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SPREAD RELATION AND VALUE DISTRIBUTION IN AN ANGULAR DOMAIN OF HOLOMORPHIC CURVES

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§ 1. Recently Baernstein [1] proved Edrei's spread conjecture [5] for meromorphic functions in $|z|<\infty$. Mutō [6] sharpened the result of Bieberbach [3] by making use of the spread relation proved in [1].

The purpose of this paper is to extend the spread relation and the results of Mutō for meromorphic functions in $|z|<\infty$ to ones for holomorphic curves in the projective space.

The standard symbols of the theory of holomorphic curves in the projective space

$$T(r),\ N(r,\ a),\ m(r,\ a),\ \delta(a),\ \cdots$$

are used throughout the paper (cf. Wu [10]. We take $\log r$ as a parameter of harmonic exhaustion.).

§ 2. Let x: $\boldsymbol{C}\rightarrow P_n\boldsymbol{C}$ be a transcendental holomorphic curve in the projective space and $T(r)$ its order function. The *order* λ and the *lower order* μ of a holomorphic curve x: $\boldsymbol{C}\rightarrow P_n\boldsymbol{C}$ are defined by

$$\lambda=\limsup_{r\rightarrow\infty}\frac{\log T(r)}{\log r} \qquad \text{and} \qquad \mu=\liminf_{r\rightarrow\infty}\frac{\log T(r)}{\log r}\ ,$$

respectively.

A positive, increasing, unbounded sequence $\{r_m\}$ is called *a sequence of Pólya peaks of order* ρ *of* x: $\boldsymbol{C}\rightarrow P_n\boldsymbol{C}$ (or $T(r)$) if it is possible to find positive sequences $\{r_m'\}$, $\{r_m''\}$ and $\{\varepsilon_m\}$ such that, as $m\rightarrow\infty$,

$$r_m'\rightarrow\infty\ ,\quad \frac{r_m}{r_m'}\rightarrow\infty\ ,\quad \frac{r_m''}{r_m}\rightarrow\infty\ ,\quad \varepsilon_m\rightarrow 0$$

and such that

$$\frac{T(t)}{T(r_m)}\leqq\left(\frac{t}{r_m}\right)^{\rho}(1+\varepsilon_m) \qquad (r_m'<t<r_m'')\ .$$

In this from, Pólya peaks were introduced by Edrei [4]. He proved that if the lower order μ is finite, then a sequence of Pólya peaks exists for every finite ρ satisfying $\mu\leqq\rho\leqq\lambda$.

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Let $x: \boldsymbol{C} \rightarrow P_n\boldsymbol{C}$ be a holomorphic curve of finite lower order μ. Fix a sequence $\{r_m\}$ of Pólya peaks of order μ of x. Let $\Lambda(r)$ be a positive function with

$$\text{(1)} \qquad \Lambda(r)=o(T(r)) \qquad (r\rightarrow\infty).$$

Define the set of arguments $E_\Lambda(r, a) \subset (-\pi, \pi]$ by

$$E_\Lambda(r, a)=\{\theta ; \ |<x(re^{i\theta}), a>|<e^{-\Lambda(r)}\}$$

for $a \in P_n\boldsymbol{C}$ satisfying $<x(z), a> \not\equiv 0$ and let

$$\sigma_\Lambda(a)=\liminf_{m\rightarrow\infty} \operatorname{meas} E_\Lambda(r_m, a),$$

$$\sigma(a)=\inf_\Lambda \sigma_\Lambda(a),$$

where the infimum is taken over all functions satisfying (1).

Then we shall prove

THEOREM 1 (Spread relation). *Let $x: \boldsymbol{C} \rightarrow P_n\boldsymbol{C}$ be a holomorphic curve of positive, finite lower order μ. Then*

$$\sigma(a) \geqq \min\left\{2\pi, \frac{4}{\mu}\sin^{-1}\left(\frac{\delta(a)}{2}\right)^{1/2}\right\}$$

for every $a \in P_n\boldsymbol{C}$ satisfying $<x(z), a> \not\equiv 0$.

Since the spread relation for meromorphic functions in $|z|<\infty$ is best possible, it is clear that our Theorem 1 is best possible.

