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SOME CHARACTERIZATIONS OF QUATERNIONIC KAEHLERIAN MANIFOLDS WITH CONSTANT Q-SECTIONAL CURVATURE

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An n-dimensional Riemannian manifold M is of constant curvature, as is well known, if and only if there exists an umbilical hypersurface with constant mean curvature passing through every point with every $(n-1)$-direction at the point (See [3]), and that M is conformally flat if and only if there exists an umbilical hypersurface passing through every point with every $(n-1)$-direction at the point (See [3]. Tachibana and Tashiro [4] proved that a $2n$-dimensional Kaehlerian manifold is of constant holomorphic sectional curvature k if and only if there exists a hypersurface with the second fundamental tensor K_{ba} of the form

$$K_{ba}=g_{ba}-ku_bu_a\,,$$

u being a unit vector field, passing through every point and being tangent to every $(2n-1)$-direction at the point. So, we may expect an analogous conditions for a $4m$-dimensional quaternionic Kaehlerian manifold $(4m\geqq 8)$ to be of constant Q-sectional curvature c. In this sense we shall prove our main theorem giving a characterization of a $4m$-dimensional quaternionic Kaehlerian manifold $(4m\geqq 8)$ with constant Q-sectional curvature, as follows:

THEOREM. *For a $4m$-dimensional quaternionic Kaehlerian manifold $\tilde{M}$ $(4m\geqq 8)$, the following three conditions are equivalent to each other*:

(A) *$\tilde{M}$ is of constant Q-sectional curvature c.*

(B) *The system of partial differential equations*

$$\nabla_j X_i=\frac{c}{4}g_{ji}+X_jX_i-F_j{}^tX_tF_i{}^sX_s-G_j{}^tX_tG_i{}^sX_s-H_j{}^tX_tH_i{}^sX_s$$

is completely integrable with unknown functions X_i, $\{F, G, H\}$ being the quaternionic Kaehlerian structure of $\tilde{M}$.

(C) *There exists a hypersurface having the second fundamental tensor K_{ba} of the form*

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$$K_{ba}=\frac{c}{4}g_{ba}-(u_b u_a+v_b v_a+w_b w_a)$$

and passing through every point with every $(4m-1)$*-direction at the point, where* u, v *and* w *are certain mutually orthogonal unit vectors appearing in* (2.1).

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1. Preliminaries

We shall recall some definitions and theorems given in [1] for later use. Let $\tilde{M}$ be a differentiable manifold of dimension n, and assume that there is a 3-dimensional vector bundle V consisting of tensors of type $(1,1)$ over $\tilde{M}$ satisfying the following conditions:

(1) In any coordinate neighborhood $\{\tilde{U}; y^h\}$, there is a local base $\{F, G, H\}$ of V such that*)

$$\begin{aligned}&F_h{}^i F_j{}^h=-\delta_j^i, \quad G_h{}^i G_j{}^h=-\delta_j^i, \quad H_h{}^i H_j{}^h=-\delta_j^i,\\ &G_h{}^i H_j{}^h=-H_h{}^i G_j{}^h=F_j{}^i, \quad H_h{}^i F_j{}^h=-F_h{}^i H_j{}^h=G_j{}^i, \quad F_h{}^i G_j{}^h=-G_h{}^i F_j{}^h=H_j{}^i.\end{aligned} \tag{1.1}$$

(2) There is a Riemannian metric g such that $g(\phi X, Y)+g(X, \phi Y)=0$ holds for any cross-section ϕ of V.

(3)

$$\begin{aligned}\nabla_j F_i{}^h&=r_j G_i{}^h-q_j H_i{}^h,\\ \nabla_j G_i{}^h&=-r_j F_i{}^h+p_j H_i{}^h,\\ \nabla_j H_i{}^h&=q_j F_i{}^h-p_j G_i{}^h,\end{aligned} \tag{1.2}$$

