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KODAI MATH. J.
29 (2006), 170–184

SPECTRAL GEOMETRY OF TOTALLY COMPLEX SUBMANIFOLDS OF QP^n

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Dedicated to the Loving Memory of My Parents

Abstract

We calculate some invariants determined by the spectrum of the Jacobi operator J of totally complex submanifolds of the quaternionic projective space QP^n. We use these invariants, and the ones determined by the spectrum of the Laplace-Beltrami operator Δ, to characterize parallel submanifolds of QP^n.

1. Introduction

Let M be a compact (connected and smooth) Riemannian manifold without boundary, isometrically immersed in a Riemannian manifold $\overline{M}$. The *Jacobi operator* J of M is a second order elliptic operator, associated to the isometric immersion of M into $\overline{M}$. J is defined on the space of smooth sections of the normal bundle $TM^{\perp}$ by the formula

$$J = D + \tilde{R} - \tilde{A},$$

where D is the rough Laplacian of the normal connection $\nabla^{\perp}$ on $TM^{\perp}$, $\tilde{R}$ and $\tilde{A}$ are linear transformations of $TM^{\perp}$ defined by means of a partial Ricci tensor of $\overline{M}$ and of the second fundamental form A, respectively. J is also called the *second variation operator*, because it naturally appears in the formula which gives the second variation for the area function of a compact minimal submanifold (see [S]). Its spectrum, denoted by

$$\mathrm{spec}(M, J) = \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots + \uparrow \infty\}$$

is discrete, as a consequence of the compactness of M.

*Author supported by funds of the University of Lecce and the G.N.S.A.G.A.

2000 *Mathematics Subject Classification.* 58J50, 53C42, 53C40, 53C20.

Key words and phrases. Jacobi operator, quaternionic projective space, totally complex submanifolds.

Received May 12, 2005; revised October 3, 2005.

The problem of characterizing (compact) submanifolds using the spectrum of the Jacobi operator, has been extensively investigated. Estimates of the first eigenvalue of J have been introduced to characterize some minimal hypersurfaces of the standard unit sphere (see for example [Pe]). Other authors applied Gilkey's results [G] to the asymptotic expansion of the partition function $Z(t)$ associated to $\operatorname{spec}(M, J)$, in order to find Riemannian invariants determined by the spectrum of J and to study spectral geometry of some minimal submanifolds. This kind of study has been made for minimal submanifolds of the Euclidean sphere ([D], [H]), Kaehler submanifolds of the complex projective space CP^n [H], Sasakian submanifolds [Sh], and by the author and D. Perrone for totally real minimal submanifolds of CP^n ([CP], [C1]) and QP^n [C2].

Moreover, an analogous investigation was made (by Urakawa [Ur] first, and recently by many other authors), about the spectral geometry determined by the Jacobi operator associated to the energy of a harmonic map.

Besides totally real submanifolds, another typical class of submanifolds of the quaternionic projective space QP^n is the class of totally complex submanifolds. They have been first introduced by Funabashi [F] and their Riemannian geometry has been studied by several authors ([CoGa], [Mr], [Mt], [T1], [T2], [X]). In particular, totally complex parallel submanifolds of QP^n have been completely classified by Tsukada in [T1].

In this paper, we consider a totally complex submanifold of QP^n, of complex dimension n, and we determine the first three terms of the asymptotic expansion for the partition function associated to the spectrum of its Jacobi operator. We then use the Riemannian invariants determined by the spectrum of J, and the ones determined by the spectrum of the Laplace-Beltrami operator Δ, to characterize totally complex parallel submanifolds of QP^n.

The paper is organized in the following way. In Section 2, we recall some basic results about QP^n and its totally complex submanifolds. In Section 3, we compute the first three terms of the asymptotic expansion for the partition function associated to $\operatorname{spec}(M, J)$, M being an n-dimensional totally complex submanifold of QP^n. Such invariants are used in Section 4 to characterize totally complex parallel submanifolds of QP^n. Further spectral characterizations of these submanifolds are given in Section 5, making also use of the invariants determined by the spectrum of the Laplace-Beltrami operator Δ.

The author wishes to express his gratitude towards the referee for his careful revision of the paper.

2. Totally complex submanifolds of QP^n

Let $(\overline{M}, g)$ be a $4n$-dimensional quaternionic Riemannian manifold and V the three-dimensional vector bundle of tensors of type $(1,1)$ with local basis of almost Hermitian structures I_1, I_2, I_3, satisfying

a) $I_1I_2 = -I_2I_1 = I_3$, $I_2I_3 = -I_3I_2 = I_1$, $I_3I_1 = -I_1I_3 = I_2$, $I_1^2 = I_2^2 = I_3^2 = -1$;

b) for any cross-section ξ of V, $\overline{\nabla}_X\xi$ is also a cross-section of V, where X is a vector field on $\overline{M}$ and $\overline{\nabla}$ is the Levi Civita connection of $\overline{M}$.

