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SUMS, PRODUCTS, AND RATIOS FOR THE GENERALIZED BIVARIATE PARETO DISTRIBUTION

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Abstract

We derive the exact distributions of $R = X + Y$, $U = XY$ and $W = X/(X + Y)$ and the corresponding moment properties when X and Y follow the generalized bivariate Pareto distribution. The expressions turn out to involve special functions.

1. Introduction

Since the 1930s, the statistics literature has seen many developments in the theory and applications of linear combinations and ratios of random variables. Some of these include:

- Ratios of normal random variables appear as sampling distributions in single equation models, in simultaneous equations models, as posterior distributions for parameters of regression models and as modeling distributions, especially in economics when demand models involve the indirect utility function (details in Yatchew, 1986).
- Weighted sums of uniform random variables—in addition to the well known application to the generation of random variables—have applications in stochastic processes which in many cases can be modeled by these weighted sums. In computer vision algorithms these weighted sums play a pivotal role (Kamgar-Parsi et al., 1995). An earlier application of the linear combinations of uniform random variables is given in connection with the distribution of errors in nth tabular differences Δ^n (Lowan and Laderman, 1939).
- Ratio of linear combinations of chi-squared random variables are part of von Neumann's (1941) test statistics (mean square successive difference divided by the variance). These ratios appear in various two-stage tests (Toyoda and Ohtani, 1986). They are also used in tests on structural coefficients of a multivariate linear functional relationship model (details in Chaubey and Nur Enayet Talukder (1983) and Provost and Rudiuk (1994)).
- Sums of independent gamma random variables have applications in queuing theory problems such as determination of the total waiting time and in

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civil engineering problems such as determination of the total excess water flow into a dam. They also appear in test statistics used to determine the confidence limits for the coefficient of variation of fiber diameters (Linhart (1965) and Jackson (1969)) and in connection with the inference about the mean of the two-parameter gamma distribution (Grice and Bain, 1980).

- Linear combinations of inverted gamma random variables are used for testing hypotheses and interval estimation based on generalized p-values, specifically for the Behrens-Fisher problem and variance components in balanced mixed linear models (Witkovský, 2001).
- As to the Beta distributions their linear combinations occur in calculations of the power of a number of tests in ANOVA (Monti and Sen, 1976) among other applications. More generally, the linear combinations are used for detecting changes in the location of the distribution of a sequence of observations in quality control problems (Lai, 1974). Pham-Gia and Turkkan (1993, 1994, 1998, 2002) and Pham-Gia (2000) provided applications of sums and ratios to availability, Bayesian quality control and reliability.
- Linear combinations of the form $T = a_1 t_{f_1} + a_2 t_{f_2}$, where t_f denotes the Student t random variable based on f degrees of freedom, represents the Behrens-Fisher statistic and—as early as the middle of the twentieth century—Stein (1945) and Chapman (1950) developed a two-stage sampling procedure involving the T to test whether the ratio of two normal random variables is equal to a specified constant.
- Weighted sums of the Poisson parameters are used in medical applications for directly standardized mortality rates (Dobson et al., 1991).

In this paper, we consider the distributions of $R = X + Y$, $U = XY$ and $W = X/(X + Y)$ when X and Y are correlated Pareto random variables with the joint pdf given by

$$f(x, y) = \frac{K(xy)^{p-1}}{(\alpha + \beta x + \lambda y + \delta xy)^{p+q}} \tag{1}$$

for $x > 0$, $y > 0$, $p > 0$ and $q > 0$, where $K = K(\alpha, \beta, \lambda, \delta, p, q)$ denotes the normalizing constant. This distribution is known as the generalized bivariate Pareto distribution. As often with the Pareto distributions, this distribution has applications in reliability studies. Inaba and Shirahata (1986) fitted this distribution to data on white blood counts and survival times of patients who died of acute myelogenous leukemia (Gross and Clark, 1975), comparing it with the bivariate normal distribution.