∇ being the Riemannian connection of $(\tilde{M}, g)$, where $F_i{}^h$, $G_i{}^h$ and $H_i{}^h$ are respectively the components of F, G and H in $\{\tilde{U}; y^h\}$, p_j, q_j and r_j certain local 1-forms defined in $\tilde{U}$, and X, Y arbitrary vector fields. Such a local base $\{F, G, H\}$ is called a *canonical local base* of the bundle V in $\tilde{U}$, and $(\tilde{M}, g, V)$ or $\tilde{M}$ is called a *quaternionic Kaehlerian manifold* and (g, V) a *quaternionic Kaehlerian structure*. Thus a quaternionic Kaehlerian manifold is necessarily of dimension $n=4m$ $(m\geqq 1)$ and orientable (See [1]). We put $F_{ih}=F_i{}^s g_{sh}$, $G_{ih}=G_i{}^s g_{sh}$ and $H_{ih}=H_i{}^s g_{sh}$, where g_{ji} are components of the Riemannian metric g. Denoting by $K_{kji}{}^h$ components of the curvature tensor of $(\tilde{M}, g)$, we put $K_{kjih}=K_{kji}{}^s g_{sh}$. The following formulas were proved in [1]:

$$\begin{aligned}K_{kjth}F_i{}^t+K_{kjit}F_h{}^t&=C_{kj}G_{ih}-B_{kj}H_{ih},\\ K_{kjth}G_i{}^t+K_{kjit}G_h{}^t&=-C_{kj}F_{ih}+A_{kj}H_{ih},\\ K_{kjth}H_i{}^t+K_{kjit}H_h{}^t&=B_{kj}F_{ih}-A_{kj}G_{ih},\end{aligned} \tag{1.3}$$

*) The indices h, i, j, k, s, t run over the range $\{1, 2, \cdots, 4m\}$ and the summation convention will be used with respect to this system of indices.

where A_{kj}, B_{kj} and C_{kj} are defined by

$$A_{jk}=\frac{k}{4m(m+2)}F_{jk}, \quad B_{kj}=\frac{k}{4m(m+2)}G_{jk} \text{ and } C_{kj}=\frac{k}{4m(m+2)}H_{jk}, \tag{1.4}$$

k being the scalar curvature of $(\tilde{M}, g)$, if $\dim \tilde{M}=4m\geqq 8$.

Let P be an arbitrary point in a quaternionic Kaehlerian manifold $(\tilde{M}, g)$ of dimension $4m$ and X a tangent vector of $\tilde{M}$ at P. Then the 4-dimensional subspace $Q(X)$ of the tangent space of $\tilde{M}$ at P defined by

$$Q(X)=\{Y=aX+bFX+cGX+dHX \mid a,\ b,\ c,\ d\in R\}$$

is called the *Q-section* determined by X.

If we denote by $\sigma(X, Y)$ the sectional curvature of $\tilde{M}$ with respect to the section spanned by X and Y at a point, then it is by definition given as

$$\sigma(X, Y)=-K_{kjih}X^kY^jX^iY^h/(\|X\|^2\|Y\|^2-g(X, Y)^2),$$

where $\|X\|$ is the length of X. Using (1.3) and (1.4) gives

$$\begin{aligned}
&\sigma(X,\ FX)=k/4m(m+2)-K(X, FX, GX, HX),\\
&\sigma(X,\ GX)=k/4m(m+2)-K(X, GX, HX, FX),\\
&\sigma(X,\ HX)=k/4m(m+2)-K(X, HX, FX, GX)
\end{aligned} \tag{1.5}$$

for a unit vector X if $\dim \tilde{M}=4m\geqq 8$ (See [1]). If the sectional curvature $\sigma(Y, Z)$ is a constant $\rho(X)$ for any $Y, Z\in Q(X)$, then $\rho(X)$ is called the *Q-sectional curvature* of $(\tilde{M}, g)$. S. Ishihara [1] proved the following Theorems A and B:

THEOREM A. *Let $Q(X)$ be a Q-section at a point P of a quaternionic Kaehlerian manifold $(\tilde{M}, g)$. Then the sectional curvature $\sigma(Y, Z)$ with respect to the section spanned by any $Y, Z\in Q(X)$ is a constant $\rho(X)$ if and only if $K(Y, Z)Y-\rho(X)Z\in Q^{\perp}(X)$ for any $Y, Z\in Q(X)$, and in such a case we have $\rho(X)=(k/4m(m+2))\|X\|^2$, where $Q^{\perp}(X)$ denotes the orthogonal complement of $Q(X)$ in the tangent space of $\tilde{M}$ at P, k the scalar curvature, and* $\dim \tilde{M}=4m$ $(\geqq 8)$.