Our extensions of the results of Mutō are stated as follows:

THEOREM 2. *Let $x: \boldsymbol{C} \rightarrow P_n\boldsymbol{C}$ be a holomorphic curve of finite lower order $\mu \geqq 1$ with $\delta(a_0)>0$ for some $a_0 \in P_n\boldsymbol{C}$ and Δ a sector defined by*

$$\text{(2)} \qquad \Delta=\left\{z ; \ |\arg z-\omega|<\pi-\frac{2}{\mu}\sin^{-1}\left(\frac{\delta(a_0)}{2}\right)^{1/2}+\eta\right\},$$

where η is an arbitrary positive number. Suppose that the solutions in Δ of $<x(z), a_0>=0$ are finite in number. Then the equation $<x(z), a>=0$ has an infinite number of solutions in the sector Δ except at most $2n$ $a \in P_n\boldsymbol{C}$ in general position.

THEOREM 3. *Let $x: \boldsymbol{C} \rightarrow P_n\boldsymbol{C}$ be a holomorphic curve of lower order μ $(1/2<\mu<1)$ with $\delta(a_0)=1$ for some $a_0 \in P_n\boldsymbol{C}$ and Δ a sector defined by*

$$\text{(3)} \qquad \Delta=\left\{z ; \ |\arg z-\omega|<\frac{\pi}{2\mu}+\eta\right\},$$

where η is an arbitrary positive number. Suppose that the solutions in Δ of

$<x(z), a_0>=0$ are finite in number. Then the equation $<x(z), a>=0$ has an infinite number of solutions in the sector $\varDelta$ except at most $2n$ $a\in P_nC$ in general position.

§2. *Proof of Theorem* 1. We shall prove our Theorem 1 by making use of an ingenious method of Baernstein [1, 2].

Let $x: C\rightarrow P_nC$ be a holomorphic curve of positive, finite lower order μ and a a point in P_nC satisfying $<x(z), a>\not\equiv 0$. Then we have a reduced representation $\tilde{x}(z)=(x_0(z), x_1(z), \cdots, x_n(z)): C\rightarrow C^{n+1}-\{0\}$ of x such that

$$<x(z), a>=\frac{x_0(z)}{|\tilde{x}(z)|}, \qquad x_0(z)\not\equiv 0.$$

Then we have

$$N(r, a)=N(r, 0, x_0). \tag{2.1}$$

Put

$$|\tilde{x}(z)|_s=\max_{0\leqq j\leqq n}|x_j(z)|,$$

$$T_s(r)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|\tilde{x}(re^{i\theta})|_s d\theta-\log|\tilde{x}(0)|_s,$$

$$E'_\varLambda(r, a)=\{\theta;\ \log|\tilde{x}(re^{i\theta})|_s-\log|x_0(re^{i\theta})|>\varLambda(r)\}.$$

Then, since $|\tilde{x}(z)|_s\leqq|\tilde{x}(z)|\leqq(n+1)^{1/2}|\tilde{x}(z)|_s$, we have

$$E'_\varLambda(r, a)\subset E_\varLambda(r, a) \tag{2.2}$$

and using a result in [10, p. 105],

$$T(r)-T_s(r)=o(1) \qquad (r\rightarrow\infty). \tag{2.3}$$

Hence $T(r)$ and $T_s(r)$ have the same Pólya peaks. Further we may assume that

$$|\tilde{x}(0)|_s=1 \quad \text{and} \quad h(0)=1, \tag{2.4}$$

where $h(z)=cz^{-k}x_0(z)$ with a suitable non-zero constant c and a non-negative integer k. Put

$$\varLambda_1(r)=\varLambda(r)+k\log r-\log|c|,$$

$$E(r)\equiv E_{\varLambda_1}(r, a, h)=\{\theta;\ \log|\tilde{x}(re^{i\theta})|_s-\log|h(re^{i\theta})|>\varLambda_1(r)\}.$$