If X is a unit vector on $\overline{M}$, the *quaternionic section* determined by X is the 4-plane $Q(X)$ spanned by X, I_1X, I_2X and I_3X. Any 2-plane in a quaternionic section is called a *quaternionic plane* and its sectional curvature is called *quaternionic sectional curvature*. A *quaternionic space form* is a quaternionic manifold of constant quaternionic sectional curvature. Troughout the paper, by QP^n (or $QP^n(c)$) we shall denote the $4n$-dimensional *quaternionic projective space* of constant quaternionic sectional curvature $c > 0$. Its curvature tensor $\overline{R}$ is given by

$$\begin{aligned}\overline{R}(X,Y,Z,W) = \frac{c}{4}\{&\bar{g}(Y,Z)\bar{g}(X,W) - \bar{g}(X,Z)\bar{g}(Y,W)\\ &+ \bar{g}(I_1Y,Z)\bar{g}(I_1X,W) - \bar{g}(I_1X,Z)\bar{g}(I_1Y,W)\\ &- 2\bar{g}(I_1X,Y)\bar{g}(I_1Z,W) + \bar{g}(I_2Y,Z)\bar{g}(I_2X,W)\\ &- \bar{g}(I_2X,Z)\bar{g}(I_2Y,W) - 2\bar{g}(I_2X,Y)\bar{g}(I_2Z,W)\\ &+ \bar{g}(I_3Y,Z)\bar{g}(I_3X,W) - \bar{g}(I_3X,Z)\bar{g}(I_3Y,W)\\ &- 2\bar{g}(I_3X,Y)\bar{g}(I_3Z,W)\}.\end{aligned}$$

By definition, an *almost Hermitian* submanifold of $\overline{M}$ is a submanifold, of even dimension $2m$ (where $m \le n$), for which there exists a section $\widetilde{I}_1$ of the induced bundle $V|_M$ such that $\widetilde{I}_1{}^2 = -\mathrm{id}$ and $\widetilde{I}_1TM = TM$. An almost Hermitian submanifold M of $\overline{M}$, together with a section $\widetilde{I}_1$ of $V|_M$, is said to be *totally complex* if, at each point $p \in M$, we have $L(T_pM) \perp T_pM$ for each $L \in V_p$ such that $L \perp \widetilde{I}_1$ [F]. Alekseevski and Marchiafava [AMa] proved that if $\overline{M}$ has nonvanishing scalar curvature, then an almost Hermitian submanifold is Kaheler if and only it is totally complex.

From now on, let (M, g) be a totally complex submanifold of QP^n. We are interested in the case when M has complex dimension n, since totally complex submanifolds with such dimension appear to be the most typical parallel submanifolds of QP^n (see [T1] and Section 4 of this paper). We shall denote by ∇ and R the Levi Civita connection and the curvature tensor of M, respectively. The normal connection is given by

$$\begin{aligned}\nabla^{\perp} : TM \times TM^{\perp} &\to TM^{\perp}\\ (X,\xi) &\mapsto \nabla^{\perp}_X\xi,\end{aligned}$$

where $\nabla^{\perp}_X\xi$ denotes the normal component of $\overline{\nabla}_X\xi$. The second fundamental form σ and the Weingarten operator A are respectively defined by

$$\sigma(X,Y) = \overline{\nabla}_XY - \nabla_XY, \quad A_\xi X = -\overline{\nabla}_X\xi + \nabla^{\perp}_X\xi$$

for all $X, Y \in TM$ and $\xi \in TM^{\perp}$. Moreover, $\bar{g}(\sigma(X,Y),\xi) = g(A_\xi X, Y)$.

Let $R^{\perp}$ denote the curvature tensor associated to the normal connection $\nabla^{\perp}$. The curvature tensors R, $\overline{R}$ and $R^{\perp}$ satisfy the Gauss and the Ricci equations:

$$R(X,Y,Z,W) = g(R(X,Y)Z,W) = \bar{R}(X,Y,Z,W) + \bar{g}(\sigma(Y,Z),\sigma(X,W)) - \bar{g}(\sigma(X,Z),\sigma(Y,W)),$$

$$R^{\perp}(X,Y,\xi,\eta) = \bar{g}(R^{\perp}(X,Y)\xi,\eta) = \bar{R}(X,Y,\xi,\eta) + g([A_\xi,A_\eta]X,Y),$$

where $[A_\xi,A_\eta] = A_\xi \circ A_\eta - A_\eta \circ A_\xi$ for all $X,Y,Z,W \in TM$ and $\xi,\eta \in TM^{\perp}$.

Following Funabashi [F], we consider on QP^n a local orthonormal frame $\{e_1,\dots,e_n,e_{\bar{1}} = I_1e_1,\dots,e_{\bar{n}} = I_1e_n,e_{1^*} = I_2e_1,\dots,e_{n^*} = I_2e_n,e_{\bar{1}^*} = I_3e_1,\dots,e_{\bar{n}^*} = I_3e_n\}$ such that, restricted to M, the vector fields $e_1,\dots,e_n,e_{\bar{1}},\dots,e_{\bar{n}}$ are tangent to M. We shall use the following convention for the range of indices:

$i,j,k,\dots = 1,\dots,n;$
$a,b,c,\dots = 1,\dots,n,\bar{1},\dots,\bar{n};$
$\alpha,\beta,\gamma,\dots = a^*,b^*,c^*,\dots = 1^*,\dots,n^*,\bar{1}^*,\dots,\bar{n}^*.$

We put $A_\alpha = A_{e_\alpha}$, $A_\alpha e_a = h^\alpha_{ab}e_b$. Note that, by definition, $h^\alpha_{ab} = h^\alpha_{ba}$ for all a, b and α. Moreover, by [F], the following identities hold:

$$h^{k^*}_{ij} = h^{\bar{k}^*}_{i\bar{j}} = h^{\bar{j}^*}_{i\bar{k}} = -h^{j^*}_{\bar{i}\bar{k}} = h^{j^*}_{ik} = h^{\bar{j}^*}_{i\bar{k}} = h^{\bar{i}^*}_{j\bar{k}} = -h^{i^*}_{\bar{j}\bar{k}}, \tag{2.1}$$

$$h^{\bar{k}^*}_{ij} = -h^{k^*}_{i\bar{j}} = -h^{j^*}_{i\bar{k}} = -h^{\bar{j}^*}_{\bar{i}\bar{k}} = h^{\bar{j}^*}_{ik} = -h^{j^*}_{i\bar{k}} = -h^{i^*}_{j\bar{k}} = -h^{\bar{i}^*}_{\bar{j}\bar{k}}. \tag{2.2}$$

Taking into account (2.1) and (2.2), it is easy to show that $\operatorname{tr} A_\alpha = 0$ for all α. Then, the mean curvature vector $H = \operatorname{trace}(\sigma) = \sum_\alpha \operatorname{tr} A_\alpha e_\alpha$ vanishes, that is, a totally complex submanifold is always minimal.