THEOREM B. *A quaternionic Kaehlerian manifold of dimension $4m\geqq 8$ is of constant Q-sectional curvature $c=c(P)$ if and only if its curvature tensor has components of the form*

$$\begin{aligned}
K_{kjih}=\frac{c}{4}(&g_{kh}g_{ji}-g_{jh}g_{ki}+F_{kh}F_{ji}-F_{jh}F_{ki}-2F_{kj}F_{ih}+G_{kh}G_{ji}\\
&-G_{jh}G_{ki}-2G_{kj}G_{ih}+H_{kh}H_{ji}-H_{jh}H_{ki}-2H_{kj}H_{ih}),
\end{aligned}$$

where $c=c(P)$ is necessarily a constant.

2. Hypersurfaces in a quaternionic Kaehlerian manifold of constant Q-sectional curvature

Let M be a real hypersurface in a quaternionic Kaehlerian manifold $\tilde{M}$ of dimension $4m$. We assume that $\tilde{M}$ is covered by a system of coordinate neighborhoods $\{\tilde{U}; y^h\}$*). Then M is covered by a system of coordinate neighborhoods $\{U; x^a\}$, where $U=\tilde{U}\cap M$. When M is represented by $y^j=y^j(x^a)$ in terms of local coordinates (y^j) in $\tilde{U}(\subset\tilde{M})$ and (x^a) in $U(\subset M)$, we denote the vectors $\partial_a y^j$ $(\partial_a=\partial/\partial x^a)$ tangent to M by $B_a{}^j$ and the unit normal vector by N^j. Then the transforms $F_h{}^iB_a{}^h$, $G_h{}^iB_a{}^h$, $H_h{}^iB_a{}^h$ and $F_h{}^iN^h$, $G_h{}^iN^h$, $H_h{}^iN^h$ of $B_a{}^h$ and N^h by the canonical local base $\{F, G, H\}$ in $\{\tilde{U}; y^j\}$ can be respectively expressed as linear combinations of $B_a{}^h$ and N^h, that is,

$$
\begin{aligned}
&F_h{}^iB_a{}^h=\phi_a{}^bB_b{}^i+u_aN^i, \qquad F_h{}^iN^h=-u^aB_a{}^i,\\
(2.1)\qquad &G_h{}^iB_a{}^h=\psi_a{}^bB_b{}^i+v_aN^i, \qquad G_h{}^iN^h=-v^aB_a{}^i,\\
&H_h{}^iB_a{}^h=\theta_a{}^bB_b{}^i+w_aN^i, \qquad H_h{}^iN^h=-w^aB_a{}^i,
\end{aligned}
$$

where $\phi_a{}^b$, $\psi_a{}^b$ and $\theta_a{}^b$ are tensor fields of type (1, 1), u_a, v_a and w_a 1-forms, $u^a=u_bg^{ab}$, $v^a=v_bg^{ab}$ and $w^a=w_bg^{ab}$, $g_{ba}=g_{ji}B_b{}^jB_a{}^i$ the Riemannian metric in M induced from that of $\tilde{M}$ and $(g^{ba})=(g_{ba})^{-1}$. Then, using (1.1) and (2.1), we can easily verify

$$
\begin{aligned}
&\phi_a{}^b\phi_c{}^a=-\delta_c^b+u_cu^b, \qquad \phi_b{}^au^b=0, \qquad u_bu^b=1,\\
(2.2)\qquad &\psi_a{}^b\psi_c{}^a=-\delta_c^b+v_cv^b, \qquad \psi_b{}^av^b=0, \qquad v_bv^b=1,\\
&\theta_a{}^b\theta_c{}^a=-\delta_c^b+w_cw^b, \qquad \theta_b{}^aw^b=0, \qquad w_bw^b=1,
\end{aligned}
$$

