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UNIQUENESS OF MEROMORPHIC FUNCTIONS CONCERNING WEAKLY WEIGHTED-SHARING

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Abstract

In this paper, we introduce the definition of weakly weighted-sharing which is between "CM" and "IM". Using the notion of weakly weighted-sharing, we study the uniqueness problems on meromorphic function and its kth order derivative $f^{(k)}$ satisfying certain sharing set properties. As consequences, we are able to answer questions posed by Kit-wing Yu, which were also studied by I. Lahiri and A. Sarkar, L. P. Liu and Y. X. Gu. Our results sharpen the above results.

1. Introduction and main results

In this paper, we shall use the standard notations in Nevanlinna's value distribution theory of meromorphic functions such as $T(r,f)$, $N(r,f)$ and $m(r,f)$ (see W. K. Hayman [1] or L. Yang [2]). The notation $S(r,f)$ is defined to be any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ possibly outside a set of r of finite linear measure. A meromorphic function a is called a small function with respect to f provided that $T(r,a) = S(r,f)$. Denote $S(f)$ the set of all small functions of f.

For any two nonconstant meromorphic functions f and g, and $a \in S(f)$, we say that f and g share a IM (CM) provided that $f-a$ and $g-a$ have the same zeros ignoring(counting) multiplicities. If $\frac{1}{f}$ and $\frac{1}{g}$ share 0 IM (CM), we say that f and g share ∞ IM (CM).

Let $N_E(r,a)$ be the counting function of all common zeros of $f-a$ and $g-a$ with the same multiplicities, and $N_0(r,a)$ be the counting functions of all common zeros of $f-a$ and $g-a$ ignoring multiplicities. Denote by $\overline{N}_E(r,a)$ and $\overline{N}_0(r,a)$ the reduced counting functions of f and g corresponding to the counting functions $N_E(r,a)$ and $N_0(r,a)$, respectively. If

$$\overline{N}\left(r,\frac{1}{f-a}\right) + \overline{N}\left(r,\frac{1}{g-a}\right) - 2\overline{N}_E(r,a) = S(r,f) + S(r,g),$$

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then we say that f and g share a "CM". If

$$\overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{g-a}\right) - 2\overline{N}_0(r, a) = S(r, f) + S(r, g),$$

then we say that f and g share a "IM".

Definition 1 [3–4]. Let k be a positive integer, and let f be a meromorphic function and $a \in S(f)$.

(i) $\overline{N}_{k)}\left(r, \frac{1}{f-a}\right)$ denotes the counting function of those a-points of f whose multiplicities are not greater than k, where each a-point is counted only once.

(ii) $\overline{N}_{(k}\left(r, \frac{1}{f-a}\right)$ denotes the counting function of those a-points of f whose multiplicities are not less than k, where each a-point is counted only once.

(iii) $N_p\left(r, \frac{1}{f-a}\right)$ denotes the counting function of those a-points of f, where an a-point of f with multiplicity m counted m times if $m \le p$ and p times if $m > p$.

Definition 2 [5]. We denote by $\delta_p(a, f)$ the quantity

$$\delta_p(a, f) = 1 - \overline{\lim_{r \to \infty}} \frac{N_p\left(r, \frac{1}{f-a}\right)}{T(r, f)},$$

where p is a positive integer.

Clearly $\delta_p(a, f) \ge \delta(a, f)$.

In 2003, Kit-wing Yu [6] considered the uniqueness problem of an entire function or meromorphic function when it shares one small function with its derivative and proved the following results.

Theorem A. *Let $k \ge 1$. Let f be a non-constant entire function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If f, $f^{(k)}$ share a CM and $\delta(0, f) > \frac{3}{4}$, then $f \equiv f^{(k)}$.*

Theorem B. *Let $k \ge 1$. Let f be a non-constant non-entire meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$, f and a do not have any common pole. If f, $f^{(k)}$ share a CM and $4\delta(0, f) + 2(8 + k)\Theta(\infty, f) > 19 + 2k$, then $f \equiv f^{(k)}$.*

In the same paper, Kit-wing Yu posed the following open questions:

Question 1. Can a CM shared value be replaced by an IM shared value in Theorem A?