The aim of this paper is to derive explicit expressions for the pdfs and moments of $R = X + Y$, $U = XY$ and $W = X/(X + Y)$. The calculations involve the Gauss hypergeometric function defined by

$$G(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. We also need the following important lemmas.

LEMMA 1. *For* $0 < \alpha < \rho$,

$$\int_0^\infty \frac{x^{\alpha-1}}{(x+z)^\rho}\, dx = z^{\alpha-\rho} B(\alpha, \rho-\alpha),$$

where

$$B(a,b) = \int_0^1 w^{a-1}(1-w)^{b-1}\, dw$$

for $a > 0$ *and* $b > 0$ *is the beta function.*

LEMMA 2. *For* $p > 0$ *and* $q > 0$,

$$\int_a^b (x-a)^{p-1}(b-x)^{q-1}(cx+d)^r\, dx$$
$$= (b-a)^{p+q-1}(ac+d)^r B(p,q) G\left(p, -r; p+q; \frac{c(a-b)}{ac+d}\right).$$

LEMMA 3. *For* $0 < \alpha < \rho + \lambda$,

$$\int_0^\infty x^{\alpha-1}(x+y)^{-\rho}(x+z)^{-\lambda}\, dx = z^{-\lambda} y^{\alpha-\rho} B(\alpha, \rho+\lambda-\alpha) G\left(\alpha, \lambda; \rho+\lambda; 1-\frac{y}{z}\right).$$

LEMMA 4. *For* $a > 0$, $b^2 < ac$ *and* $0 < p < 2\rho$,

$$\int_0^\infty \frac{x^{p-1}}{(ax^2+2bx+c)^\rho}\, dx = a^{-p/2} c^{p/2-\rho} B(p, 2\rho-p) G\left(\frac{p}{2}, \rho-\frac{p}{2}; \rho+\frac{1}{2}; 1-\frac{b^2}{ac}\right).$$

LEMMA 5. *For* $a > b$ *and* $0 < \alpha < 2\rho$,

$$\int_0^\infty \frac{x^{\alpha-1}}{\{(x+a)^2-b^2\}^\rho}\, dx$$
$$= B(\alpha, 2\rho-\alpha)(a^2-b^2)^{\alpha/2-\rho} G\left(\alpha, 2\rho-\alpha; \rho+\frac{1}{2}; \frac{1}{2}\left(1-\frac{a}{\sqrt{a^2-b^2}}\right)\right).$$

LEMMA 6. *For* $0 < |\eta| < \pi$ *and* $1 < \alpha < 2\rho$,

$$\int_0^\infty \frac{x^{\alpha-1}}{(x^2+y^2+2xy\cos\eta)^\rho}\, dx$$
$$= \frac{1}{2y^{2\rho-\alpha}} \left\{ B\left(\frac{\alpha}{2}, \rho-\frac{\alpha}{2}\right) G\left(\frac{\alpha}{2}, \rho-\frac{\alpha}{2}; \frac{1}{2}; \cos^2\eta\right)\right.$$
$$\left. - (\alpha-1)|\cos\eta| B\left(\frac{\alpha-1}{2}, \rho-\frac{\alpha-1}{2}\right) G\left(\frac{\alpha-1}{2}, \rho-\frac{\alpha-1}{2}; \frac{3}{2}; \cos^2\eta\right)\right\}.$$

The properties of the Gauss hypergeometric function can be found in Prudnikov et al. (1986) and Gradshteyn and Ryzhik (2000).

2. PDFS

Theorems 1 to 5 derive the pdfs of $R = X + Y$, $U = XY$ and $W = X/(X + Y)$ when X and Y are distributed according to (1).

THEOREM 1. *If X and Y are jointly distributed according to* (1) *and if* $\beta = \lambda$ *then*

$$f_R(r) = K 2^{1-2p} B(p, 1/2) \frac{r^{2p-1}}{(\alpha + \beta r)^{p+q}} G\left(p, p+q; p + \frac{1}{2}; -\frac{\delta r^2}{4(\alpha + \beta r)} \right) \tag{2}$$

for $0 < r < \infty$.

