvature tensor K is defined in a standard way [2]:

$$s(X, Y) = g^{ij} K(e_i, X, Y, e_j),$$

or in local holomorphic coordinates

$$s_{\bar\beta\gamma} = K^\alpha_{\alpha\bar\beta\gamma}.$$

Taking into account (4.11), the relation

$$K^\lambda_{\alpha\bar\beta\gamma} - K^\lambda_{\gamma\bar\beta\alpha} = -D_{\bar\beta} T^\lambda_{\alpha\gamma}$$

implies that

$$s_{\gamma\bar\beta} - s_{\bar\beta\gamma} = d\theta_{\gamma\bar\beta}. \tag{4.12}$$

From (4.6) and (4.8) it follows that

$$\tilde\Gamma^\lambda_{\alpha\beta} = \frac{1}{2}(D^\lambda_{\alpha\beta} + D^\lambda_{\beta\alpha})$$

and

$$\tilde R^\lambda_{\alpha\bar\beta\gamma} = \frac{1}{2}(K^\lambda_{\alpha\bar\beta\gamma} + K^\lambda_{\gamma\bar\beta\alpha}). \tag{4.13}$$

Taking into account (4.7), (4.5), (4.12) and (4.13) we obtain the relations between the Ricci tensor $\tilde\rho$ and the basic tensors of type $(0, 2)$:

$$\begin{aligned}
&\tilde\rho_{\alpha\beta} = \frac{1}{2}(\rho_{\alpha\beta} + \rho^*_{\alpha\beta}),\\
&\tilde\rho_{\alpha\beta} + \tilde\rho_{\beta\alpha} = \rho_{\alpha\beta},\\
&\tilde\rho_{\alpha\beta} - \tilde\rho_{\beta\alpha} = \rho^*_{\alpha\beta} = \frac{1}{2}\, d\theta_{\alpha\beta},\\
&\tilde\rho_{\alpha\bar\beta} - \tilde\rho_{\bar\beta\alpha} = \frac{1}{2}(s_{\alpha\bar\beta} - s_{\bar\beta\alpha}) = \frac{1}{2}\, d\theta_{\alpha\bar\beta}.
\end{aligned} \tag{4.14}$$

From (4.14) it follows that

$$\tilde\rho(X, Y) - \tilde\rho(Y, X) = \frac{1}{2}\, d\theta(X, Y), \quad X, Y \in \mathfrak{X}M. \tag{4.15}$$

5. A local description of Hermitian manifolds with flat associated connection

In this section we study the class of Hermitian manifolds with flat associated connection $\tilde{\nabla}$.

Let $\tilde{\Gamma}^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta}$ be the essential components of $\tilde{\nabla}$ with respect to the local holomorphic coordinates $z^1, \dots, z^n$. The classical problem to find new holomorphic coordinate functions $w^{\alpha} = w^{\alpha}(z^1, \dots, z^n)$, $\alpha = 1, \dots, n$ satisfying the condition that the components of $\tilde{\nabla}$ with respect to $w^1, \dots, w^n$ are zero (or $\dfrac{\partial}{\partial w^{\alpha}}$ are parallel with respect to $\tilde{\nabla}$) leads to the geometric system of PDE

$$\frac{\partial^2 w^{\gamma}}{\partial z^{\alpha} \partial z^{\beta}} = \Gamma^{\sigma}_{\alpha\beta} \frac{\partial w^{\gamma}}{\partial z^{\sigma}}. \tag{5.1}$$

This system is completely integrable if $\tilde{R} = 0$, i.e. if

$$R^{\lambda}_{\alpha\beta\gamma} = 0, \quad \partial_{\bar{\beta}} \Gamma^{\lambda}_{\alpha\gamma} = 0. \tag{5.2}$$

Thus we have the standard statement

LEMMA 5.1. *Let (M, g, J) be a Hermitian manifold with flat associated connection $\tilde{\nabla}$. Then any point $p \in M$ has a neighborhood with holomorphic coordinates $z^1, \dots, z^n$ satisfying the condition*

$$\tilde{\Gamma}^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta} = 0.$$

We call the local holomorphic coordinates from Lemma 5.1 *special* holomorphic coordinates.