QUESTION 2. Is the condition $\delta(0,f) > \frac{3}{4}$ sharp in Theorem A?

QUESTION 3. Is the condition $4\delta(0,f) + 2(8+k)\Theta(\infty,f) > 19 + 2k$ sharp in Theorem B?

QUESTION 4. Can the condition "f and a do not have any common pole" be deleted in Theorem B?

In 2004, L. P. Liu and Y. X. Gu [7] applied a different method and obtained the following results.

THEOREM C. *Let f be a non-constant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If f, $f^{(k)}$ share a CM, $f^{(k)}$ and a do not have any common pole of same multiplicity and $2\delta(0,f) + 4\Theta(\infty,f) > 5$, then $f \equiv f^{(k)}$.*

THEOREM D. *Let f be a non-constant entire function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If f, $f^{(k)}$ share a CM and $\delta(0,f) > \frac{1}{2}$, then $f \equiv f^{(k)}$.*

In this paper, we introduce the definition of weakly weighted-sharing. By the new definition, we obtain uniqueness theorems which answer the questions posed by Kit-wing Yu. Moreover, our results improve Theorem A, B, C, D mentioned above.

Next, we introduce some notations for our definition.

DEFINITION 3. Let f and g be two nonconstant meromorphic functions sharing a "IM", for $a \in S(f) \cap S(g)$, and a positive integer k or ∞.

(i) $\overline{N}^E_{k)}(r,a)$ denotes the counting function of those a-points of f whose multiplicities are equal to the corresponding a-points of g, both of their multiplicities are not greater than k, where each a-point is counted only once.

(ii) $\overline{N}^o_{(k}(r,a)$ denotes the reduced counting function of those a-points of f which are a-points of g, both of their multiplicities are not less than k, where each a-point is counted only once.

DEFINITION 4. For $a \in S(f) \cap S(g)$, if k is a positive integer or ∞, and

$$\overline{N}_{k)}\left(r,\frac{1}{f-a}\right) - \overline{N}^E_{k)}(r,a) = S(r,f), \quad \overline{N}_{k)}\left(r,\frac{1}{g-a}\right) - \overline{N}^E_{k)}(r,a) = S(r,g);$$

$$\overline{N}_{(k+1}\left(r,\frac{1}{f-a}\right) - \overline{N}^o_{(k+1}(r,a) = S(r,f), \quad \overline{N}_{(k+1}\left(r,\frac{1}{g-a}\right) - \overline{N}^o_{(k+1}(r,a) = S(r,g).$$

Or if $k = 0$ and

$$\overline{N}\left(r,\frac{1}{f-a}\right) - \overline{N}_0(r,a) = S(r,f), \quad \overline{N}\left(r,\frac{1}{g-a}\right) - \overline{N}_0(r,a) = S(r,g),$$

then we say f and g weakly share a with weight k. Here, we write f, g share "(a,k)" to mean that f, g weakly share a with weight k.

Obviously, if f and g share "(a,k)", then f and g share "(a,p)" for any p $(0 \le p \le k)$. Also, we note that f and g share a "IM" or "CM" if and only if f and g share "$(a,0)$" or "(a,∞)", respectively.

Now, we state the main results of this paper.

Theorem 1. *Let $k \ge 1$ and $2 \le m \le \infty$. Let f be a non-constant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If f, $f^{(k)}$ share "(a,m)" and $2\delta_{2+k}(0,f) + 4\Theta(\infty,f) > 5$, then $f \equiv f^{(k)}$.*

Theorem 2. *Let $k \ge 1$, and let f be a non-constant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If f, $f^{(k)}$ share "$(a,1)$" and $\frac{5}{2}\delta_{2+k}(0,f) + \frac{k+9}{2}\Theta(\infty,f) > \frac{k}{2} + 6$, then $f \equiv f^{(k)}$.*

Theorem 3. *Let $k \ge 1$, and let f be a non-constant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If f, $f^{(k)}$ share a "IM" and $5\delta_{2+k}(0,f) + (2k+7)\Theta(\infty,f) > 2k+11$, then $f \equiv f^{(k)}$.*

If f is a nonconstant entire function, then $\Theta(\infty,f) = 1$. So we have the following results.