M is said to be *totally geodesic* if $\sigma = 0$, *parallel* (or *with parallel second fundamental form*) if $\nabla\sigma = 0$, where

$$(\nabla_X\sigma)(Y,Z) = \nabla^{\perp}_X(\sigma(Y,Z)) - \sigma(\nabla_XY,Z) - \sigma(Y,\nabla_XZ).$$

From the Gauss equation, using (2.1) and (2.2), we get (see also [F])

$$R_{ijkl} = R_{ij\bar{k}\bar{l}} = R_{\bar{i}\bar{j}\bar{k}\bar{l}} = \frac{c}{4}(\delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl}) + \sum_\alpha (h^\alpha_{il}h^\alpha_{jk} - h^\alpha_{ik}h^\alpha_{jl}), \tag{2.3}$$

$$R_{\bar{i}jk\bar{l}} = \frac{c}{4}(\delta_{jk}\delta_{il} + \delta_{ik}\delta_{jl} + 2\delta_{ij}\delta_{kl}) - \sum_\alpha (h^\alpha_{il}h^\alpha_{jk} + h^\alpha_{ik}h^\alpha_{jl}), \tag{2.4}$$

$$R_{ijk\bar{l}} = \sum_r (h^{r^*}_{il}h^{\bar{r}^*}_{jk} - h^{r^*}_{jk}h^{\bar{r}^*}_{il} + h^{r^*}_{ik}h^{\bar{r}^*}_{jl} - h^{r^*}_{jl}h^{\bar{r}^*}_{ik}), \tag{2.5}$$

$$R_{i\bar{j}\bar{k}\bar{l}} = \sum_r (h^{r^*}_{jk}h^{\bar{r}^*}_{il} - h^{r^*}_{il}h^{\bar{r}^*}_{jk} + h^{r^*}_{ik}h^{\bar{r}^*}_{jl} - h^{r^*}_{jl}h^{\bar{r}^*}_{ik}) \tag{2.6}$$

From the Gauss equation it follows that the Ricci tensor ϱ of M is given by

$$\varrho(X,Y) = \frac{n+1}{2}cg(X,Y) - \sum_\alpha g(A_\alpha X, A_\alpha Y) \tag{2.7}$$

and for the scalar curvature τ of M, we have

$$\tau = n(n+1)c - \|\sigma\|^2, \tag{2.8}$$

where $\|\sigma\|^2 = \sum \operatorname{tr} A_\alpha^2 = \sum (h_{ab}^\alpha)^2$.

We now prove the following

LEMMA 2.1. *Let M be a totally complex submanifold of QP^n, of complex dimension n. Then*

$$\|R\|^2 = 2n(n+1)c^2 - 4c\|\sigma\|^2 - \sum_{\alpha,\beta} \operatorname{tr}[A_\alpha, A_\beta]^2, \tag{2.9}$$

$$\|\varrho\|^2 = \frac{n}{2}(n+1)^2c^2 - (n+1)c\|\sigma\|^2 + \sum_\alpha (\operatorname{tr} A_\alpha^2)^2, \tag{2.10}$$

$$\frac{1}{2}\Delta\|\sigma\|^2 = \|\nabla\sigma\|^2 - \|R\|^2 - \|\varrho\|^2 - \frac{n+7}{2}c\|\sigma\|^2 + \frac{n}{2}(n+1)(n+5)c^2. \tag{2.11}$$

Proof. The components R_{abcd} of the curvature tensor of M are described by formulas (2.3)–(2.6). Taking into account that

$$\sum_\alpha \Big(\sum (h_{bd}^\alpha h_{cf}^\alpha - h_{bf}^\alpha h_{cd}^\alpha)\Big)^2 = -\sum_{\alpha,\beta} \operatorname{tr}[A_\alpha, A_\beta]^2$$

and

$$\sum_{\alpha,b,c} (h_{bb}^\alpha h_{cc}^\alpha - (h_{bc}^\alpha)^2) = \|H\|^2 - \|\sigma\|^2 = -\|\sigma\|^2,$$

we obtain formula (2.9).

Next, from (2.7) we have

$$\varrho_{ab} = \frac{n+1}{2}c\delta_{ab} - \sum_\alpha g(A_\alpha e_a, A_\alpha e_b), \tag{2.12}$$

for all a, b. Note that $\sum_\alpha g(A_\alpha e_a, A_\alpha e_a) = \|\sigma\|^2$. Moreover, we have

$$\sum_\alpha (\operatorname{tr} A_\alpha^2)^2 = \sum_{\alpha,\beta} (\operatorname{tr} A_\alpha A_\beta)^2 = \sum_\alpha g(A_\alpha e_a, A_\alpha e_b)^2 \tag{2.13}$$

(see also formula (3.28) in [F]). Using (2.12) and (2.13), we get (2.10) for $\|\varrho\|^2$.