The aim of this section is to describe locally the Hermitian manifolds satisfying (5.2).

According to Lemma 5.1 (5.2) implies that $\Gamma^{\sigma}_{\alpha\beta} g_{\sigma\bar{\gamma}} = 0$, or

$$\partial_{\alpha} g_{\beta\bar{\gamma}} + \partial_{\beta} g_{\alpha\bar{\gamma}} = 0 \tag{5.3}$$

with respect to special holomorphic coordinates.

Further we consider the derivative $\partial_{\alpha}\partial_{\beta} g_{\gamma\bar{\delta}}$ in special holomorphic coordinates. Since this derivative is symmetric in (α, β) and skew symmetric in (β, γ), then we have

$$\begin{aligned} \partial_{\alpha}\partial_{\beta} g_{\gamma\bar{\delta}} &= -\partial_{\alpha}\partial_{\gamma} g_{\beta\bar{\delta}} = -\partial_{\gamma}\partial_{\alpha} g_{\beta\bar{\delta}} = \partial_{\gamma}\partial_{\beta} g_{\alpha\bar{\delta}} \\ &= \partial_{\beta}\partial_{\gamma} g_{\alpha\bar{\delta}} = -\partial_{\beta}\partial_{\alpha} g_{\gamma\bar{\delta}} = -\partial_{\alpha}\partial_{\beta} g_{\gamma\bar{\delta}}. \end{aligned}$$

Hence

$$\partial_{\alpha}\partial_{\beta} g_{\gamma\bar{\delta}} = 0 \quad (\text{and } \partial_{\bar{\alpha}}\partial_{\bar{\beta}} g_{\gamma\bar{\delta}} = 0). \tag{5.4}$$

On the other hand the derivative $\partial_{\alpha}\partial_{\bar{\beta}} g_{\gamma\bar{\delta}}$ has the following symmetries

$$\partial_{\alpha}\partial_{\bar{\beta}} g_{\gamma\bar{\delta}} = \partial_{\bar{\beta}}\partial_{\alpha} g_{\gamma\bar{\delta}} = \partial_{\alpha}\partial_{\bar{\beta}} g_{\bar{\delta}\gamma}.$$

Now we can prove the basic statement in this section.

THEOREM 5.2. *Let (M, g, J) be a Hermitian manifold with flat associated connection. Then in special holomorphic coordinates $z^1, \dots, z^n$ the metric g has the following form*

$$g_{\alpha\bar{\beta}} = a_{\alpha\bar{\beta}\lambda\bar{\mu}} z^{\lambda} z^{\bar{\mu}} + b_{\alpha\bar{\beta}\lambda} z^{\lambda} + b_{\alpha\bar{\beta}\bar{\mu}} z^{\bar{\mu}} + h_{\alpha\bar{\beta}},$$

where $a_{\alpha\bar{\beta}\lambda\bar{\mu}}$, $b_{\alpha\bar{\beta}\lambda}$, $b_{\alpha\bar{\beta}\bar{\mu}}$ and $h_{\alpha\bar{\beta}}$ are constants.

Proof. As a corollary of (5.4) we have

$$g_{\alpha\bar{\beta}} = A_{\alpha\bar{\beta}\lambda}(z^{\bar{1}}, \dots, z^{\bar{n}}) z^{\lambda} + B_{\alpha\bar{\beta}}(z^{\bar{1}}, \dots, z^{\bar{n}}), \tag{5.5}$$

for some anti-holomorphic functions $A_{\alpha\bar{\beta}\lambda}$, $B_{\alpha\bar{\beta}}$ and

$$g_{\alpha\bar{\beta}} = A'_{\alpha\bar{\beta}\bar{\mu}}(z^1, \dots, z^n) z^{\bar{\mu}} + B'_{\alpha\bar{\beta}}(z^1, \dots, z^n), \tag{5.6}$$

for some holomorphic functions $A'_{\alpha\bar{\beta}\bar{\mu}}$, $B'_{\alpha\bar{\beta}}$.