Then it follows from (1), (2.1) and (2.3) that

$$\varLambda_1(r)=o(T(r))=o(T_s(r)) \qquad (r\rightarrow\infty), \tag{2.5}$$

$$E(r)=E'_\varLambda(r, a), \tag{2.6}$$

$$N(r, 0, h)=N(r, a)-k\log r,$$

and so

$$\delta(a)=1-\limsup_{r\to\infty}\frac{N(r,a)}{T(r)}=1-\limsup_{r\to\infty}\frac{N(r,0,h)}{T_s(r)}. \tag{2.7}$$

Therefore in order to prove our Theorem 1, from (2.2), (2.5) and (2.6), it is sufficient to prove that

$$\liminf_{m\to\infty}\text{ meas } E(r_m)\geqq \min\left\{2\pi,\ \frac{4}{\mu}\sin^{-1}\left(\frac{\delta(a)}{2}\right)^{1/2}\right\}. \tag{2.8}$$

Now define

$$T^*(z)=\sup_E\frac{1}{2\pi}\int_E\{\log|\tilde{x}(re^{i\omega})|_s-\log|h(re^{i\omega})|\}d\omega$$
$$+N(r,0,h),\quad (z=re^{i\theta},\quad 0<r<\infty,\quad 0\leqq\theta\leqq\pi),$$

where the supremum is taken over all measurable sets $E\subset(-\pi,\pi]$ whose measure equals 2θ. Then $T^*(z)$ is defined on $\{z;\ \mathrm{Im}\, z\geqq 0\}$ and for $0<r<\infty$,

$$T^*(re^{i\pi})=T^*(-r)=T_s(r), \tag{2.9}$$

$$T^*(r)=N(r,0,h)\equiv N(r). \tag{2.10}$$

Further we have for $r\geqq r_0>0$

$$T^*(re^{i\theta})\leqq T_s(r), \tag{2.11}$$

because $|\tilde{x}(z)|_s\geqq|x_0(z)|$ and if $k=0$ then we may assume that $c=1$. Since $\log|\tilde{x}(z)|_s$ and $\log|h(z)|$ are subharmonic, it follows from Theorem A′ in Baernstein [2] and (2.4) that $T^*(z)$ is subharmonic in $\{z;\ \mathrm{Im}\, z>0\}$ and is continuous on $\{z;\ \mathrm{Im}\, z\geqq 0\}$.

If $\delta(a)=0$ there is nothing to prove, so from now on we assume that $\delta(a)>0$. Put

$$\gamma=\frac{1}{2\pi}\min\left\{2\pi,\ \frac{4}{\mu}\sin^{-1}\left(\frac{\delta(a)}{2}\right)^{1/2}\right\}$$

Then we have

$$0<\gamma\leqq 1,\quad 0<\gamma\mu\leqq 1/2\quad\text{and}\quad 1-\delta(a)\leqq\cos\pi\gamma\mu. \tag{2.12}$$

Define

$$v(z)=T^*(z^\gamma)\qquad (z=re^{i\theta},\ 0<r<\infty,\ 0\leqq\theta\leqq\pi).$$

Then $v(z)$ is subharmonic in $\{z;\ \mathrm{Im}\, z>0\}$. Therefore from the reasoning of Baernstein [1, pp. 430-433] and taking (2.7) and (2.9)-(2.12) into account, we obtain the following result:

$$v(s_m e^{i\theta})\leqq T_s(r_m)(\cos(\pi-\theta)\gamma\mu+\alpha_m)\qquad (m=1,2,\cdots;0<\theta<\pi), \tag{2.13}$$

where $s_m=r_m^{1/\gamma}$ and $\{\alpha_m\}$ is a sequence tending to zero. Let

$$\sigma_m = \text{meas } E(r_m) .$$

Then (2.8) is equivalent to the inequality

$$\liminf_{m\to\infty} \sigma_m \geqq 2\pi\gamma . \tag{2.14}$$

We have

$$\begin{aligned} T_s(r_m) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \log |\tilde{x}(r_m e^{i\omega})|_s d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \{\log |\tilde{x}(r_m e^{i\omega})|_s - \log |h(r_m e^{i\omega})|\} d\omega + N(r_m) \\ &\leqq \frac{1}{2\pi}\int_{E(r_m)} \{\log |\tilde{x}(r_m e^{i\omega})|_s - \log |h(r_m e^{i\omega})|\} d\omega + \Lambda_1(r_m) + N(r_m) \\ &\leqq T^*(r_m e^{i\sigma_m/2}) + \Lambda_1(r_m) . \end{aligned}$$

Dividing by $T_s(r_m)$ and remembering (2.11), we find that

$$\lim_{m\to\infty} \frac{T^*(r_m e^{i\sigma_m/2})}{T_s(r_m)} = 1 . \tag{2.15}$$

Let

$$M = \{m ; \ \sigma_m < 2\pi\gamma\} .$$

If M is a finite set, then (2.14) holds and we have finished, so we assume that M is infinite.