and

$$
\begin{aligned}
&\psi_a{}^c\phi_b{}^a=-\theta_b{}^c+u_bv^c, \qquad \phi_b{}^av_a=-w_b,\\
&\phi_a{}^c\psi_b{}^a=\theta_b{}^c+v_bu^c, \qquad \psi_b{}^au_a=w_b,\\
(2.3)\qquad &\theta_a{}^c\psi_b{}^a=-\phi_b{}^c+v_bw^c, \qquad \psi_b{}^aw_a=-u_b,\\
&\psi_a{}^c\theta_b{}^a=\phi_b{}^c+w_bv^c, \qquad \theta_b{}^av_a=u_b,\\
&\phi_a{}^c\theta_b{}^a=-\psi_b{}^c+w_bu^c, \qquad \theta_b{}^au_a=-v_b,\\
&\theta_a{}^c\phi_b{}^a=\psi_b{}^c+u_bw^c, \qquad \phi_b{}^aw_a=v_b.
\end{aligned}
$$

We denote by ∇_a the operator of covariant differentiation with respect to the Christoffel symbols $\left\{ {a \atop cd} \right\}$ formed with g_{ba}. Then the equations of Gauss and

*) Here and in the sequel the indices h, i, j, k, s, t run over the range $\{1, 2, \cdots, 4m\}$ and the indices a, b, c, d, e over the range $\{1, 2, \cdots, 4m-1\}$. The summation convention will be used with respect to these two systems of indices.

Weingarten are respectively

$$\nabla_c B_b{}^i = K_{cb} N^i, \qquad \nabla_c N^i = -K_c{}^b B_b{}^i, \tag{2.4}$$

where

$$\nabla_c B_b{}^i = \partial_c B_b{}^i - \left\{ {a \atop cb} \right\} B_a{}^i + \left\{ {i \atop kj} \right\} B_c{}^k B_b{}^j, \qquad \nabla_c N^i = \partial_c N^i + \left\{ {i \atop kj} \right\} B_c{}^k N^j,$$

$\left\{ {i \atop kj} \right\}$ being the Christoffel symbols formed with g_{ji}, and where K_{cb} is the second fundamental tensor of M with respect to the normal vector N^i and $K_c{}^a = K_{cb} g^{ba}$.

Differentiating (2.1) covariantly along M and taking account of (2.4) we get

$$\begin{aligned} &\nabla_c \phi_b{}^a = r_c \psi_b{}^a - q_c \theta_b{}^a - K_{cb} u^a + K_c{}^a u_b, \\ &\nabla_c \psi_b{}^a = -r_c \phi_b{}^a + p_c \theta_b{}^a - K_{cb} v^a + K_c{}^a v_b, \\ &\nabla_c \theta_b{}^a = q_c \phi_b{}^a - p_c \psi_b{}^a - K_{cb} w^a + K_c{}^a w_b; \\ &\nabla_c u_b = r_c v_b - q_c w_b - K_{ca} \phi_b{}^a, \quad \nabla_c v_b = -r_c u_b + p_c w_b - K_{ca} \psi_b{}^a, \\ &\nabla_c w_b = q_c u_b - p_c v_b - K_{ca} \theta_b{}^a, \end{aligned} \tag{2.5}$$

where $p_c = p_j B_c{}^j$, $q_c = q_j B_c{}^j$ and $r_c = r_j B_c{}^j$.

When the ambient manifold $\tilde{M}$ is of constant Q-sectional curvature, substituting (1.6) in the equation of Codazzi

$$\nabla_c K_{ba} - \nabla_b K_{ca} = K_{kjih} B_c{}^k B_b{}^j B_a{}^i N^h \tag{2.6}$$

gives

$$\begin{aligned} \nabla_c K_{ba} - \nabla_b K_{ca} = \frac{c}{4} (&u_c \phi_{ba} - u_b \phi_{ca} - 2\phi_{cb} u_a + v_c \psi_{ca} - v_b \psi_{ca} \\ &- 2\psi_{cb} v_a + w_c \theta_{ba} - w_b \theta_{ca} - 2\theta_{cb} w_a), \end{aligned} \tag{2.7}$$

where $\phi_{ba} = \phi_b{}^c g_{ca}$, $\psi_{ba} = \psi_b{}^c g_{ca}$ and $\theta_{ba} = \theta_b{}^c g_{ca}$. Thus we have

PROPOSITION. *There is no umbilical real hypersurface in a quaternionic Kaehlerian manifold* $\tilde{M}$ ($\dim \tilde{M} \geq 8$) *of constant Q-sectional curvature c, if* $c \neq 0$.