Differentiating (5.5) and (5.6) we get

$$\begin{aligned}
\partial_{\bar{\delta}} g_{\alpha\bar{\beta}} &= \partial_{\bar{\delta}} A_{\alpha\bar{\beta}\lambda}(z^{\bar{1}}, \dots, z^{\bar{n}}) z^{\lambda} + \partial_{\bar{\delta}} B_{\alpha\bar{\beta}}(z^{\bar{1}}, \dots, z^{\bar{n}}) \\
&= A'_{\alpha\bar{\beta}\bar{\delta}}(z^1, \dots, z^n), \\
\partial_{\gamma} g_{\alpha\bar{\beta}} &= A_{\alpha\bar{\beta}\gamma}(z^{\bar{1}}, \dots, z^{\bar{n}}) \\
&= \partial_{\gamma} A'_{\alpha\bar{\beta}\bar{\mu}}(z^1, \dots, z^n) z^{\bar{\mu}} + \partial_{\gamma} B'_{\alpha\bar{\beta}}(z^1, \dots, z^n).
\end{aligned}$$

These equalities imply $\partial_{\bar{\delta}} A_{\alpha\bar{\beta}\gamma} = \partial_{\gamma} A'_{\alpha\bar{\beta}\bar{\delta}} = const$, $\partial_{\bar{\delta}} B_{\alpha\bar{\beta}} = const$, $\partial_{\gamma} B'_{\alpha\bar{\beta}} = const$.

Putting

$$\begin{aligned}
a_{\alpha\bar{\beta}\gamma\bar{\delta}} = \partial_{\bar{\delta}} A_{\alpha\bar{\beta}\gamma} = \partial_{\gamma} A'_{\alpha\bar{\beta}\bar{\delta}} = const, \\
b_{\alpha\bar{\beta}\bar{\delta}} = \partial_{\bar{\delta}} B_{\alpha\bar{\beta}} = const, \\
b_{\alpha\bar{\beta}\gamma} = \partial_{\gamma} B'_{\alpha\bar{\beta}} = const
\end{aligned}$$

we obtain

$$g_{\alpha\bar{\beta}} = a_{\alpha\bar{\beta}\lambda\bar{\mu}} z^{\lambda} z^{\bar{\mu}} + b_{\alpha\bar{\beta}\bar{\mu}} z^{\bar{\mu}} + C_{\alpha\bar{\beta}}(z^1, \dots, z^n)$$

for some holomorphic functions $C_{\alpha\bar{\beta}}$ and similarly

$$g_{\alpha\bar{\beta}} = a_{\alpha\bar{\beta}\lambda\bar{\mu}} z^{\lambda} z^{\bar{\mu}} + b_{\alpha\bar{\beta}\lambda} z^{\lambda} + C'_{\alpha\bar{\beta}}(z^{\bar{1}}, \dots, z^{\bar{n}})$$

for some anti-holomorphic functions $C'_{\alpha\bar{\beta}}$. These equalities give that

$$C_{\alpha\bar{\beta}} = b_{\alpha\bar{\beta}\lambda} z^{\lambda} + h_{\alpha\bar{\beta}},$$

where $h_{\alpha\bar{\beta}} = const$.

Hence

$$g_{\alpha\bar{\beta}} = a_{\alpha\bar{\beta}\lambda\bar{\mu}} z^{\lambda} z^{\bar{\mu}} + b_{\alpha\bar{\beta}\lambda} z^{\lambda} + b_{\alpha\bar{\beta}\bar{\mu}} z^{\bar{\mu}} + h_{\alpha\bar{\beta}}.$$ QED