The following formula holds for an n-dimensional totally complex submanifold of QP^n:

$$\frac{1}{2}\Delta\|\sigma\|^2 = \|\nabla\sigma\|^2 + \sum_{\alpha,\beta} \operatorname{tr}[A_\alpha, A_\beta]^2 - \sum_\alpha (\operatorname{tr} A_\alpha^2)^2 + \frac{n+3}{2}c\|\sigma\|^2 \tag{2.14}$$

(see [F]). Using (2.9) and (2.10) in (2.14), we get (2.11). □

3. Spectral invariants of the Jacobi operator

Let M be an n-dimensional Riemannian manifold immersed in a Riemannian manifold $\overline{M}$ of dimension $\bar{n} = n + r$. The *normal bundle* $TM^\perp$ is a real r-dimensional vector bundle on M, with inner product induced by the metric $\bar{g}$ of $\overline{M}$. We denote by D the *rough Laplacian* associated to the normal connection $\nabla^\perp$ of $TM^\perp$, that is,

$$D\xi = -\nabla^\perp_{e_i}\nabla^\perp_{e_i}\xi + \nabla^\perp_{\nabla_{e_i}e_i}\xi,$$

where ξ is a section of $TM^\perp$. Next, let $\tilde{A}$ be the *Simons operator* defined by

$$\bar{g}(\tilde{A}\xi, \eta) = \operatorname{tr}(A_\xi \circ A_\eta),$$

for $\xi, \eta \in TM^\perp$ [S]. Moreover, we consider the operator $\tilde{R}$ defined by

$$\tilde{R}(\xi) = \sum_{i=1}^{n} (\overline{R}(e_i, \xi)e_i)^\perp,$$

where $(\overline{R}(e_i, \xi)e_i)^\perp$ denotes the normal component of $\overline{R}(e_i, \xi)e_i)$.

The *Jacobi operator* (or *second variation operator*), acting on cross-sections of $TM^\perp$, is the second order elliptic differential operator J defined by (see [S])

$$J : TM^\perp \to TM^\perp$$
$$\xi \mapsto (D - \tilde{A} + \tilde{R})\xi.$$

When M is compact, we can define an inner product for cross-sections on $TM^\perp$, by

$$\langle \xi, \eta \rangle = \int_M \bar{g}(\xi, \eta)\, dv$$

and J is self-adjoint with respect to this product. Moreover, J is strongly elliptic and it has an infinite sequence of eigenvalues, with finite multiplicities, denoted by

$$\operatorname{spec}(M, J) = \{\lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots + \uparrow\infty\}.$$

The partition function $Z(t) = \sum_{i=1}^{\infty} \exp(-\lambda_i t)$ has the asymptotic expansion

$$Z(t) \sim (4\pi t)^{-n/2}\{a_0(J) + a_1(J)t + a_2(J)t^2 + \cdots\}.$$

By Gilkey's results [G] (see also [D] and [H]), it follows that the coefficients a_0, a_1 and a_2 are given by the following

THEOREM 3.1 [G]. *We have*

$$a_0 = r\,\operatorname{vol}(M),$$

$$a_1 = \frac{r}{6}\int_M \tau\, dv + \int_M \operatorname{tr} \tilde{E}\, dv,$$

$$a_2 = \frac{r}{360}\int_M \{2\|R\|^2 - 2\|\varrho\|^2 + 5\tau^2\}\, dv + \frac{1}{360}\int_M \{-30\|R^\perp\|^2 + \operatorname{tr}(60\tau\tilde{E} + 180\tilde{E}^2)\}\, dv,$$

where $\tilde{E} = \tilde{A} - \tilde{R}$.

We now consider a totally complex submanifold M of $QP^n(c)$ of complex dimension n and we compute explicitly the coefficients a_0, a_1 and a_2 in terms of invariants depending on the curvature of M and its isometric immersion in QP^n.

Proposition 3.2. *Let M be a totally complex submanifold of QP^n, of complex dimension n. Then*

$$\|R^\perp\|^2 = \|R\|^2 + 6c\|\sigma\|^2 - 4nc^2, \tag{3.1}$$

$$\operatorname{tr} \tilde{E} = \|\sigma\|^2 + n(n+3)c, \tag{3.2}$$

$$\operatorname{tr} \tilde{E}^2 = \|\varrho\|^2 + 2(n+2)c\|\sigma\|^2 + 2n(n+2)c^2. \tag{3.3}$$

Proof. From the Ricci equation, we get

$$R^\perp_{abc^*d^*} = \bar{R}_{abc^*d^*} - \langle [A_{c^*}A_{d^*}]e_a, e_b\rangle$$

and so,

$$\|R^\perp\|^2 = \sum_{a,b,c,d} (R^\perp_{abc^*d^*})^2 = R_1 + R_2 + R_3, \tag{3.4}$$

with

$$\begin{aligned} R_1 &= \sum \bar{R}^2_{abc^*d^*} = 4\sum(\bar{R}^2_{ijk^*l^*} + \bar{R}^2_{\bar{i}jk^*\bar{l}^*}) \\ &= \frac{c^2}{4}\sum((\delta_{il}\delta_{jk} - \delta_{jl}\delta_{ik})^2 + (\delta_{il}\delta_{jk} + \delta_{jl}\delta_{ik} - \delta_{ij}\delta_{kl})^2) \\ &= 2n(n-1)c^2, \end{aligned} \tag{3.5}$$