Corollary 1. *Let $k \ge 1$ and $2 \le m \le \infty$. Let f be a non-constant entire function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If f, $f^{(k)}$ share "(a,m)" and $\delta_{2+k}(0,f) > \frac{1}{2}$, then $f \equiv f^{(k)}$.*

Corollary 2. *Let $k \ge 1$, and let f be a non-constant entire function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If f and $f^{(k)}$ share "$(a,1)$" and $\delta_{2+k}(0,f) > \frac{3}{5}$, then $f \equiv f^{(k)}$.*

Corollary 3. *Let $k \ge 1$, and let f be a non-constant entire function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If f, $f^{(k)}$ share a "IM" and $\delta_{2+k}(0,f) > \frac{4}{5}$, then $f \equiv f^{(k)}$.*

Remark 1: Theorem 1 and Corollary 1 improve Theorem A–D. Theorem 2 and Corollary 2 improve Theorem A, B. Theorem 3 and Corollary 3 answer question 1. Meanwhile, we give an affirmative answer to the forth question.

2. Some lemmas

Lemma 1 [3]. *Let f be a nonconstant meromorphic function and let k be a positive integer. Then*

(i) $N\left(r, \frac{1}{f^{(k)}}\right) \le N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f)$.

(ii) $N\left(r, \frac{1}{f^{(k)}}\right) \le T(r, f^{(k)}) - T(r, f) + N\left(r, \frac{1}{f}\right) + S(r, f)$.

Next, we introduce some notations for the following lemma.

When f and g share 1 "IM", $\overline{N}^L\left(r, \frac{1}{f-1}\right)$ denotes the counting function of the 1-points of f whose multiplicities are greater than 1-points of g, where each zero is counted only once. Similarly, we have $\overline{N}^L\left(r, \frac{1}{g-1}\right)$. $N_{11}\left(r, \frac{1}{f-1}\right)$ denotes the counting function of common simple 1-points of f and g.

Lemma 2. *Let f be a nonconstant meromorphic function and let k be a positive integer. Then*

(i) $N_2\left(r, \frac{1}{f^{(k)}}\right) \le N_{2+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f)$.

(ii) $N_2\left(r, \frac{1}{f^{(k)}}\right) \le T(r, f^{(k)}) - T(r, f) + N_{2+k}\left(r, \frac{1}{f}\right) + S(r, f)$.

Proof. By Lemma 1(i), we have

$$N_2\left(r, \frac{1}{f^{(k)}}\right) + \sum_{p=3}^{\infty} \overline{N}_{(p}\left(r, \frac{1}{f^{(k)}}\right) \le N_{2+k}\left(r, \frac{1}{f}\right) + \sum_{p=3+k}^{\infty} \overline{N}_{(p}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f),$$

i.e.

$$\begin{aligned} N_2\left(r, \frac{1}{f^{(k)}}\right) &\le N_{2+k}\left(r, \frac{1}{f}\right) + \sum_{p=3+k}^{\infty} \overline{N}_{(p}\left(r, \frac{1}{f}\right) - \sum_{p=3}^{\infty} \overline{N}_{(p}\left(r, \frac{1}{f^{(k)}}\right) \\ &\quad + k\overline{N}(r, f) + S(r, f) \\ &\le N_{2+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f). \end{aligned}$$

(ii) can be followed by using part (i) directly.

