M. OZAWA
KŌDAI MATH. SEM. REP
28 (1977), 342—360

ON UNIQUELY FACTORIZABLE ENTIRE FUNCTIONS

BY MITSURU OZAWA

1. Introduction

A meromorphic function $F(z)=f(g(z))$ is said to have f and g as left and right factors respectively, provided that f is meromorphic and g is entire (g may be meromorphic when f is rational). $F(z)$ is said to be prime (pseudoprime) if every factorization of the above form into factors implies either f is linear or g is linear (either f is rational or g is a polynomial). If $F(z)$ is representable as $f_1(f_2 \cdots (f_n(z)) \cdots)$ and $g_1(g_2 \cdots (g_n(z)) \cdots)$ and if with suitable linear transformations λ_j, $j=1, \cdots, n-1$

$$f_1=g_1(\lambda_1),\ f_2=\lambda_1^{-1}(g_2(\lambda_2)),\ \cdots,\ f_n=\lambda_{n-1}^{-1}(g_n)$$

hold, then two factorizotions are called to be equivalent.

e^z and $\cos z$ occupy a special situation in the factorization theory in the above sense [7]. They are really pseudoprime and admit infinitely many non-equivalent factorizations. Further $F(z)=z^2e^{2z^2}$ has two non-equivalent factorizations $F(z)=w^2\circ(ze^{z^2})=(we^{2w})\circ z^2$. This shows that factorization into two primes is not unique in general.

In this paper we shall consider some sufficient conditions guaranteeing a unique factorization into two primes up to equivalence. So far as the present author knows there is no systematic research in this tendency up to the present time. The work of this paper does not mean any systematic one either. As an example of a uniquely factorizable function z^2e^{2z} was listed in Gross' book [4], p. 133.

2. Lemmas

LEMMA 1 (Edrei [2]). *Let $f(z)$ be an entire function. Assume that there exists an unbounded sequence $\{a_n\}_{n=1}^{\infty}$ such that all the roots of $f(z)=a_n (n=1, 2, \cdots)$ lie on a single straight line. Then $f(z)$ is a polynomial of degree at most two.*

LEMMA 2. *Let $f(z)$ be entire function. Let w_1 and w_2 be two different finite numbers. Then*

$$N(r, w_1, f)+N(r, w_2, f)$$

Received May 4, 1976

has the same order as the one of f.

This is nothing but the well-known theorem due to Borel. See Nevanlinna [5].

LEMMA 3. *Let* $f(z)$ *be an entire function. Then*

$$\sum_{a\neq\infty}\Theta(a)\leqq 1\,.$$

Here

$$1-\Theta(a)=\varlimsup_{r\to\infty}\frac{\bar{N}(r,a,f)}{m(r,f)}\,.$$

This is nothing but Nevanlinna's famous theorem for entire functions [5]. This includes

$$\sum_{a\neq\infty}\left(1-\frac{1}{\nu(a)}\right)\leqq 1\,.$$

Here $\nu(a)$ indicates the least order of almost all a-points of $f(z)$.

LEMMA 4. *Let* $f(z)$ *be an entire function whose* 1*-points and* -1*-points are multiple of order* $2n$, $n\geqq 1$. *Then* $f(z)=\cos L(z)$, *where* $L(z)$ *is an entire function.*

Proof. By the assumption

$$f(z)^2-1=g(z)^2$$

with entire $g(z)$. Hence

$$f-g=e^{iL},\qquad f+g=e^{-iL}\,,$$

where L is entire. Thus

$$f=\frac{e^{iL}+e^{-iL}}{2}=\cos L\,.$$

LEMMA 5. [1] *Let* $f(z)$ *be an entire function of finite order* $\rho_f\geqq 1$. *Then* $f(z)$ *takes every value infinitely often with at most one exception in every sector whose aperture is greater than* $\pi(2-1/\rho_f)$.