$$\begin{aligned} R_2 &= \sum g([A_{c^*}, A_{d^*}]e_a, e_b)^2 = -\sum \operatorname{tr}[A_{c^*}, A_{d^*}]^2 \\ &= -\sum \operatorname{tr}[A_\alpha, A_\beta]^2, \end{aligned} \tag{3.6}$$

where we used the fact that $[A_\alpha, A_\beta]$ is skew-symmetric, and

$$\begin{aligned} R_3 &= -2\sum \bar{R}_{abc^*d^*}g([A_{c^*}, A_{d^*}]e_a, e_b) \\ &= -2c\sum g(A_{a^*}e_a, A_{b^*}e_b) + 2c\sum g(A_{b^*}e_a, A_{a^*}e_b) \\ &= -2c\|H\|^2 + 2c\|\sigma\|^2 = 2c\|\sigma\|^2. \end{aligned} \tag{3.7}$$

Using (3.5)–(3.7) in (3.4), we then get (3.1).

From the expression of the curvature tensor of QP^n, it easily follows that

$$\tilde{R}(\xi) = -\frac{n+3}{2}c\xi.$$

Hence,

$$\operatorname{tr} \tilde{R} = -n(n+3)c, \tag{3.8}$$

$$\operatorname{tr} \tilde{R}^2 = \frac{n(n+3)^2}{2} c^2, \tag{3.9}$$

$$\operatorname{tr} \tilde{R} \circ \tilde{A} = -\frac{n+3}{2} c \operatorname{tr} \tilde{A}. \tag{3.10}$$

Next, by the definition of $\tilde{A}$, we get

$$\operatorname{tr} \tilde{A} = \sum_{\alpha} \bar{g}(\tilde{A}e_\alpha, e_\alpha) = \sum_{a,\alpha} \bar{g}(A_\alpha e_a, A_\alpha e_a) = \sum (h_{ab}^\alpha)^2 = \|\sigma\|^2 \tag{3.11}$$

and

$$\operatorname{tr} \tilde{A}^2 = \sum_{\alpha} \bar{g}(\tilde{A}e_\alpha, \tilde{A}e_\alpha) = \sum_{\alpha,\beta} (\bar{g}(A_\alpha, A_\beta))^2 = \sum_{\alpha,\beta} (\operatorname{tr}(A_\alpha A_\beta))^2. \tag{3.12}$$

Note that $\operatorname{tr} \tilde{E} = \operatorname{tr} \tilde{A} - \operatorname{tr} \tilde{R}$ and $\operatorname{tr} \tilde{E}^2 = \operatorname{tr}(\tilde{A}^2 - 2\tilde{R} \circ \tilde{A} + \tilde{R}^2)$. So, using (3.8)–(3.12) and taking into account (2.10) and (2.13), we get (3.2) and (3.3). □

The following result follows from Theorem 3.1 and Proposition 3.2.

THEOREM 3.3. *On a totally complex submanifold M of $QP^n(c)$, of complex dimension n, the first three coefficients of the asymptotic expansion of the partition function of the Jacobi operator are given by*

$$a_0 = 2n \operatorname{vol}(M), \tag{3.13}$$

$$a_1 = \frac{n-3}{3} \int_M \tau \, dv + 2n(n+2)c \operatorname{vol}(M), \tag{3.14}$$

$$\begin{aligned} a_2 = {} & \frac{2n-15}{180} \int_M \|R\|^2 \, dv + \frac{45-n}{90} \int_M \|\varrho\|^2 \, dv + \frac{n-6}{36} \int_M \tau^2 \, dv \\ & + k_1(n)c \int_M \tau \, dv + k_2(n)c^2 \operatorname{vol}(M), \end{aligned} \tag{3.15}$$

where $k_1 = \dfrac{2n^2 - 2n - 9}{6}$ and $k_2 = \dfrac{n(6n^2 + 21n + 23)}{6}$ are constants depending on n. □

4. Totally complex parallel submanifolds of QP^n and the spectrum of J

Parallel submanifolds of QP^n have been classified by K. Tsukada in [T1]. He proved that there are four types of parallel not totally geodesic submanifolds in QP^n:

(R–R) totally real submanifolds contained in a totally real totally geodesic submanifold,

$(R–C)$ totally real submanifolds contained in a totally complex totally geodesic submanifold,

$(C–C)$ totally complex submanifolds contained in a totally complex totally geodesic submanifold,

$(C–Q)$ totally complex submanifolds contained in an invariant totally geodesic submanifold.

The immersion of a submanifold of type $(R–R)$ (respectively, $(R–C)$) into QP^n, is given by the composition of its immersion into the real projective space RP^k (respectively, the complex projective space CP^k), with the standard totally geodesic immersion of RP^k (respectively, CP^k) into QP^k. In the same way, the immersion of a submanifold of type $(C–C)$ into QP^n, is the composition of its Kaehler immersion into CP^k with the standard totally geodesic immersion of CP^k into QP^k. For this reason, totally complex submanifolds of type $(C–Q)$ appear to be the most specific parallel submanifolds of QP^n. Moreover, up to our knowledge, the known results about totally complex submanifolds of QP^n, except for Tsukada's works [T1] and [T2], mostly concern submanifolds of type $(C–C)$ (see for example [CoGa], [Mr], [Mt], [X]).

Tsukada [T1] proved that a totally complex parallel submanifold of QP^n, of type $(C–Q)$, has complex dimension n. Moreover, associated with a totally complex parallel immersion into QP^n, there exists a Kaehler immersion into a $(2n+1)$-dimensional complex projective space CP^{2n+1}, whose composition with the projection of CP^{2n+1} onto QP^n coincides with the given totally complex immersion. Therefore, the classification of totally complex parallel submanifolds of QP^n is related to the one of Kaehler imbeddings of Hermitian symmetric spaces into the complex projective space, given By Nakagawa and Takagi in [NTa].