Proof. From (1), the joint pdf of $(R, W) = (X + Y, X/R)$ becomes

$$f(r, w) = \frac{K r^{2p-1} \{w(1-w)\}^{p-1}}{\{\alpha + \beta r w + \lambda r (1-w) + \delta r^2 w (1-w)\}^{p+q}}. \tag{3}$$

Thus, the pdf of R can be written as

$$\begin{aligned} f_R(r) &= K r^{2p-1} \int_0^1 \frac{\{w(1-w)\}^{p-1}}{\{\alpha + \beta r w + \lambda r(1-w) + \delta r^2 w(1-w)\}^{p+q}} \, dw \\ &= K r^{2p-1} \int_0^{1/4} \frac{u^{p-1} (1/4 - u)^{-1/2}}{(\alpha + \beta r + \delta r^2 u)^{p+q}} \, du \end{aligned} \tag{4}$$

after substituting $u = w(1 - w)$. The result of the theorem following applying Lemma 2 to calculate the integral in (4). ▲

THEOREM 2. *If X and Y are jointly distributed according to* (1) *then*

$$\begin{aligned} f_U(u) &= K(\beta\lambda)^{-(p+q)/2} B(p+q, p+q) u^{p-1} \\ &\quad \times G\left(\frac{p+q}{2}, \frac{p+q}{2}; p+q+\frac{1}{2}; 1 - \frac{(\alpha + \delta u)^2}{4\beta\lambda} \right) \end{aligned} \tag{5}$$

for $(\alpha + \delta u)^2 < 4\beta\lambda$, *and*

$$\begin{aligned} f_U(u) &= K(\beta\lambda)^{-(p+q)/2} B(p+q, p+q) u^{(p-q)/2-1} \\ &\quad \times G\left(p+q, p+q; p+q+\frac{1}{2}; \frac{1}{2}\left(1 - \frac{\alpha + \delta u}{2\sqrt{\beta\lambda u}} \right) \right) \end{aligned} \tag{6}$$

for $(\alpha + \delta u)^2 > 4\beta\lambda u$.

Proof. From (1), the joint pdf of $(X, U) = (X, XY)$ becomes

$$f(x,u) = \frac{Ku^{p-1}x^{p+q-1}}{\{\beta x^2 + (\alpha+\delta u)x + \lambda u\}^{p+q}}. \tag{7}$$

If $(\alpha+\delta u)^2 > 4\beta\lambda u$ then one can rewrite (7) as

$$f(x,u) = \frac{Ku^{p-1}x^{p+q-1}}{\beta^{p+q}\{(x+a)^2 - b^2\}^{p+q}},$$

where $a = (\alpha+\delta u)/(2\beta)$ and $b^2 = a^2 - \lambda u/\beta$. Integrating with respect to x using Lemma 5 one obtains the form in (6). On the other hand, if $(\alpha+\delta u)^2 < 4\beta\lambda$ then integrating (7) with respect to x using Lemma 4 one obtains the form in (5). ▲

THEOREM 3. *If X and Y are jointly distributed according to* (1) *then*

$$f_W(w) = K\alpha^{-q}\delta^{-p}B(2p,2q)\{w(1-w)\}^{-1}G\left(p,q;p+q+\frac{1}{2};1-\frac{\{\beta w+\lambda(1-w)\}^2}{4\alpha\delta w(1-w)}\right) \tag{8}$$

for $\{\beta w + \lambda(1-w)\}^2 < 4\alpha\delta w(1-w)$, *and*

$$f_W(w) = K\alpha^{-q}\delta^{q}B(2p,2q)\{w(1-w)\}^{p+q-1} \times G\left(2p,2q;p+q+\frac{1}{2};\frac{1}{2}\left(1-\frac{\beta w+\lambda(1-w)}{2\sqrt{\alpha\delta w(1-w)}}\right)\right) \tag{9}$$

for $\{(\beta-\lambda)^2 + 4\delta\alpha\}^2 w^2 + \{2(\beta-\lambda) - 4\alpha\delta\}w + \lambda^2 > 0$.