The point

$$(r_m e^{i\sigma_m/2})^{1/\gamma} = s_m e^{i\sigma_m/2\gamma}$$

belongs to the domain of $v(z)$, i. e. the upper half-plane, if and only if $m \in M$, in which case we have

$$T^*(r_m e^{i\sigma_m/2}) = v(s_m e^{i\sigma_m/2\gamma}) \qquad (m \in M) .$$

Using this in (2.15), comparing with (2.13), and remembering (2.12), we deduce that

$$\lim_{\substack{m\to\infty \\ m\in M}} \sigma_m/2\gamma = \pi ,$$

which shows that (2.14) holds in this case also.

Thus the proof of Theorem 1 is complete.

§ 3. In order to prove our Theorem 2 and Theorem 3 we need some preliminary results.

Let U be the unit disc, i. e. $U = \{w ; \ |w| < 1\}$. U admits a finite harmonic exhaustion. Hence, from Corollary 2 in Toda [8] (cf. [7, Theorem B]) we deduce

LEMMA 1. *Let* $y: U\to P_nC$ *be a holomorphic curve in the projective space satisfying*

$$\limsup_{t\to 1} T(t)\Big/\log\frac{1}{1-t}=\infty .$$

If $a_j\in P_nC$, $j=1, 2, \cdots, 2n+1$, *are in general position and* $<y, a_j>\not\equiv 0$ *for all* j, *then*

$$\sum_{j=1}^{2n+1}\delta(a_j)\leqq 2n .$$

Put $V=\{z;\ |z|<R\leqq\infty\}$. Then we prove

LEMMA 2. *Let* $x: V\to P_nC$ *be a holomorphic curve in the projective space. Suppose that there is* $a\in P_nC$ *such that* $<x(z), a>$ *has no zero in* V. *Then for* $0<r<t<R$ *and* $|z|=r$,

$$T(t)\geqq\frac{t-r}{t+r}\Big\{\log\frac{1}{|<x(z), a>|}-\frac{1}{2}\log(n+1)\Big\}-A ,$$

where A *is a constant.*

Proof. Let $x: V\to P_nC$ be a holomorphic curve with $\langle x(z), a\rangle\neq 0$ in V. Then there is a reduced representation $\tilde{x}(z)=(x_0(z), x_1(z), \cdots, x_n(z)): V\to C^{n+1}-\{0\}$ such that $x_0(z)\equiv 1$ and $\langle x(z), a\rangle=1/|\tilde{x}(z)|$. Hence we have

$$\log\frac{1}{|\langle x(z), a\rangle|}=\log|\tilde{x}(z)|\leqq\max_{1\leqq j\leqq n}\overset{+}{\log}|x_j(z)|+\frac{1}{2}\log(n+1) . \tag{3.1}$$

Since $x_j(z)$ is holomorphic in V, it is well known that for $0<r<t<R$ and $|z|=r$,

$$\log|x_j(z)|\leqq\frac{t+r}{t-r}\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}|x_j(te^{i\theta})|d\theta$$

$$\leqq\frac{t+r}{t-r}\frac{1}{2\pi}\int_0^{2\pi}\log|\tilde{x}(te^{i\theta})|d\theta$$

and so

$$\max_{1\leqq j\leqq n}\overset{+}{\log}|x_j(z)|\leqq\frac{t+r}{t-r}\frac{1}{2\pi}\int_0^{2\pi}\log|\tilde{x}(te^{i\theta})|d\theta .$$

Hence, combining this with (3.1) and a result in [10, p. 105], we obtain with a constant A

$$\log\frac{1}{|\langle x(z), a\rangle|}-\frac{1}{2}\log(n+1)\leqq\frac{t+r}{t-r}(T(t)+A)$$

and consequently, for $|z|=r<t<R$

$$T(t)\geqq\frac{t-r}{t+r}\Big\{\log\frac{1}{|\langle x(z), a\rangle|}-\frac{1}{2}\log(n+1)\Big\}-A .$$

This proves our Lemma 2.