Lemma 3. *Let m be a nonnegative integer or ∞. Let F and G be two nonconstant meromorphic functions, and F, G share "$(1, m)$". Let*

$$H = \left(\frac{F''}{F'} - 2\frac{F'}{F-1}\right) - \left(\frac{G''}{G'} - 2\frac{G'}{G-1}\right).$$

If $H \not\equiv 0$, then

(i) *If* $2 \le m \le \infty$, *then*

$$T(r,F) \le N_2(r,F) + N_2\left(r,\frac{1}{F}\right) + N_2(r,G) + N_2\left(r,\frac{1}{G}\right) + S(r,F) + S(r,G).$$

(ii) *If* $m = 1$, *then*

$$T(r,F) \le N_2(r,F) + N_2\left(r,\frac{1}{F}\right) + N_2(r,G) + N_2\left(r,\frac{1}{G}\right) + \overline{N}^L\left(r,\frac{1}{F-1}\right) + S(r,F) + S(r,G).$$

(iii) *If* $m = 0$, *then*

$$T(r,F) \le N_2(r,F) + N_2\left(r,\frac{1}{F}\right) + N_2(r,G) + N_2\left(r,\frac{1}{G}\right) + 2\overline{N}^L\left(r,\frac{1}{F-1}\right) + \overline{N}^L\left(r,\frac{1}{G-1}\right) + S(r,F) + S(r,G).$$

The same inequalities holds for $T(r,G)$.

Proof. (i) If $2 \le m \le \infty$, then by the Second Fundamental Theorem, we have

$$T(r,F) \le \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-1}\right) - N_0\left(r,\frac{1}{F'}\right) + S(r,F), \tag{2.1}$$

where $N_0\left(r,\frac{1}{F'}\right)$ is the counting function in $|z| < r$ of the zeros of F' that are not the zeros of F and $F-1$. $N_0\left(r,\frac{1}{G'}\right)$ can be defined similarly.

By a simple calculation, any pole of F is not a pole of $\frac{F''}{F'} - 2\frac{F'}{F-1}$, any pole of G is not a pole of $\frac{G''}{G'} - 2\frac{G'}{G-1}$. Furthermore, let z_1 be a common zero of $F-1$ and $G-1$ with multiplicity t, where $1 \le t \le 2$. We know that H is analytic at z_1. Therefore, by $H \not\equiv 0$, we have

$$N_{1)}\left(r,\frac{1}{F-1}\right) \le \overline{N}\left(r,\frac{1}{H}\right) + S(r,F) + S(r,G) \le T(r,H) + S(r,F) + S(r,G). \tag{2.2}$$

Note that $m(r,H) = S(r,F) + S(r,G)$ and

$$N(r,H) \le \overline{N}_{(2}(r,F) + \overline{N}_{(2}(r,G) + \overline{N}_{(2}\left(r,\frac{1}{F}\right) + \overline{N}_{(2}\left(r,\frac{1}{G}\right) + N_0\left(r,\frac{1}{F'}\right) + N_0\left(r,\frac{1}{G'}\right) + \overline{N}^L\left(r,\frac{1}{F-1}\right) + \overline{N}^L\left(r,\frac{1}{G-1}\right) + S(r,F) + S(r,G)$$

By (2.2), we have

$$N_{1)}\left(r,\frac{1}{F-1}\right) \le \overline{N}_{(2}(r,F) + \overline{N}_{(2}(r,G) + \overline{N}_{(2}\left(r,\frac{1}{F}\right) + \overline{N}_{(2}\left(r,\frac{1}{G}\right) + N_0\left(r,\frac{1}{F'}\right) + N_0\left(r,\frac{1}{G'}\right) + \overline{N}^L\left(r,\frac{1}{F-1}\right) + \overline{N}^L\left(r,\frac{1}{G-1}\right) + S(r,F) + S(r,G). \tag{2.3}$$

Since F and G share "$(1,m)$", we have

$$\overline{N}_{(2}\left(r,\frac{1}{G-1}\right) + \overline{N}^L\left(r,\frac{1}{F-1}\right) + \overline{N}^L\left(r,\frac{1}{F-1}\right) + N_0\left(r,\frac{1}{G'}\right) + N\left(r,\frac{1}{G}\right) - \overline{N}\left(r,\frac{1}{G}\right) \le N\left(r,\frac{1}{G'}\right).$$

It follows from Lemma 1 that

$$\overline{N}_{(2}\left(r,\frac{1}{G-1}\right) + \overline{N}^L\left(r,\frac{1}{F-1}\right) + \overline{N}^L\left(r,\frac{1}{G-1}\right) + N_0\left(r,\frac{1}{G'}\right) \le \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + S(r,G). \tag{2.4}$$