Proof. If $\{(\beta-\lambda)^2 + 4\delta\alpha\}^2 w^2 + \{2(\beta-\lambda) - 4\alpha\delta\}w + \lambda^2 > 0$ then one can rewrite (3) as

$$f(r,w) = \frac{Kr^{2p-1}}{\delta^{p+q}\{w(1-w)\}^{q+1}\{(r+a)^2 - b^2\}^{p+q}},$$

where $a = \{\beta w + \lambda(1-w)\}/\{2\delta w(1-w)\}$ and $b^2 = a^2 - \alpha/\{\delta w(1-w)\}$. Integrating with respect to r using Lemma 5 one obtains the form in (9). On the other hand, if $\{\beta w + \lambda(1-w)\}^2 < 4\alpha\delta w(1-w)$ then integrating (3) with respect to r using Lemma 4 one obtains the form in (8). ▲

Alternative and equivalent formulas for (5)–(6) and (8)–(9) are given in the following theorems.

THEOREM 4. *If X and Y are jointly distributed according to* (1) *then*

$$
\begin{aligned}
f_U(u) = \frac{K u^{(p-q)/2-1}}{2(\beta\lambda)^{(p+q)/2}} \Bigg[& B\left(\frac{p+q}{2}, \frac{p+q}{2}\right) G\left(\frac{p+q}{2}, \frac{p+q}{2}; \frac{1}{2}; \frac{(\alpha+\delta u)^2}{4\beta\lambda u}\right) \\
& - \frac{(p+q-1)(\alpha+\delta u)}{2\sqrt{\beta\lambda u}} B\left(\frac{p+q-1}{2}, \frac{p+q+1}{2}\right) \\
& \times G\left(\frac{p+q-1}{2}, \frac{p+q+1}{2}; \frac{3}{2}; \frac{(\alpha+\delta u)^2}{4\beta\lambda u}\right) \Bigg]
\end{aligned}
$$

for $0 < u < \infty$ *and* $|\alpha + \delta u| \leq 2\sqrt{\beta\lambda u}$.

Proof. Using (7), one can write

$$
f_U(u) = \frac{K u^{p-1}}{\beta^{p+q}} \int_0^\infty \frac{x^{p+q-1}}{\{x^2 + 2xy\cos\eta + y^2\}^{p+q}}\, dx,
$$

where $y = \sqrt{\lambda u/\beta}$ and $2\cos\eta = \{\alpha + \delta u\}/\sqrt{\beta\lambda u}$. Application of Lemma 6 to calculate this integral yields the result of the theorem. ▲

THEOREM 5. *If X and Y are jointly distributed according to* (1) *then*

$$
\begin{aligned}
f_W(w) = \frac{K}{2\alpha^q\delta^p w(1-w)} \Bigg[& B(p,q) G\left(p, q; \frac{1}{2}; \frac{\{\beta w + \lambda(1-w)\}^2}{4\alpha\delta w(1-w)}\right) \\
& - (2p-1) B\left(p - \frac{1}{2}, q + \frac{1}{2}\right) \frac{\beta w + \lambda(1-w)}{2\alpha\sqrt{\delta w(1-w)}} \\
& \times G\left(p - \frac{1}{2}, q + \frac{1}{2}; \frac{3}{2}; \frac{\{\beta w + \lambda(1-w)\}^2}{4\alpha\delta w(1-w)}\right) \Bigg]
\end{aligned}
$$

for $0 < w < 1$ *and* $|\beta w + \lambda(1-w)| \leq 2\sqrt{\alpha\delta w(1-w)}$.