The following table describes explicitly all n-dimensional compact totally complex parallel submanifolds, embedded into $QP^n(c)$.

Table 1

M	dim	τ
$CP^n(c)$	n	$n(n+1)c$
$Sp(3)/U(3)$	6	$24c$
$SU(6)/S(U(3)\times U(3))$	9	$54c$
$SO(12)/U(6)$	15	$150c$
$E_7/E_6\cdot T$	27	$486c$
$CP^1(c)\times CP^1(c)\times CP^1(c)$	3	$6c$
$CP^1(c)\times CP^1\left(\frac{c}{2}\right)$	2	$3c$
$CP^1(c)\times Q^{n-1}$	n	$(2+(n-1)^2)c$

The first six manifolds listed in Table 1 are irreducible symmetric spaces and so, Einstein spaces. It is easy to check that, for two different manifolds in the Table 1, having the same complex dimension n, the scalar curvature τ never attains the same value. Therefore, we have the following

THEOREM 4.1. *Each n-dimensional totally complex parallel submanifold M_0 of $QP^n(c)$ is uniquely determined by its scalar curvature.*

Taking into account formulas (3.13)–(3.15) and Theorem 4.1, we can now prove the following

THEOREM 4.2. *Each n-dimensional totally complex parallel submanifold M_0 of $QP^n(c)$ is uniquely determined by its* spec(J).

Proof. We treat the cases $n \neq 3$ and $n = 3$ separately.

a) If $n \neq 3$, then, by (3.13) and (3.14) it follows that spec J determines the dimension and the scalar curvature of a totally complex parallel submanifold M_0 of $QP^n(c)$. The conclusion then follows from Theorem 4.1.

b) If $n = 3$, a totally complex parallel submanifold of $QP^3(c)$ is either isometric to $CP^3(c)$ or to $CP^1(c) \times CP^1(c) \times CP^1(c)$ (see Table 1). We will show that they do not have the same spec J. In fact, let M_0 be an n-dimensional totally complex parallel Einstein submanifold of QP^n. Then $\|\sigma_0\|^2$ is constant and $\|\varrho_0\|^2 = \dfrac{\tau_0^2}{2n}$. Taking also into account (2.8), from (2.11) it follows

$$\|R_0\|^2 = \frac{n+7}{2} c\tau_0 - \frac{1}{2n}\tau_0^2 - n(n+1)c^2. \tag{4.1}$$

Suppose now that $M_0 = CP^3(c)$ and $M_0' = CP^1(c) \times CP^1(c) \times CP^1(c)$ have the same spec J. In particular, $a_0(M_0) = a_0(M_0')$ and $a_2(M_0) = a_2(M_0')$. So, by (3.13) and (3.15), $\mathrm{vol}(M_0) = \mathrm{vol}(M_0')$, and

$$\begin{aligned} &\int_{M_0} \left\{ -\frac{1}{20}\|R_0\|^2 + \frac{7}{15}\|\varrho_0\|^2 - \frac{1}{12}\tau_0^2 \right\} dv + \frac{1}{2}c \int_{M_0} \tau_0 \, dv \\ &\quad = \int_{M_0'} \left\{ -\frac{1}{20}\|R_0'\|^2 + \frac{7}{15}\|\varrho_0'\|^2 - \frac{1}{12}(\tau_0')^2 \right\} dv + \frac{1}{2}c \int_{M_0'} \tau_0' \, dv. \end{aligned} \tag{4.2}$$

We know that $\tau_0 = 12c$ and $\tau_0' = 6c$ (see Table 1). Since both M_0 and M_0' are Einstein, $\|\varrho_0\|^2 = \dfrac{\tau_0^2}{6} = 24c^2$ and $\|\varrho_0'\|^2 = 6c^2$. Moreover, from (4.1) we get $\|R_0\|^2 = 24c^2$ and $\|R_0'\|^2 = 12c^2$. Using this information in (4.2), we obtain

$$4c^2 \, \mathrm{vol}(M_0) = \frac{11}{5} c^2 \, \mathrm{vol}(M_0'),$$

which can not occur, since $\mathrm{vol}(M_0) = \mathrm{vol}(M_0')$ and $c > 0$. $\square$

We now characterize totally complex parallel Einstein submanifolds of $QP^n(c)$, in the class of all totally complex submanifolds, by proving the following

THEOREM 4.3. *Let M be a compact totally complex submanifold of $QP^n(c)$ and M_0 a totally complex parallel Einstein submanifold of $QP^n(c)$. If* $\operatorname{spec}(M,J) = \operatorname{spec}(M_0,J)$ *and* $8 \le \dim M_0 \le 26$, *then M is isometric to M_0.*

Proof. Since $\operatorname{spec}(M,J) = \operatorname{spec}(M_0,J)$, M and M_0 have the same complex dimension n and, from Theorem 3.3, since $n \neq 3$, we get

$$\operatorname{vol}(M) = \operatorname{vol}(M_0), \tag{4.3}$$

$$\int_M \tau \, dv = \int_{M_0} \tau_0 \, dv, \tag{4.4}$$

$$\begin{aligned}\frac{2n-15}{180}\int_M \|R\|^2 \, dv + \frac{45-n}{90}\int_M \|\varrho\|^2 \, dv + \frac{n-6}{36}\int_M \tau^2 \, dv \\ = \frac{2n-15}{180}\int_{M_0} \|R_0\|^2 \, dv + \frac{45-n}{90}\int_{M_0} \|\varrho_0\|^2 \, dv + \frac{n-6}{36}\int_{M_0} \tau_0^2 \, dv.\end{aligned} \tag{4.5}$$