In addition, we have

$$\overline{N}\left(r,\frac{1}{F-1}\right) = N_{1)}\left(r,\frac{1}{F-1}\right) + \overline{N}_{(2}\left(r,\frac{1}{F-1}\right) = N_{1)}\left(r,\frac{1}{F-1}\right) + \overline{N}_{(2}\left(r,\frac{1}{G-1}\right). \tag{2.5}$$

Combining (2.1), (2.3), (2.4) and (2.5), we obtain

$$T(r,F) \le N_2(r,F) + N_2\left(r,\frac{1}{F}\right) + N_2(r,G) + N_2\left(r,\frac{1}{G}\right) + S(r,F) + S(r,G).$$

(ii) If $m = 1$, then (2.4) is replaced by

$$\overline{N}_{(2}\left(r,\frac{1}{G-1}\right) + \overline{N}^L\left(r,\frac{1}{G-1}\right) + N_0\left(r,\frac{1}{G'}\right) \le \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + S(r,G).$$

Similar to the arguments in (i), we see that (ii) holds.

(iii) If $m = 0$, then by a simple calculation, any common simple zero of $F-1$ and $G-1$ is zero of H. Therefore, by $H \not\equiv 0$, we have

$$N_{11}\left(r,\frac{1}{F-1}\right) \le \overline{N}\left(r,\frac{1}{H}\right) + S(r,F) + S(r,G) \le T(r,H) + S(r,F) + S(r,G). \tag{2.6}$$

Thus

$$N_{11}\left(r,\frac{1}{F-1}\right)\le \overline{N}_{(2}(r,F)+\overline{N}_{(2}(r,G)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)$$
$$+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+\overline{N}^L\left(r,\frac{1}{F-1}\right)$$
$$+\overline{N}^L\left(r,\frac{1}{G-1}\right)+S(r,F)+S(r,G). \tag{2.7}$$

By the Second Fundament Theorem, we have

$$T(r,F)+T(r,G)\le \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-1}\right)-N_0\left(r,\frac{1}{F'}\right)$$
$$+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)-N_0\left(r,\frac{1}{G'}\right)$$
$$+S(r,F)+S(r,G), \tag{2.8}$$

In addition, we have

$$\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)=2\overline{N}\left(r,\frac{1}{G-1}\right)+S(r,F)+S(r,G)$$
$$\le N_{11}\left(r,\frac{1}{F-1}\right)+\overline{N}^L\left(r,\frac{1}{F-1}\right)+N\left(r,\frac{1}{G-1}\right)$$
$$+S(r,F)+S(r,G)$$
$$\le N_{11}\left(r,\frac{1}{F-1}\right)+\overline{N}^L\left(r,\frac{1}{F-1}\right)+T(r,G)$$
$$+S(r,F)+S(r,G).$$

Combining (2.6), (2.7) and (2.8), we obtain

$$T(r,F)\le N_2(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2(r,G)+N_2\left(r,\frac{1}{G}\right)+2\overline{N}^L\left(r,\frac{1}{F-1}\right)$$
$$+\overline{N}^L\left(r,\frac{1}{G-1}\right)+S(r,F)+S(r,G).$$

Lemma 4 [8]. *Let f be a transcendental meromorphic function and $\alpha(\not\equiv 0,\infty)$ be a meromorphic function such that $T(r,\alpha)=S(r,f)$. Let b and c are any two finite nonzero distinct complex numbers. If $\psi=\alpha(f)^n(f^{(k)})^p$, where $n(\ge 0)$, $p(\ge 1)$ and $k(\ge 1)$ are integers, then*

$$(p+n)T(r,f)\le (p+n)N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{\psi-b}\right)+N\left(r,\frac{1}{\psi-c}\right)-N(r,f)$$
$$-N\left(r,\frac{1}{\psi'}\right)+S(r,f).$$

3. Proofs of main theorems

Proof of Theorem 1.
Let

$$F = \frac{f}{a}, \quad G = \frac{f^{(k)}}{a}. \tag{3.1}$$

Then it is easy to verify F and G share "$(1, m)$".