The constants in the above theorem determine the tensors a_{ijkl}, b_{ijk} and h_{ij} at the initial point with the following properties:

$$a_{\alpha\bar{\beta}\gamma\bar{\delta}} = a_{\bar{\beta}\alpha\gamma\bar{\delta}} = a_{\alpha\bar{\beta}\bar{\delta}\gamma} = -a_{\gamma\bar{\beta}\alpha\bar{\delta}} = -a_{\alpha\bar{\delta}\gamma\bar{\beta}},$$
$$b_{\alpha\bar{\beta}\gamma} = b_{\bar{\beta}\alpha\gamma} = -b_{\gamma\bar{\beta}\alpha}, \quad b_{\alpha\bar{\beta}\bar{\delta}} = b_{\bar{\beta}\alpha\bar{\delta}} = -b_{\alpha\bar{\delta}\bar{\beta}},$$
$$h_{\alpha\bar{\beta}} = h_{\bar{\beta}\alpha}.$$

6. Conformal invariants with respect to the associated curvature tensor

In this section we study the relation between the conformal transformations of the metric of a Hermitian manifold and the holomorphically projective transformations of the associated connection. We prove that some holomorphically projective invariants can be considered as conformal invariants.

Let (M, g, J) be a Hermitian manifold and $g' = e^{2u}g$ be a conformal metric on M determined by the C^∞-function u on M. It is well known that the corresponding Lee forms θ and θ' of the Hermitian structures (g, J) and (g', J) are related as follows:

$$\theta' = \theta + 2(n-1)\,du.$$

Hence $d\theta$ is a conformal invariant, i.e. $d\theta' = d\theta$.

If ∇' is the Levi-Civita connection of the metric g' and $\tilde{\nabla}'$ is the connection associated with ∇', then because of (4.7) we have

$$\tilde{\Gamma}'^{\lambda}_{\alpha\beta} = \Gamma'^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\alpha\beta} + u_\alpha\delta^\lambda_\beta + u_\beta\delta^\lambda_\alpha = \tilde{\Gamma}^{\lambda}_{\alpha\beta} + u_\alpha\delta^\lambda_\beta + u_\beta\delta^\lambda_\alpha, \tag{6.1}$$

where δ^β_α are the Kronecker's deltas and $u_\alpha = du(\partial/\partial z^\alpha)$.

This equality shows that the conformal transformation $g' = e^{2u}g$ of the metric g generates the holomorphically projective transformation (6.1) of the associated connection $\tilde{\nabla}$ with closed 1-form ($2\omega = du$), i.e. a special holomorphically projective transformation in the sense of Section 3.

Conversely, let

$$\tilde{\Gamma}'^{\lambda}_{\alpha\beta} = \tilde{\Gamma}^{\lambda}_{\alpha\beta} + 2\omega_\alpha\delta^\lambda_\beta + 2\omega_\beta\delta^\lambda_\alpha \tag{6.2}$$

be a special holomorphically projective transformation of $\tilde{\nabla}$. There exists locally a function u so that $2\omega = du$. Then the conformal change $g' = e^{2u}g$ of the metric g generates the given special holomorphically projective transformation (6.2) of the associated connection $\tilde{\nabla}$. Hence, the special holomorphically projective transformation (6.2) determines locally (up to a homothety) a conformal change $g' = e^{2u}g$.

Thus we have the following statement.

PROPOSITION 6.1. *Every conformal transformation $g' = e^{2u}g$ of the metric g of a Hermitian manifold generates a special holomorphically projective transformation* (3.1) *of the associated connection $\tilde{\nabla}$ and vice versa, every special holomorphically projective transformation* (3.1) *of $\tilde{\nabla}$ generates locally a conformal change $g' = e^{2u}g$ of the metric g.*

Now we study conformal invariants with respect to the associated connection $\tilde{\nabla}$, by using their relation to the corresponding special holomorphically projective invariants.

Taking into account (3.9), (3.10), (3.11) and Proposition 6.1 we obtain the following statement.