Proof. If there is an umbilical real hypersurface in $\tilde{M}$, we may put $K_{cb} = \mu g_{cb}$. Substituting this into (2.7) and transvecting with g^{ba}, we can easily see that $\nabla_c \mu = 0$ because of (2.2). Hence $c = 0$.

Remark. If $\{u, v, w\}$ appearing in (2.2) is a Sasakian 3-structure (See [2]), that is, if u, v and w are three Sasakian structures which are mutually orthogonal and satisfy the relations $[u, v] = 2w$, $[v, w] = 2u$ and $[w, u] = 2v$, then from (2.5) we have

$$u_b \delta_c^a - u^a g_{bc} = r_c \psi_b{}^a - q_c \theta_b{}^a - K_{cb} u^a + K_c{}^a u_b, \tag{2.8}$$

(2.9) $$\phi_{cb}=r_c v_b-q_c w_b-K_{ca}\phi_b{}^a,$$

(2.10) $$v_b\delta_c^a-v^a g_{bc}=-r_c\phi_b{}^a+p_c\theta_b{}^a-K_{cb}v^a+K_c{}^a v_b,$$

(2.11) $$\psi_{cb}=-r_c u_b+p_c w_b-K_{ca}\psi_b{}^a.$$

Transvecting (2.9) with v^b and w^b respectively gives

$$-w_c=r_c-K_{ca}w^a, \qquad v_c=-q_c+K_{ca}w^a$$

because of (2.3). If we transvect (2.8) with $\psi_a{}^b$ and $\theta_a{}^b$ respectively and take account of (2.2), (2.3) and the equations above, we get $(4m-3)r_c=0$ and $(4m-3)q_c=0$, and consequently $q_c=r_c=0$. Similarly if we transvect (2.10) with $\theta_a{}^b$ and transvect w^b to (2.11) respectively, we can also see $p_c=0$ because of (2.2), (2.3) and $q_c=r_c=0$. Therefore (2.8) and (2.10) become

(2.12) $$u_b\delta_c^a-u^a g_{bc}=-K_{cb}u^a+K_c{}^a u_b,$$

(2.13) $$v_b\delta_c^a-v^a g_{bc}=-K_{cb}v^a+K_c{}^a v_b.$$

Contracting (2.12) with respect to a and c gives $K_{cb}u^b=\alpha u_c$ for certain function α in M, and consequently transvecting (2.12) with u_a gives

$$K_{cb}=g_{cb}+(\alpha-1)u_c u_b.$$

By a quite same advice we also have from (2.13)

$$K_{cb}=g_{cb}+(\beta-1)v_c v_b,$$

where β is also a function in M. Since u and v are mutually orthogonal unit vectors, we find $K_{cb}=g_{cb}$. Hence by means of the previous proposition, when $c\neq 0$, $\{u, v, w\}$ cannot be a Sasakian 3-structure.

3. Proof of the main theorem

We first consider the system of partial differential equations

(3.1) $$\nabla_j X_i=\frac{c}{4}g_{ji}+X_j X_i-F_j{}^t X_t F_i{}^s X_s-G_j{}^t X_t G_i{}^s X_s-H_j{}^t X_t H_i{}^s X_s$$

for an arbitrary vector X. Then by using (1.2) and (3.1) itself we can easily verify

$$\begin{aligned}\nabla_k\nabla_j X_i-\nabla_j\nabla_k X_i=-\frac{c}{4}(&\delta_k^t g_{ji}-\delta_j^t g_{ki}+F_k{}^t F_{ji}-F_j{}^t F_{ki}-2F_{kj}F_i{}^t\\&+G_k{}^t G_{ji}-G_j{}^t G_{ki}-2G_{kj}G_i{}^t+H_k{}^t H_{ji}-H_j{}^t H_{ki}\\&-2H_{kj}H_i{}^t)X_t.\end{aligned}$$

Thus the necessary and sufficient condition for the system (3.1) to be completely integrable is that the curvature tensor K_{kjih} of $\tilde{M}$ has the form (1.6) because of the integrability condition

$$\nabla_k\nabla_j X_i - \nabla_j\nabla_k X_i = -K_{kji}{}^t X_t$$

of (3.1). Hence by using Theorem B we find the equivalence (A)↔(B).