Let H be defined as in Lemma 3. Suppose that $H \not\equiv 0$. It follows from Lemma 3 that

$$T(r, G) \le N_2(r, F) + N_2\left(r, \frac{1}{F}\right) + N_2(r, G) + N_2\left(r, \frac{1}{G}\right) + S(r, F) + S(r, G).$$

Using Lemma 2, we have

$$\begin{aligned} T(r, f^{(k)}) &\le N_2(r, f) + N_2\left(r, \frac{1}{f}\right) + N_2(r, f^{(k)}) + N_2\left(r, \frac{1}{f^{(k)}}\right) + S(r, f) \\ &\le N_{2+k}\left(r, \frac{1}{f}\right) + T(r, f^{(k)}) - T(r, f) + N_{2+k}\left(r, \frac{1}{f}\right) + 4\overline{N}(r, f) + S(r, f), \end{aligned}$$

i.e.

$$T(r, f) \le 2N_{2+k}\left(r, \frac{1}{f}\right) + 4\overline{N}(r, f) + S(r, f).$$

It follows that $2\delta_{2+k}(0, f) + 4\Theta(\infty, f) \le 5$, which contradicts $2\delta_{2+k}(0, f) + 4\Theta(\infty, f) > 5$. Therefore $H \equiv 0$. That is

$$\frac{F''}{F'} - 2\frac{F'}{F-1} \equiv \frac{G''}{G'} - 2\frac{G'}{G-1}.$$

It follows that

$$\frac{1}{F-1} = \frac{A}{G-1} + B,$$

where $A (\neq 0)$ and B are constants. Therefore,

$$F = \frac{(B+1)G + (A-B-1)}{BG + (A-B)}. \tag{3.2}$$

and

$$T(r, F) = T(r, G) + S(r, f).$$

Now we distinguish the following two cases.

Case 1. Suppose that $B \neq -1, 0$.

If $A - B - 1 \neq 0$, then from (3.2), we have $\overline{N}\left(r, \dfrac{1}{G + \dfrac{A-B-1}{B+1}}\right) = \overline{N}\left(r, \dfrac{1}{F}\right)$.

By the Second Fundamental Theorem, we have $T(r,G) < \overline{N}(r,G) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}\left(r,\frac{1}{G+\frac{A-B-1}{B+1}}\right) + S(r,G)$, i.e.

$$\begin{aligned}T(r,f^{(k)}) &< \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f^{(k)}}\right) + \overline{N}\left(r,\frac{1}{f}\right) + S(r,f)\\ &\le \overline{N}(r,f) + T\left(r,f^{(k)}\right) - T(r,f) + N_{2+k}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f}\right) + S(r,f),\end{aligned}$$

and so

$$T(r,f) < \overline{N}(r,f) + 2N_{2+k}\left(r,\frac{1}{f}\right) + S(r,f).$$

It follows that $2\delta_{2+k}(0,f) + \Theta(\infty,f) \le 2$, which contradicts $2\delta_{2+k}(0,f) + 4\Theta(\infty,f) > 5$. Therefore, $A-B-1=0$. From (3.2), we obtain $\overline{N}\left(r,\frac{1}{G+\frac{1}{B}}\right) = \overline{N}(r,F)$.

Similar to the arguments in the above, we also have a contradiction.

Case 2. Suppose that $B=-1$.
If $A+1 \ne 0$. Then from (3.2), we have $\overline{N}\left(r,\frac{1}{G-(A+1)}\right) = \overline{N}(r,F)$.

Similar to the arguments in Case 1, we can get a contradiction. Therefore, $A+1=0$, then from (3.2), we have $FG \equiv 1$. From (3.1), we have

$$ff^{(k)} \equiv a^2. \tag{3.3}$$

In the following, we distinguish two subcases.

a) If f is a rational function, then a becomes a nonzero constant. So from (3.1), we see that f has no zero and pole. Since f is nonconstant, this is a contradiction.

b) If f is transcendental then by Lemma 4, we get in view of (3.1)

$$\begin{aligned}2T(r,f) &\le 2N\left(r,\frac{1}{f}\right) + 2T(r,ff^{(k)}) + S(r,f)\\ &\le 2N\left(r,\frac{1}{f}\right) + S(r,f)\\ &\le 2N\left(r,\frac{1}{a^2}\right) + S(r,f) = S(r,f),\end{aligned}$$

This is a contradiction.