PROPOSITION 6.2. *Let* (M, g, J) $(\dim M = 2n \geq 4)$ *be a Hermitian manifold with associated curvature tensor* $\tilde{R}$. *If* $\tilde{S}(X, Y) = \dfrac{\tilde{\rho}(X, Y) + \tilde{\rho}(Y, X)}{2}$, *then the tensors*

$$\begin{aligned}\tilde{W}(X, Y)Z = \tilde{R}(X, Y)Z &- \frac{n}{2(n^2-1)}\{\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y - \tilde{S}(Y, JZ)JX + \tilde{S}(X, JZ)JY\} \\ &+ \frac{1}{2(n^2-1)}\{\tilde{S}(JY, JZ)X - \tilde{S}(JX, JZ)Y \\ &\qquad + \tilde{S}(JY, Z)JX - \tilde{S}(JX, Z)JY\} \\ &- \frac{1}{2(n+1)}\{\tilde{S}(X, JY)JZ - \tilde{S}(JX, Y)JZ\}\end{aligned}$$

and

$$\tilde{\rho}(X, Y) - \tilde{\rho}(Y, X)$$

are conformal invariants.

We call the tensor $\tilde{W}$ from Proposition 6.2 *the associated conformal curvature tensor*.

The associated conformal curvature tensor $\tilde{W}$ has two types of components with respect to local holomorphic coordinates:

$$\begin{aligned}\tilde{W}^{\lambda}_{\alpha\beta\gamma} &= \tilde{R}^{\lambda}_{\alpha\beta\gamma} - \frac{1}{n-1}\left(\frac{\tilde{\rho}_{\beta\gamma} + \tilde{\rho}_{\gamma\beta}}{2}\delta^{\lambda}_{\alpha} - \frac{\tilde{\rho}_{\alpha\gamma} + \tilde{\rho}_{\gamma\alpha}}{2}\delta^{\lambda}_{\beta}\right) \\ \tilde{W}^{\lambda}_{\alpha\bar{\beta}\gamma} &= \tilde{R}^{\lambda}_{\alpha\bar{\beta}\gamma} - \frac{1}{n+1}\left(\frac{\tilde{\rho}_{\alpha\bar{\beta}} + \tilde{\rho}_{\bar{\beta}\alpha}}{2}\delta^{\lambda}_{\gamma} + \frac{\tilde{\rho}_{\gamma\bar{\beta}} + \tilde{\rho}_{\bar{\beta}\gamma}}{2}\delta^{\lambda}_{\alpha}\right).\end{aligned} \tag{6.3}$$

The components $\tilde{W}^{\lambda}_{\alpha\beta\gamma}$ and $\tilde{W}^{\lambda}_{\alpha\bar{\beta}\gamma}$ give rise to two tensors, which are again conformal invariants.

The usual Weyl conformal curvature tensor W of the metric g and $\tilde{W}$ have a common part. Namely, taking into account (4.7) and (4.14) we obtain from (6.3) that

$$\tilde{W}^{\lambda}_{\alpha\beta\gamma} = R^{\lambda}_{\alpha\beta\gamma} - \frac{1}{2(n-1)}(\rho_{\beta\gamma}\delta^{\lambda}_{\alpha} - \rho_{\alpha\gamma}\delta^{\lambda}_{\beta}) = W^{\lambda}_{\alpha\beta\gamma}$$

Thus we have:

PROPOSITION 6.3. *Let* (M, g, J) $(\dim M = 2n \geq 4)$ *be a Hermitian manifold with conformal curvature tensor* W *and associated conformal curvature tensor* $\tilde{W}$. *Then the following conditions are equivalent*:

(i) $\tilde{W} = 0$;

(ii) $W^{\lambda}_{\alpha\beta\bar{\gamma}} = 0$ *and* $\tilde{W}^{\lambda}_{\alpha\beta\bar{\gamma}} = 0$.

Now, let $\tilde{W} = 0$. After a contraction this equality implies that $\tilde{\rho}(X, Y) - \tilde{\rho}(Y, X) = 0$, i.e. the Ricci tensor $\tilde{\rho}$ is symmetric. Because of (4.15) it follows that

$$d\theta(X, Y) = 0,$$

i.e. the Lee form θ is closed.

Further we study the properties of the associated conformal curvature tensor in the class of Hermitian manifolds satisfying the condition:

$$d\Phi = \frac{1}{n-1}\theta \wedge \Phi. \tag{6.4}$$

These manifolds are known as W_4-manifolds according to the classification in [4]. Every four dimensional Hermitian manifold (Hermitian surface) satisfies (6.4). In dimension $2n \geq 6$ every W_4-manifold is locally conformal Kähler and vice versa.