Since τ_0 is constant and $\operatorname{vol}(M) = \operatorname{vol}(M_0)$, we have

$$\begin{aligned}\int_M \tau^2 \, dv - \int_{M_0} \tau_0^2 \, dv &= \int_M \tau^2 \, dv - 2\tau_0 \int_{M_0} \tau_0 \, dv + \int_{M_0} \tau_0^2 \, dv \\ &= \int_M (\tau - \tau_0)^2 \, dv \ge 0\end{aligned} \tag{4.6}$$

where the equality holds if and only if $\tau = \tau_0$.

Next, let $E = \varrho - \frac{\tau}{2n} g$ denote the *Einstein curvature tensor* of (M,g) ($2n$ being the real dimension of M). Since $\|E\|^2 = \|\varrho\|^2 - \frac{\tau^2}{2n}$ and $E_0 = 0$ because M_0 is an Einstein space, (4.5) becomes

$$\begin{aligned}\frac{2n-15}{180}\left(\int_M \|R\|^2 \, dv - \int_{M_0} \|R_0\|^2 \, dv\right) + \frac{45-n}{90}\int_M \|E\|^2 \, dv \\ + \frac{5n^2 - 31n + 45}{180n}\left(\int_M \tau^2 \, dv - \int_{M_0} \tau_0^2 \, dv\right) = 0.\end{aligned} \tag{4.7}$$

Moreover, integrating (2.11) over M and using $\|E\|^2$, we get

$$\begin{aligned}\int_M \|\nabla\sigma\|^2 \, dv = \int_M \|R\|^2 \, dv + \int_M \|E\|^2 \, dv + \frac{1}{2n}\int_M \tau^2 \, dv \\ - \frac{n+7}{2} c \int_M \tau \, dv + n(n+1)c^2 \operatorname{vol}(M).\end{aligned} \tag{4.8}$$

Formula (4.8) also holds for M_0, with $\nabla'\sigma_0 = E_0 = 0$. We use (4.8) to calculate $\int_M \|R\|^2\, dv - \int_{M_0} \|R_0\|^2\, dv$. Taking into account (4.3) and (4.4), we get

$$\int_M \|\nabla\sigma\|^2\, dv = \left(\int_M \|R\|^2\, dv - \int_{M_0} \|R_0\|^2\, dv\right) + \int_M \|E\|^2\, dv + \frac{1}{2n}\left(\int_M \tau^2\, dv - \int_{M_0} \tau_0^2\, dv\right).$$

Hence, (4.7) becomes

$$\alpha(n)\int_M \|\nabla\sigma\|^2\, dv + \beta(n)\int_M \|E\|^2\, dv + \gamma(n)\left(\int_M \tau^2\, dv - \int_{M_0} \tau_0^2\, dv\right) = 0,$$

where

$$\alpha(n) = \frac{2n-15}{180},$$
$$\beta(n) = \frac{105-4n}{180},$$
$$\gamma(n) = \frac{10n^2 - 64n + 105}{360n}.$$

Since $8 \le n \le 26$, whe have $\alpha(n), \beta(n), \gamma(n) > 0$ and so, $\nabla'\sigma = 0$, $E = 0$ and $\tau = \tau_0$. Thus, M is an Einstein (compact) totally real parallel submanifold of $QP^n(c)$, with the same spec(J) of M_0, and Theorem 4.2 implies that M is isometric to M_0. □

As a consequence of Theorem 4.3, we have at once the following

Corollary 4.4. *Totally complex submanifolds $SU(6)/S(U(3)\times U(3))$ of QP^9 and $SO(12)/U(6)$ of QP^{15} are completely characterized by their* spec(J).

In the case of a totally complex totally geodesic submanifold $CP^n(c)$, we can improve the result given in Theorem 4.3. In fact, let M be a totally complex submanifold of complex dimension $n \neq 3$. Then, (3.13) and (3.14) imply that $\int_M \tau\, dv$ is a spectral invariant. So, by (2.8), also $\int_M \|\sigma\|^2\, dv$ is a spectral invariant. In particular, since $CP^n(c)$ is the only totally complex submanifold of $QP^n(c)$ with $\sigma = 0$, we have the following

Theorem 4.5. *In the class of all compact totally complex submanifolds of $QP^n(c)$, the complex projective space $CP^n(c)$ is characterized by its* spec(J) *for all $n \neq 3$.*

5. Spectral geometry of the Laplace operator for totally complex submanifolds of QP^n

The problem of characterizing a (compact) Riemannian manifold through the spectrum of the Laplace-Beltrami operator Δ acting on functions, is a well-known classical problem in Riemannian geometry. In general, a Riemannian manifold M is not completely characterized by the spectrum of Δ [BGM, p. 154]. The well-known asymptotic expansion of Minakshisundaram-Pleijel expresses the partition function associated to $\mathrm{spec}(M, \Delta)$. The coefficients $a_i(\Delta)$ of this asymptotic expansion are Riemannian invariants of M [BGM]. In particular,

$$a_0(\Delta) = \mathrm{vol}(M), \tag{5.1}$$

$$a_1(\Delta) = \frac{1}{6}\int_M \tau \, dv, \tag{5.2}$$

$$a_2(\Delta) = \frac{1}{360}\int_M \{2\|R\|^2 - 2\|\varrho\|^2 + 5\tau^2\} \, dv. \tag{5.3}$$

In [U], Udagawa used these invariants to characterize Hermitian symmetric submanifolds of degree 3 among all Kaehler-Einstein submanifolds of the complex projective space. By a direct calculation, we can prove a similar result for totally complex parallel submanifolds of QP^n.