Proof. Using (3), one can write

$$
f_W(w) = \frac{K}{\delta^{p+q}\{w(1-w)\}^{q+1}} \int_0^\infty \frac{r^{2p-1}}{\{r^2 + 2ry\cos\eta + y^2\}^{p+q}}\, dr,
$$

where $y = \sqrt{\alpha}/\sqrt{\delta w(1-w)}$ and $2\cos\eta = \{\beta w + \lambda(1-w)\}/\sqrt{\alpha\delta w(1-w)}$. Application of Lemma 6 to calculate this integral yields the result of the theorem. ▲

Figures 1 to 3 illustrate the shape of the pdfs (2), (5)–(6) and (8)–(9) for selected values of p and q. Each plot contains four curves corresponding to selected values of q. The effect of the parameters is evident.

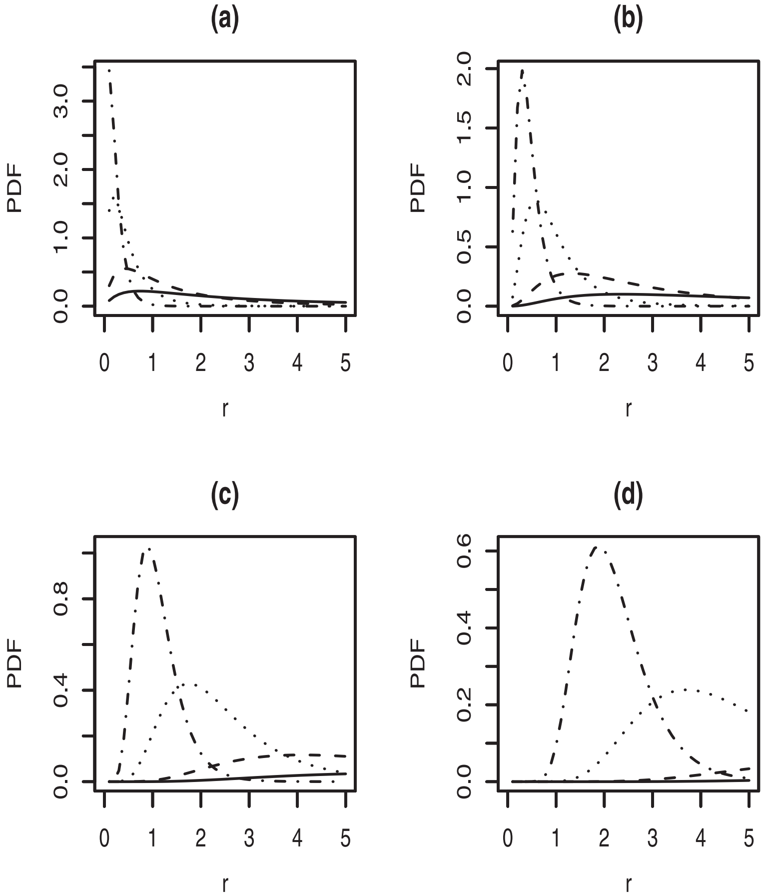

FIGURE 1. Plots of the pdf of (2) for $\alpha = 1$, $\beta = 1$, $\lambda = 1$, $\delta = 1$ and (a): $p = 1$; (b): $p = 2$; (c): $p = 5$; and, (d): $p = 10$. The four curves in each plot are: the solid curve ($q = 1$), the curve of lines ($q = 2$), the curve of dots ($q = 5$), and the curve of lines and dots ($q = 10$).

3. Moments

Here, we derive the moments of $R = X + Y$ and $U = XY$ when X and Y are distributed according to (1). We need the following lemma.

LEMMA 7. *If X and Y are jointly distributed according to* (1) *then*

$$E(X^m Y^n) = \frac{KB(n+p, q-n)B(m+p, q-m)\alpha^{m+n+p-q}}{\beta^{m+p}\lambda^{n+p}} \times G\left(m+p, n+p; p+q; 1 - \frac{\alpha\delta}{\beta\lambda}\right)$$

for $m \geq 1$, $n \geq 1$, $q > m$ and $q > n$.