Remark. Any solution X_i of differential equation (3.1) is not globally defined evenwhen the manifold is simply connected if the manifold is complete. We shall now show this fact in the followings. From (3.1) we easily obtain $X^j\nabla_j X_i = (\|X\|^2 + c/4)X_i$, which means that any integral curve γ of the vector field X^h is a geodesic. Denote by γ^h the components of the tangent vector of γ (with arclength as its parameter). Then transvecting (3.1) with $\dot{\gamma}^j\dot{\gamma}^i$ implies that

$$d\|X\|/ds = \|X\|^2 + c/4$$

holds along γ. Integrating the differential equation above gives along γ

$$\|X\| = \begin{cases} -(\sqrt{c}/2)\cot\{(\sqrt{c}\,s/2) + (\pi/2) + \alpha\}\,, & (c>0) \\ -(\sqrt{|c|}/2)\coth\{(\sqrt{|c|}\,s/2) + \alpha\}\,, & (c<0) \\ -1/(s+\alpha)\,, & (c=0) \end{cases}$$

where α is an arbitrary constant. This equation shows that X^h is not globally defined if the manifold is complete.

Next we assume that there exist hypersurfaces satisfying the condition (C) stated in the theorem. Then, using the equation (2.6) of Codazzi, we have

$$K_{kjih}B_c{}^k B_b{}^j B_a{}^i N^h = -\frac{c}{4}(2\phi_{cb}u_a + u_b\phi_{ca} - u_c\phi_{ba} + 2\psi_{cb}v_a + v_b\psi_{ca} - v_c\psi_{ba} + 2\theta_{cb}w_a + w_b\theta_{ca} - w_c\theta_{ba})\,. \tag{3.2}$$

Transvecting (3.2) with $\phi^{cb}u^a$ and taking account of (2.1), (2.2) and (2.3), we find

$$2(m-1)c = \{F^{kj} - N^k(F_t{}^j N^t) + (F_t{}^k N^t)N^j\}(F_s{}^i N^s)N^h K_{kjih}\,.$$

Since $F^{kj}K_{kjih} = (k/2(m+2))F_{hi}$ which is a consequence of (1.3) and (1.4) (See [1]), the above equation implies

$$2(m-1)c = k/2(m+2) - 2\sigma(N, FN)\,. \tag{3.3}$$

Transvecting (3.2) with $\psi^{cb}v^a$ and $\theta^{cb}w^a$ respectively, we can similarly find

$$\begin{aligned} 2(m-1)c &= k/2(m+2) - 2\sigma(N, GN)\,, \\ 2(m-1)c &= k/2(m+2) - 2\sigma(N, HN)\,. \end{aligned} \tag{3.4}$$

On putting $c = k/4m(m+2)$, (1.5), (3.3) and (3.4) give

(3.5) $\qquad K(N, FN, GN, HN)=K(N, GN, HN, FN)=K(N, HN, FN, GN)=0.$

On the other hand, from the first equation of (1.3), we have

$$K_{kjth}N^kG_s{}^jN^sF_i{}^tN^iG_r{}^hN^r+K_{kjit}N^kG_s{}^jN^sN^iH_r{}^tN^r=0$$

because of (1.1) and (1.4). Thus we obtain

(3.6) $$K(N, GN, N, HN)=0$$

since $K_{kjth}N^kG_s{}^jN^sF_i{}^tN^iG_r{}^hN^r=(K_{kjth}B_c{}^kB_b{}^jB_a{}^tN^h)u^cv^bv^a=0$.