First we consider the case $2n \geq 6$.

THEOREM 6.4. *Let* (M, g, J) $(\dim M = 2n \geq 6)$ *be a locally conformal Kähler manifold with associated conformal curvature tensor* $\tilde{W}$. *Then the following conditions are equivalent*:

(i) $\tilde{W} = 0$;

(ii) $\tilde{W}^{\lambda}_{\alpha\beta\bar{\gamma}} = 0$;

(iii) *there exists locally a conformal metric with flat associated connection*;

(iv) *the metric* g *is locally conformally equivalent to a Kähler metric of constant holomorphic sectional curvatures.*

Proof. Since (M, g, J) is locally conformal Kähler, then it follows that $W^{\lambda}_{\alpha\beta\bar{\gamma}} = 0$. According to Proposition 6.3 the condition $\tilde{W}^{\lambda}_{\alpha\beta\bar{\gamma}} = 0$ is equivalent to the condition $\tilde{W} = 0$, which proves (i) $\Leftrightarrow$ (ii).

Let $\tilde{W} = 0$. This implies that $\tilde{\rho}$ is symmetric and the holomorphically projective curvature tensor P_H of the connection $\tilde{\nabla}$ vanishes. Then there exists locally a 1-form ω such that the connection $\tilde{\nabla}'$ given by (6.2) is flat. Since $\tilde{\rho}$ is symmetric, then ω is closed, i.e. (6.2) is a special holomorphically projective transformation. Putting locally $2\omega = du$, it follows from Proposition 6.1 that the metric $g' = e^{2u}g$ is with flat associated connection, which proves the implication (i) $\Rightarrow$ (iii). The inverse implication follows from the conformal invariance of $\tilde{W}$.

To prove (i) $\Rightarrow$ (iv), let $\tilde{W} = 0$. Under the conditions of the theorem we can consider locally a Kähler metric g' conformal to g. Since $\tilde{W}$ is a conformal invariant, then the Kähler manifold (M, g', J) has vanishing associated conformal curvature tensor. Taking into account that the curvature tensor of a Kähler manifold coincides with its associated curvature tensor and using the second equality of (6.3) we conclude that (M, g', J) is a Kähler manifold of constant holomorphic sectional curvatures, which proves the implication.

The inverse implication (iv) $\Rightarrow$ (i) follows from Proposition 6.2 and the fact that the associated conformal curvature tensor of any Kähler manifold of constant holomorphic sectional curvature vanishes. QED

Finally we shall consider the four dimensional case, i.e. Hermitian surfaces.

Let (M, g, J) be a Hermitian surface. We have the following standard formulas:

$$F_{\bar{\gamma}\alpha\beta} = i(\theta_\alpha g_{\beta\bar{\gamma}} - \theta_\beta g_{\alpha\bar{\gamma}}),$$

$$\nabla_\gamma F_{\bar{\delta}\alpha\beta} = i(\nabla_\gamma \theta_\alpha g_{\beta\bar{\delta}} - \nabla_\gamma \theta_\beta g_{\alpha\bar{\delta}}),$$

$$R^\lambda_{\alpha\beta\gamma} = \frac{1}{2}\left(\nabla_\gamma \theta_\alpha + \frac{1}{2}\theta_\alpha\theta_\gamma\right)\delta^\lambda_\beta - \frac{1}{2}\left(\nabla_\gamma\theta_\beta + \frac{1}{2}\theta_\beta\theta_\gamma\right)\delta^\lambda_\alpha, \tag{6.5}$$

$$\rho_{\beta\gamma} = -\frac{1}{2}(\nabla_\beta\theta_\gamma + \nabla_\gamma\theta_\beta) - \frac{1}{2}\theta_\beta\theta_\gamma, \tag{6.6}$$

$$\rho^*_{\beta\gamma} = \frac{1}{2}(\nabla_\beta\theta_\gamma - \nabla_\gamma\theta_\beta) = \frac{1}{2}\, d\theta_{\beta\gamma}, \tag{6.7}$$