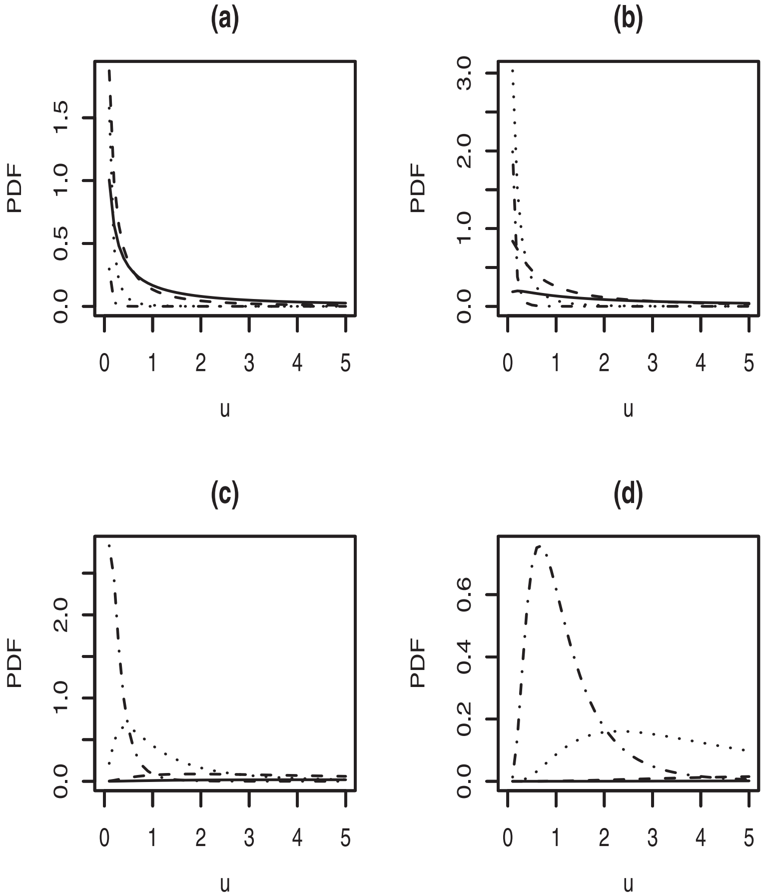

FIGURE 2. Plots of the pdf of (5)–(6) for $\alpha = 1$, $\beta = 1$, $\lambda = 1$, $\delta = 1$ and (a): $p = 1$; (b): $p = 2$; (c): $p = 5$; and, (d): $p = 10$. The four curves in each plot are: the solid curve ($q = 1$), the curve of lines ($q = 2$), the curve of dots ($q = 5$), and the curve of lines and dots ($q = 10$).

Proof. The result follows by the arguments

$$
\begin{aligned}
E(X^m Y^n) &= \int_0^\infty \int_0^\infty \frac{K x^{m+p-1} y^{n+p-1}}{(\alpha + \beta x + \lambda y + \delta x y)^{p+q}} \, dy dx \\
&= K \int_0^\infty \frac{x^{m+p-1}}{(\lambda + \delta x)^{p+q}} \int_0^\infty \frac{y^{n+p-1}}{\{y + (\alpha + \beta x)/(\lambda + \delta x)\}^{p+q}} \, dy dx \\
&= K \int_0^\infty \frac{x^{m+p-1}}{(\lambda + \delta x)^{p+q}} \left(\frac{\alpha + \beta x}{\lambda + \delta x} \right)^{n-q} B(n + p, q - n) \, dx
\end{aligned}
$$