Case 3. Suppose that $B=0$.
If $A-1 \ne 0$, then from (3.2), we have $\overline{N}\left(r,\frac{1}{G+(A-1)}\right) = \overline{N}\left(r,\frac{1}{F}\right)$.

Similar to the arguments in case 1, we also have a contradiction. Therefore, $A - 1 = 0$. From (3.2), we have $F \equiv G$, this implies $f \equiv f^{(k)}$.

This completes the proof of the Theorem 1.

Proof of Theorem 2. With the same notations, since f, $f^{(k)}$ share "$(a,1)$", we obtain that F, G share "$(1,1)$".

Let H be defined as in Lemma 3. Suppose that $H \not\equiv 0$. It follows from Lemma 3 that

$$T(r,G) \leq N_2(r,F) + N_2\left(r,\frac{1}{F}\right) + N_2(r,G) + N_2\left(r,\frac{1}{G}\right) + \overline{N}^L\left(r,\frac{1}{G-1}\right) + S(r,F) + S(r,G).$$

Since

$$\begin{aligned}
\overline{N}^L\left(r,\frac{1}{G-1}\right) &\leq \frac{1}{2}N\left(r,\frac{G}{G'}\right) \\
&\leq \frac{1}{2}N\left(r,\frac{G'}{G}\right) + S(r,f) \\
&\leq \frac{1}{2}\overline{N}(r,G) + \frac{1}{2}\overline{N}\left(r,\frac{1}{G}\right) + S(r,f) \\
&\leq \frac{1}{2}\overline{N}(r,f) + \frac{1}{2}\overline{N}\left(r,\frac{1}{f^{(k)}}\right) + S(r,f).
\end{aligned}$$

Using Lemma 2, we have

$$\begin{aligned}
T(r,f^{(k)}) &\leq N_2(r,f) + N_2\left(r,\frac{1}{f}\right) + N_2(r,f^{(k)}) + N_2\left(r,\frac{1}{f^{(k)}}\right) + \frac{1}{2}\overline{N}(r,f) \\
&\quad + \frac{1}{2}\overline{N}\left(r,\frac{1}{f^{(k)}}\right) + S(r,f) \\
&\leq N_{2+k}\left(r,\frac{1}{f}\right) + T(r,f^{(k)}) - T(r,f) + N_{2+k}\left(r,\frac{1}{f}\right) + \frac{1}{2}N_{2+k}\left(r,\frac{1}{f}\right) \\
&\quad + \frac{k+9}{2}\overline{N}(r,f) + S(r,f).
\end{aligned}$$

i.e.

$$T(r,f) \leq \frac{5}{2}N_{2+k}\left(r,\frac{1}{f}\right) + \frac{k+9}{2}\overline{N}(r,f) + S(r,f),$$

It follows that $\frac{5}{2}\delta_{2+k}(0,f) + \frac{k+9}{2}\Theta(\infty,f) \leq \frac{k}{2} + 6$, which contradicts $\frac{5}{2}\delta(0,f) + \frac{k+9}{2}\Theta(\infty,f) > \frac{k}{2} + 6$.

Similar to the arguments in Theorem 1, we see that Theorem 2 holds.

Proof of Theorem 3. Using Lemma 3(iii), note that

$$\overline{N}^L\left(r, \frac{1}{F-1}\right) \le N\left(r, \frac{F}{F'}\right) \le \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + S(r, f),$$

and

$$\begin{aligned}\overline{N}^L\left(r, \frac{1}{G-1}\right) &< \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}(r, f) + S(r, f)\\ &< N_{2+k}\left(r, \frac{1}{f}\right) + (k+1)\overline{N}(r, f) + S(r, f).\end{aligned}$$

Similar to the arguments in Theorem 1, we see that Theorem 3 holds.

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