From (6.5), (6.6) and (6.7) it follows that

$$R^\lambda_{\alpha\beta\gamma} = \frac{1}{2}(\rho_{\beta\gamma}\delta^\lambda_\alpha - \rho_{\alpha\gamma}\delta^\lambda_\beta) + \frac{1}{2}(\rho^*_{\beta\gamma}\delta^\lambda_\alpha - \rho^*_{\alpha\gamma}\delta^\lambda_\beta)$$

and

$$W^\lambda_{\alpha\beta\gamma} = \frac{1}{2}(\rho^*_{\beta\gamma}\delta^\lambda_\alpha - \rho^*_{\alpha\gamma}\delta^\lambda_\beta). \tag{6.8}$$

LEMMA 6.5. *On a Hermitian surface the following conditions are equivalent*:
(i) $W^\lambda_{\alpha\beta\gamma} = 0$;
(ii) $\rho^*_{\alpha\beta} = \frac{1}{2}\, d\theta_{\alpha\beta} = 0$;
(iii) $\tilde{\rho}_{\alpha\beta} = \tilde{\rho}_{\beta\alpha}$.

Proof. The statement follows from the equality $\tilde{W}^\lambda_{\alpha\beta\gamma} = W^\lambda_{\alpha\beta\gamma}$ and the formulas (6.8), (6.7), (4.15). QED

Further we find a corollary from any of the conditions in Lemma 6.5.

LEMMA 6.6. *On a Hermitian surface the condition* $d\theta_{\alpha\beta} = 0$ *implies that*

$$\nabla_\alpha \rho_{\beta\gamma} - \nabla_\beta \rho_{\alpha\gamma} = 0.$$

Proof. From the given condition we have that $\nabla_\alpha \theta_\beta - \nabla_\beta \theta_\alpha = 0$. Differentiating (6.6) we find

$$\nabla_\alpha \rho_{\beta\gamma} = -\nabla_\alpha \nabla_\beta \theta_\gamma - \frac{1}{2}\theta_\gamma \nabla_\alpha \theta_\beta - \frac{1}{2}\theta_\beta \nabla_\alpha \theta_\gamma.$$

Hence

$$\begin{aligned}\nabla_\alpha \rho_{\beta\gamma} - \nabla_\beta \rho_{\alpha\gamma} &= -(\nabla_\alpha \nabla_\beta \theta_\gamma - \nabla_\beta \nabla_\alpha \theta_\gamma) - \frac{1}{2}(\theta_\beta \nabla_\alpha \theta_\gamma - \theta_\alpha \nabla_\beta \theta_\gamma) \\ &= R^\sigma_{\alpha\beta\gamma}\theta_\sigma - \frac{1}{2}(\theta_\beta \nabla_\alpha \theta_\gamma - \theta_\alpha \nabla_\beta \theta_\gamma).\end{aligned}$$

Now, taking into account (6.5), we obtain the assertion. QED

THEOREM 6.7. *Let* (M, g, J) *be a Hermitian surface with associated conformal curvature tensor* $\tilde{W}$. *Then the following conditions are equivalent*:

(i) $\tilde{W} = 0$;

(ii) *there exists locally a conformal metric with flat associated connection*;

(iii) *the metric g is locally conformally equivalent to a Kähler metric of constant holomorphic sectional curvatures.*

Proof. The condition $\tilde{W} = 0$ implies that $\tilde{\rho}$ is symmetric, which by virtue of (4.14) gives $\tilde{\rho}_{\alpha\beta} = \frac{1}{2}\rho_{\alpha\beta}$. Using the equalities $\tilde{\Gamma}^\lambda_{\alpha\beta} = \Gamma^\lambda_{\alpha\beta}$ and $\Gamma^{\bar{\lambda}}_{\alpha\beta} = 0$ we find

$$\tilde{\nabla}_\alpha \tilde{\rho}_{\beta\gamma} - \tilde{\nabla}_\beta \tilde{\rho}_{\alpha\gamma} = \frac{1}{2}(\nabla_\alpha \rho_{\beta\gamma} - \nabla_\beta \rho_{\alpha\gamma}).$$

Applying successively Lemma 6.6 and Theorem 3.1 we obtain that there exists locally a flat complex symmetric connection $\tilde{\nabla}'$ satisfying (3.1) or equivalently (6.2). Since ρ is symmetric, then (6.2) is a special holomorphically projective transformation. Applying Proposition 6.1 we obtain the implication (i) $\Rightarrow$ (ii).