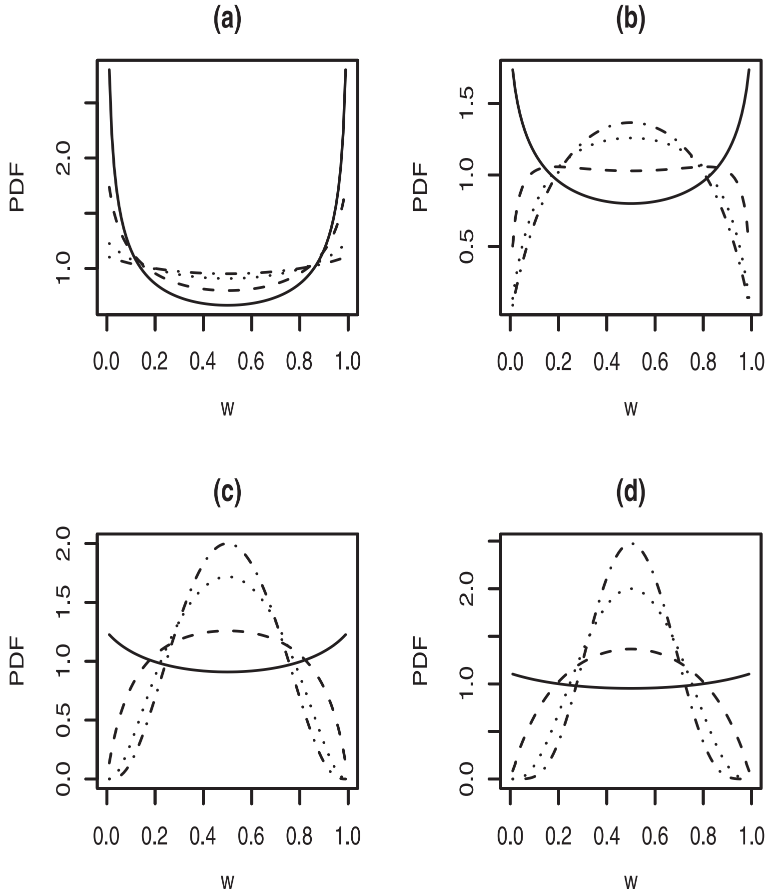

FIGURE 3. Plots of the pdf of (8)–(9) for $\alpha = 1$, $\beta = 1$, $\lambda = 1$, $\delta = 1$ and (a): $p = 1$; (b): $p = 2$; (c): $p = 5$; and, (d): $p = 10$. The four curves in each plot are: the solid curve ($q = 1$), the curve of lines ($q = 2$), the curve of dots ($q = 5$), and the curve of lines and dots ($q = 10$).

$$
\begin{aligned}
&= KB(n+p, q-n) \int_0^{\infty} \frac{x^{m+p-1}(\alpha + \beta x)^{n-q}}{(\lambda + \delta x)^{n+p}}\, dx \\
&= \frac{KB(n+p, q-n) B(m+p, q-m) \alpha^{m+n+p-q}}{\beta^{m+p} \lambda^{n+p}} \\
&\quad \times G\left(m+p, n+p; p+q; 1 - \frac{\alpha\delta}{\beta\lambda}\right),
\end{aligned}
$$

where we have applied Lemmas 1 and 3 (in the third and the fifth steps, respectively). ▲

The moments of $R = X + Y$ and $U = XY$ are now simple consequences of this lemma as illustrated in Theorems 6 and 7.

THEOREM 6. *If X and Y are jointly distributed according to* (1) *then*

$$E(R^n) = K\alpha^{n+p-q} \sum_{k=0}^{n} \binom{n}{k} \frac{B(k+p, q-k)B(n-k+p, q-n+k)}{\beta^{n-k+p}\lambda^{k+p}} \times G\left(n-k+p, k+p; p+q; 1-\frac{\alpha\delta}{\beta\lambda}\right)$$

for $n \geq 1$ *and* $q > n$.

Proof. the result follows by writing

$$E((X+Y)^n) = \sum_{k=0}^{n} \binom{n}{k} E(X^{n-k}Y^k)$$

and applying Lemma 7 to each expectation in the sum. ▲

THEOREM 7. *If X and Y are jointly distributed according to* (1) *then*

$$E(U^n) = \frac{KB(n+p, q-n)B(n+p, q-n)\alpha^{2n+p-q}}{\beta^{n+p}\lambda^{n+p}} G\left(n+p, n+p; p+q; 1-\frac{\alpha\delta}{\beta\lambda}\right)$$

for $n \geq 1$ *and* $q > n$.

Proof. follows by writing $E(U^n) = E(X^nY^n)$ and applying Lemma 7 with $m = n$. ▲

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