Theorem 5.1. *Let M be a compact totally complex Einstein submanifold of QP^n and M_0 a totally complex parallel Einstein submanifold of QP^n. If* $\mathrm{spec}(M, \Delta) = \mathrm{spec}(M_0, \Delta)$, *then M is isometric to M_0.*

Proof. Since $\mathrm{spec}(M, \Delta) = \mathrm{spec}(M_0, \Delta)$, $\dim M = \dim M_0 = n$. Moreover, from (5.1)–(5.3) we have

$$\mathrm{vol}(M) = \mathrm{vol}(M_0), \tag{5.4}$$

$$\int_M \tau \, dv = \int_{M_0} \tau_0 \, dv, \tag{5.5}$$

$$\int_M \{2\|R\|^2 - 2\|\varrho\|^2 + 5\tau^2\} \, dv = \int_{M_0} \{2\|R_0\|^2 - 2\|\varrho_0\|^2 + 5\tau_0^2\} \, dv. \tag{5.6}$$

Using (5.4) and (5.5) (instead of (4.3) and (4.4)) and proceeding as in the proof of Theorem 4.3, we can show that (4.6) also holds when $\mathrm{spec(M, \Delta) = spec(M_0, \Delta)}$ (for all n). Moreover, we can use (4.8), which holds for any totally complex submanifold of QP^n, to express $\int_M \|R\|^2 \, dv - \int_{M_0} \|R_0\|^2 \, dv$. Note that in this case, $E = E_0 = 0$. Therefore, using (5.6) and (4.8), we obtain

$$2\int_M \|\nabla\sigma\|^2 \, dv = \left(\frac{2}{n} - 5\right)\left(\int_M \tau^2 \, dv - \int_{M_0} \tau_0^2 \, dv\right). \tag{5.7}$$

Since $\dfrac{2}{n} - 5 < 0$ for all n and, by (4.6), $\int_M \tau^2 \, dv - \int_{M_0} \tau_0^2 \geq 0$, from (5.7) it follows that $\nabla\sigma = 0$ and $\tau = \tau_0$. Thus, by Theorem 4.1, M is isometric to M_0. $\square$

The idea of combining the information coming from $\mathrm{spec}\,\Delta$ and $\mathrm{spec}\,J$ for a submanifold, has already been used by H. Donnelly [D] to characterize totally geodesic submanifolds of a real space form. In the case of totally complex submanifolds of QP^n, we can prove the following

Theorem 5.2. *For all $n \geq 2$, let M be a compact totally complex submanifold of QP^n and M_0 a totally complex parallel submanifold of QP^n* (not necessarily Einstein). *If* $\mathrm{spec}(M, J) = \mathrm{spec}(M_0, J)$ *and* $\mathrm{spec}(M, \Delta) = \mathrm{spec}(M_0, \Delta)$, *then M is isometric to M_0.*

Proof. Since $\mathrm{spec}(M, \Delta) = \mathrm{spec}(M_0, \Delta)$, (5.4)–(5.6) hold and, as a consequence, also (4.6). Moreover, from $\mathrm{spec}(M, J) = \mathrm{spec}(M_0, J)$ it follows

$$\begin{aligned}(5.8)\quad & \frac{2n-15}{180}\int_M \|R\|^2\,dv + \frac{45-n}{90}\int_M \|\varrho\|^2\,dv + \frac{n-6}{36}\int_M \tau^2\,dv \\ &= \frac{2n-15}{180}\int_{M_0} \|R_0\|^2\,dv + \frac{45-n}{90}\int_{M_0} \|\varrho_0\|^2\,dv + \frac{n-6}{36}\int_{M_0} \tau_0^2\,dv.\end{aligned}$$

Next, integrating (2.11) on M, we get

$$\int_M \|\nabla\sigma\|^2\,dv = \int_M \|R\|^2\,dv + \int_M \|\varrho\|^2\,dv - \frac{n+7}{2}c\int_M \tau\,dv + n(n+1)c^2\,\mathrm{vol}(M),$$

and a corresponding formula holds for M_0 with $\nabla\sigma_0 = 0$. So,

$$\int_M \|\nabla\sigma\|^2\,dv = \left(\int_M \|R\|^2\,dv - \int_{M_0} \|R_0\|^2\,dv\right) + \left(\int_M \|\varrho\|^2\,dv - \int_{M_0} \|\varrho_0\|^2\,dv\right).$$

We can use (5.6) and (5.8) to express $\int_M \|R\|^2\,dv - \int_{M_0} \|R_0\|^2\,dv$ and $\int_M \|\varrho\|^2\,dv - \int_{M_0} \|\varrho_0\|^2\,dv$ in function of $\int_M \tau^2\,dv - \int_{M_0} \tau_0^2\,dv$. Hence, the last formula becomes

$$\int_M \|\nabla\sigma\|^2\,dv = -\frac{27}{10}\left(\int_M \tau^2\,dv - \int_{M_0} \tau_0^2\,dv\right) = 0,$$

from which whe can conclude that $\nabla\sigma = 0$ and $\tau = \tau_0$. Therefore, Theorem 4.1 implies that M is isometric to M_0. $\square$

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