The inverse follows from the conformal invariance of $\tilde{W}$.

To prove (i) $\Rightarrow$ (iii) we note that $\tilde{W} = 0$ implies θ is closed and hence the manifold is locally conformal Kähler. The rest of the proof is similar to the proof of the corresponding equivalence in Theorem 6.4. QED

Theorem 6.4 and Theorem 6.7 allow us to find the metrics conformal to the standard Kähler metrics of constant holomorphic sectional curvatures whose associated connections are flat.

Let $(\mathbf{C}^n, g_0, J)$ be the complex space with the standard complex structure J and flat Kähler metric g_0. For any $Z(z^1, \ldots, z^n) \in \mathbf{C}^n$ the distance function r^2 from the origin in $\mathbf{C}^n$ is given by

$$r^2 = g_0(Z,Z) = |z^1|^2 + \cdots + |z^n|^2 = \delta_{\alpha\bar{\beta}} z^\alpha z^{\bar{\beta}},$$

where

$$\delta_{\alpha\bar{\beta}} = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

Putting $dr = r_\alpha\, dz^\alpha + r_{\bar{\alpha}}\, dz^{\bar{\alpha}}$ we have

$$r_\alpha = \frac{1}{2r}\delta_{\alpha\bar{\beta}} z^{\bar{\beta}}$$

and the standard Fubini-Study metric g in $\mathbf{C}^n$ is given by

$$g_{\alpha\bar{\beta}} = \frac{2}{c(1+r^2)}\left(\delta_{\alpha\bar{\beta}} - \frac{4r^2}{1+r^2} r_\alpha r_{\bar{\beta}}\right), \quad c > 0.$$

The local components $\Gamma^\lambda_{\alpha\beta}$ of the Levi-Civita connection of g are

$$\Gamma^\lambda_{\alpha\beta} = -\frac{2r}{1+r^2}(r_\alpha \delta^\lambda_\beta + r_\beta \delta^\lambda_\alpha).$$

Now, let $g' = e^{2u} g$ be a conformal metric with $u = u(r^2)$. If $\tilde{\Gamma}'^\lambda_{\alpha\beta}$ are the local components of the associated connection with the Levi-Civita connection of g', then

$$\tilde{\Gamma}'^\lambda_{\alpha\beta} = -\frac{2r}{1+r^2}(r_\alpha \delta^\lambda_\beta + r_\beta \delta^\lambda_\alpha) + 2ru'(r_\alpha \delta^\lambda_\beta + r_\beta \delta^\lambda_\alpha).$$

Therefore $\tilde{\Gamma}'^\lambda_{\alpha\beta} = 0$ if and only if

$$u' = \frac{1}{1+r^2} \quad \Leftrightarrow \quad u = \ln(1+r^2) + const.$$

Thus we obtained the following

Example 1. Let $(\mathbf{C}^n, g, J)$ be the complex space with the Fubini-Study metric g. Then $(\mathbf{C}^n, g', J)$, where $g' = (1+r^2)^2 g$, is a locally conformal Kähler manifold with flat associated connection.

In a similar way we obtain

Example 2. Let $(\mathbf{D}^n, g, J)$ be the unit disc in $\mathbf{C}^n$ with the Kähler metric g of constant holomorphic sectional curvatures $-c < 0$. Then the manifold $(\mathbf{D}^n, g', J)$, where $g' = (1-r^2)^2 g$ is a locally conformal Kähler manifold with flat associated connection.

More precisely, the holomorphic coordinates $(z^1, \ldots, z^n)$ in $\mathbf{C}^n$ are special holomorphic coordinates in the sense of Lemma 5.1.

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