Next we consider the region

$$\Omega=\left\{z;\ |\arg z-\omega|<\frac{\pi}{2\gamma},\ |z|>K\right\} \tag{3.2}$$

and the conformal transformation

$$w=\frac{u^{\gamma}-u^{-\gamma}-1}{u^{\gamma}-u^{-\gamma}+1}=\phi^{-1}(u)\,. \tag{3.3}$$

Then the function

$$z=Ke^{i\omega}u=Ke^{i\omega}\phi(w) \tag{3.4}$$

maps the unit disc U onto the region Ω. (3.3) and (3.4) imply

$$|w|^{2}=\frac{1-B}{1+B}\,,$$

where

$$B=\frac{2(s^{\gamma}-s^{-\gamma})\cos(\gamma(\theta-\omega))}{s^{2\gamma}+s^{-2\gamma}+3-4\cos^{2}(\gamma(\theta-\omega))}\qquad (z=re^{i\theta},\ s=|u|=r/K)\,. \tag{3.5}$$

Hence

$$(1-|w|)/2<B<2(1-|w|)\,,$$

so that, for sufficiently large values of r, (3.5) yields

$$\frac{K^{\gamma}}{2r^{\gamma}}\cos(\gamma(\theta-\omega))<1-|w|<8\frac{K^{\gamma}}{r^{\gamma}}\,. \tag{3.6}$$

Now we have the following lemma which corresponds to Lemma in Mutō [6]:

LEMMA 3. *Let $x: C\to P_nC$ be a transcendental holomorphic curve of finite lower order μ. Suppose that for $a_j\in P_nC$, $j=0,1,\cdots,2n$, in general position, $\langle x(z),a_j\rangle$ has no zero in Ω defined by* (3.2) *and that there exists an unbounded sequence $\{r_me^{i\theta_m}\}$ such that*

$$\log\frac{1}{|\langle x(r_me^{i\theta_m}),a_0\rangle|}>\frac{T(r_m)}{\log r_m}\,, \tag{3.7}$$

$$|\theta_m-\omega|<\frac{\pi}{2\gamma}-\varepsilon\qquad(\varepsilon>0)\,. \tag{3.8}$$

Then $\mu\leqq\gamma$.

Proof. Put

$$y(w)=x(Ke^{i\omega}\phi(w))\,,$$

where $\phi(w)$ is the function defined by (3.3). Then $y: U\to P_nC$ is a holomorphic curve such that $\langle y(w),\ a_j\rangle \neq 0$ in U $(j=0, 1, \cdots, 2n)$, and so $\sum_{j=0}^{2n} \delta(a_j, y)=2n+1$. Hence it follows from Lemma 1 that

$$T(t,\ y)=o\left(\log\frac{1}{1-t}\right) \qquad (t\rightarrow\infty). \tag{3.9}$$

Put

$$w_m=\phi^{-1}((r_m/K)e^{i(\theta_m-\omega)}).$$

Then combining this with (3.7), we have

$$\log\frac{1}{|\langle y(w_m),\ a_0\rangle|}=\log\frac{1}{|\langle x(r_m e^{i\theta_m}),\ a_0\rangle|}>\frac{T(r_m,\ x)}{\log r_m} \tag{3.10}$$

and from (3.8)

$$\cos(\gamma(\theta_m-\omega))\geqq\xi>0 \qquad \text{for all} \quad m.$$

Hence (3.6) implies

$$\frac{\xi}{2}\left(\frac{K}{r_m}\right)^{\gamma}<1-|w_m|<8\left(\frac{K}{r_m}\right)^{\gamma} \tag{3.11}$$

for sufficiently large m. Taking

$$t_m=|w_m|+(1-|w_m|)/2 \tag{3.12}$$

and using Lemma 2, we have

$$T(t_m,\ y)\geqq\frac{t_m-|w_m|}{t_m+|w_m|}\left\{\log\frac{1}{|\langle y(w_m),\ a_0\rangle|}-\frac{1}{2}\log(n+1)\right\}-A. \tag{3.13}$$