Similarly, using the second and third equations of (1.3), we also obtain

(3.7) $$K(N, FN, N, GN)=0, \qquad K(N, HN, N, FN)=0.$$

Combining (3.5), (3.6) and (3.7), we get $K(Y, Z)Y-(k/4m(m+2))Z\in Q^{\perp}(N)$ for any $Y, Z\in Q(N)$, and consequently Theorem A implies the implication (C)→(A).

Finally let $\tilde{M}$ be a $4m$-dimensional $(4m\geqq 8)$ quaternionic Kaehlerian manifold of constant Q-sectional curvature c. Then the system (3.1) of partial differential equations with unknown vector field X is completely integrable. Let P be an arbitrary point and consider a solution of (3.1) with arbitrary initial value $(X_i)_P$ at P satisfying $(g_{ji}X^jX^i)_P=1$. Putting $\omega=X_idy^i$, we find $d\omega=0$ because of $\nabla_jX_i=\nabla_iX_j$. Thus the pfaffian equation $\omega=0$ is completely integrable. Let M be the integral manifold of $\omega=0$ passing through P. Since X^i is normal to M, we can put $X^i=\mu N^i$, where μ is a function in M. Substituting (3.1) and $X_j=\mu N_j$ into the equation of Weingarten $B_b{}^jB_a{}^i\nabla_jX_i=-\mu K_{ba}$ for M, we have

(3.8) $$K_{ba}=\alpha g_{ba}+\mu(u_bu_a+v_bv_a+w_bw_a),$$

where we have put $\alpha=-c/4\mu$. Thus, differentiating covariantly the both sides of (3.8) along M, we can find

$$\begin{aligned}\nabla_cK_{ba}-\nabla_bK_{ca}=&(\nabla_c\alpha)g_{ba}-(\nabla_b\alpha)g_{ca}+(\nabla_c\mu)(u_bu_a+v_bv_a+w_bw_a)\\&-(\nabla_b\mu)(u_cu_a+v_cv_a+w_cw_a)-\frac{c}{4}(2\phi_{cb}u_a+u_b\phi_{ca}-u_c\phi_{ba}\\&+2\psi_{cb}v_a+v_b\psi_{ca}-v_c\psi_{ba}+2\theta_{cb}w_a+w_b\theta_{ca}-w_c\theta_{ba}),\end{aligned}$$

from which and (2.7),

$$(\nabla_c\alpha)g_{ba}-(\nabla_b\alpha)g_{ca}+(\nabla_c\mu)(u_bu_a+v_bv_a+w_bw_a)-(\nabla_b\mu)(u_cu_a+v_cv_a+w_cw_a)=0.$$

Transvecting the above equation with g^{ba} and $u^bu^a+v^bv^a+w^bw^a$, we find respectively

$$(4m-2)\nabla_c\alpha+3\nabla_c\mu-(u^a\nabla_a\mu)u_c-(v^a\nabla_a\mu)v_c-(w^a\nabla_a\mu)w_c=0,$$

$$3\nabla_c\alpha-(u^a\nabla_a\alpha)u_c-(v^a\nabla_a\alpha)v_c-(w^a\nabla_a\alpha)w_c$$

$$+3\nabla_c\mu-(u^a\nabla_a\mu)u_c-(v^a\nabla_a\mu)v_c-(w^a\nabla_a\mu)w_c=0\,.$$

Combining the last two equations, we get

$$(4m-5)\nabla_c\alpha+(u^a\nabla_a\alpha)u_c+(v^a\nabla_a\alpha)v_c+(w^a\nabla_a\alpha)w_c=0\,,$$

which implies that

$$u^c\nabla_c\alpha=v^c\nabla_c\alpha=w^c\nabla_c\alpha=0$$

and consequently that $\nabla_c\alpha=0$. Hence, taking account of $\alpha\mu=-c/4$, we have $\mu=$ const. By means of the initial condition we find $\mu^2=1$. Thus we may suppose that $\mu=-1$, which implies

$$K_{ba}=\frac{c}{4}g_{ba}-(u_bu_a+v_bv_a+w_bw_a)$$

because of (3.8) and $\alpha\mu=-c/4$. As P and the initial value $(X_\iota)_P$ of direction at P are arbitrary, we complete the proof for the implication (A)→(C).

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