Hence (3.10)-(3.13) imply

$$T(t_m,\ y)\geqq\frac{\xi}{9}\left(\frac{K}{r_m}\right)^{\gamma}\frac{T(r_m,\ x)}{\log r_m}-A \tag{3.14}$$

for sufficiently large m. (3.9), (3.11) and (3.12) yield

$$T(t_m,\ y)\leqq A_1\log\frac{4}{\xi}\left(\frac{r_m}{K}\right)^{\gamma} \tag{3.15}$$

with a positive constant A_1. Combining (3.14) with (3.15), we obtain

$$\frac{T(r_m,\ x)}{\log r_m}\leqq\frac{9}{\xi}\left(\frac{r_m}{K}\right)^{\gamma}\left\{A+A_1\log\frac{4}{\xi}\left(\frac{r_m}{K}\right)^{\gamma}\right\}$$

and hence, for any positive number δ,

$$T(r_m,\ x)\leqq r_m^{\gamma+\delta}$$

provided m is large enough. This relation implies $\mu\leqq\gamma$. Q. E. D.

§4. Now we prove our Theorem 2 and Theorem 3.

Proof of Theorem 2. Let $x: \boldsymbol{C} \to P_n\boldsymbol{C}$ be a holomorphic curve of finite lower order $\mu \geqq 1$ with $\delta(a_0) > 0$ for some $a_0 \in P_n\boldsymbol{C}$.

Suppose, to the contrary, that there exist ω and η $(\eta > 0)$ such that for the sector Δ defined by (2) and $a_j \in P_n\boldsymbol{C}$, $j = 0, 1, \cdots, 2n$, in general position, solutions in Δ of the equations $\langle x(z), a_j \rangle = 0$ $(j = 0, \cdots, 2n)$ are finite in number. Then there is a number K such that $\langle x(z), a_j \rangle \neq 0$ in Ω defined by

$$\Omega = \left\{ z ;\ |\arg z - \omega| < \pi - \frac{2}{\mu} \sin^{-1}\left(\frac{\delta(a_0)}{2} \right)^{1/2} + \eta,\ |z| > K \right\}.$$

Let $\{r_m\}$ be a sequence of Pólya peaks of order μ of x. Put $\Lambda(r) = (\log r)^{-1} \cdot T(r)$. Then $\Lambda(r)$ satisfies (1). Hence it follows from Theorem 1 (Spread relation) that

$$\liminf_{m \to \infty} \operatorname{meas} E_\Lambda(r_m, a_0) \geqq \frac{4}{\mu} \sin^{-1}\left(\frac{\delta(a_0)}{2} \right)^{1/2}.$$

Therefore, taking

$$\frac{\pi}{\gamma} = 2\pi - \frac{4}{\mu} \sin^{-1}\left(\frac{\delta(a_0)}{2} \right)^{1/2} + 2\eta,$$

we obtain an unbounded sequence $\{r_m e^{i\theta_m}\}$ such that

$$\log \frac{1}{|\langle x(r_m e^{i\theta_m}), a_0 \rangle|} > \frac{T(r_m)}{\log r_m},$$

$$|\theta_m - \omega| < \frac{\pi}{2\gamma} - \frac{\eta}{2}$$

hold for sufficiently large m. Hence Lemma 3 implies $\mu \leqq \gamma$. From the definition of γ, however, we have $\gamma < 1$ and so $\mu < 1$, which contradicts our assumption $\mu \geqq 1$.

Thus the proof of Theorem 2 is complete.

Proof of Theorem 3. Let $x: \boldsymbol{C} \to P_n\boldsymbol{C}$ be a holomorphic curve of lower order μ $(1/2 < \mu < 1)$ such that $\delta(a_0) = 1$ for some $a_0 \in P_n\boldsymbol{C}$.

Suppose, to the contrary, that there exist ω and η (>0) such that for the sector Δ defined by (3) and $a_j \in P_n\boldsymbol{C}$, $j = 0, 1, \cdots, 2n$, in general position, solutions in Δ of the equations $\langle x(z), a_j \rangle = 0$ $(j = 0, \cdots, 2n)$ are finite in number. Then from the argument used in the proof of Theorem 2, remarking $\pi < \pi/\mu < 2\pi$, we deduce that $\mu \leqq \mu\pi/(\pi + 2\mu\eta)$. This is a contradiction.

Thus the proof of Theorem 3 is complete.

§5. Our argument for holomorphic curves $x: \boldsymbol{C} \to P_n\boldsymbol{C}$ is applicable to the case of algebroid functions.

Let $f(z)$ be an n-valued algebroid function of finite lower order μ defined by an irreducible equation

$$A_0(z)f^n+A_1(z)f^{n-1}+\cdots+A_{n-1}(z)f+A_n(z)=0,$$

where $A_0(z), \cdots, A_n(z)$ are entire functions without common zeros. Fix a sequence $\{r_m\}$ of Pólya peaks of order μ of $f(z)$ (or $T(r, f)$). Let $f_j(z)$ be the j-th determination of $f(z)$ and $\Lambda(r)$ a positive function with

$$\Lambda(r)=o(T(r, f)) \qquad (r\to\infty). \tag{5.1}$$

Define the set of arguments $E_\Lambda(r, \tau)\subset(-\pi, \pi]$ by

$$E_\Lambda(r, \tau)=\{\theta; \min_{1\leqq j\leqq n}|f_j(re^{i\theta})-\tau|<e^{-\Lambda(r)}\} \qquad (\tau\neq\infty),$$

$$E_\Lambda(r, \infty)=\{\theta; \max_{1\leqq j\leqq n}|f_j(re^{i\theta})|>e^{\Lambda(r)}\}$$

and let

$$\sigma_\Lambda(\tau)=\liminf_{m\to\infty} \operatorname{meas} E_\Lambda(r_m, \tau),$$

$$\sigma(\tau)=\inf_\Lambda \sigma_\Lambda(\tau),$$

where the infimum is taken over all functions $\Lambda(r)$ satisfying (5.1). Then from the reasoning in § 2, taking results of Valiron [9, pp. 21, 22] into account, we deduce

THEOREM 1′ (Spread relation). *Let $f(z)$ be an n-valued algebroid function of positive, finite lower order μ. Then*

$$\sigma(\tau)\geqq \min\left\{2\pi, \frac{4}{\mu}\sin^{-1}\left(\frac{\delta(\tau)}{2}\right)^{1/2}\right\}.$$

For algebroid functions we are able to have lemmas which correspond to our Lemma 1, Lemma 2 and Lemma 3 in the case of holomorphic curves. Hence we deduce

THEOREM 2′. *Let $f(z)$ be an n-valued algebroid function of finite lower order $\mu\geqq 1$. Suppose that $f(z)$ has a deficient value τ. Let Δ be a sector defined by*

$$\Delta=\left\{z;\ |\arg z-\omega|<\pi-\frac{2}{\mu}\sin^{-1}\left(\frac{\delta(\tau)}{2}\right)^{1/2}+\eta\right\},$$

where η is an arbitrary positive number. Suppose that the solutions in Δ of $f(z)=\tau$ are finite in number. Then the equation $f(z)=\alpha$ has an infinite number of solutions in the sector Δ except at most $2n-1$ values of $\alpha(\alpha\neq\tau)$.

THEOREM 3′. *Let $f(z)$ be an n-valued algebroid function of lower order μ $(1/2<\mu<1)$. Suppose that $f(z)$ has a deficient value τ satisfying $\delta(\tau)=1$. Let Δ*

be a sector defined by

$$\Delta=\left\{z;\ |\arg z-\omega|<\frac{\pi}{2\mu}+\eta\right\},$$

where η is an arbitrary positive number. Suppose that the solutions in Δ of $f(z)=\tau$ are finite in number. Then the equation $f(z)=\alpha$ has an infinite number of solutions in the sector Δ except at most $2n-1$ values of α $(\alpha\neq\tau)$.

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