LEMMA 6. [9] *Let* $f(z)$ *be an entire function of order less than one and let* $\{w_n\}$ *be an unbounded sequence for which* $f(z)=w_n$ *has roots only in the sector* $|\pi-\arg z|\leqq\omega$, $0<\omega<\pi/2$ *for every* n. *Then* $f(z)$ *is linear.*

3. Theorems

THEOREM 1. *Let* $F(z)$ *be* $g_1(z)e^{H(g_1(z))}$, *where* $H(w)$ *is a polynomial and* $g_1(z)$ *is a prime transcendental entire function of finite order having only infinitely many real zeros. Then* $F(z)$ *is uniquely factorizable into two primes.*

THEOREM 2. *Let* $g_1(z)$ *be a prime transcendental entire function of finite* *order, which has infinitely many zeros. Assume that almost all zeros of $g_1(z)$ lie in $\Re z \geqq x$ for every x. Then $\{g_1(z)\}^2$ is uniquely factorizable into two primes.*

THEOREM 3. *Let $\{a_n\}$ be a set of complex numbers and $\{\nu_n\}$ be a set of positive integers satisfying $\nu_1 < \cdots < \nu_n < \nu_{n+1} < \cdots$. Assume that*

$$\sum_{n=1}^{\infty} \frac{\nu_n}{|a_n|^s} < \infty$$

for $s<1$. Further assume that ν_1 and ν_2 are coprime. Let $g_1(z)$ be

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)^{\nu_n}.$$

Then $\{g_1(z)\}^2$, $g_1(z) \exp H(g_1(z))$ are uniquely factorizable into two primes, where $H(w)$ is a polynomial.

THEOREM 4. *Let $g_1(z)$ be a prime transcendental entire function of order $\rho_{g_1} < \infty$. Assume that all zeros of $g_1(z)$ lie in the sector*

$$|\pi - \arg z| \leqq \begin{cases} \dfrac{\pi}{2(\rho_{g_1}+\varepsilon)} & (\rho_{g_1} \geqq 1,\ \varepsilon > 0), \\ \omega < \dfrac{\pi}{2} & (\rho_{g_1} < 1). \end{cases}$$

Then $g_1 e^{H(g_1)}$ is uniquely factorizable into two primes.

Several related results were stated in [6].

THEOREM 5. *Let $F(z)$ be an entire function of order less than 1, which has the form $f(g(z))$, where*

$$f(w) = \prod_{j=1}^{\infty}\left(1-\frac{w}{a_j}\right)^{\nu_j}$$

with prime numbers $\nu_j (3 \leqq \nu_n < \nu_{n+1})$ and $g(z)$ is a prime polynomial. Then $F(z)$ is uniquely factorizable into two primes.

THEOREM 6. *Let $F(z)$ be an entire function of order zero, which has the form $f(g(z))$, where*

$$f(w) = w\prod_{j=1}^{\infty}\left(1-\frac{w}{a_j}\right)^{\nu_j},$$

$$g(z) = \prod_{l=1}^{\infty}\left(1-\frac{z}{b_l}\right)^{n_l}$$

with prime numbers ν_j, $3 \leqq \nu_n < \nu_{n+1}$ and $n_1 = 1$ and prime numbers n_l, $3 \leqq n_l < n_{l+1}$ for $l \geqq 2$ and further $\nu_j \neq n_l$ for all j and l. Further assume that $g(z) = a_j$ has only simple roots for all j. Then $F(z)$ is uniquely factorizable into two primes.

It seems to the present author that there are several extensions of the above theorem.

4. Proof of Theorem 1. Let $F(z)$ be $f(g(z))$.

Case 1). f and g are transcendental entire.

(a). Suppose that $f(w)=0$ has infinitely many roots $\{w_n\}_{n=1}^{\infty}$. Let us consider $g(z)=w_n$. Then all the roots of $g(z)=w_n$ for all n lie on the real axis, whence follows that $g(z)$ is a polynomial of degree at most two by Lemma 1. This contradicts the transcendency of g.

(b). Suppose that $f(w)=0$ has only finitely many roots $\{w_j\}_{j=1}^{n}$. Further suppose that $w_1 \neq w_2$. In this case

$$f(w)=P(w)e^{L(w)}$$

with a polynomial $P(w)$ and non-constant entire $L(w)$. Hence

$$g_1(z)e^{H(g_1(z))}=P(g(z))e^{L(g(z))}.$$

By $w_1 \neq w_2$ we can make use of Lemma 2 and conclude that the order ρ_g of g is finite. Hence with a polynomial $M(z)$

$$g_1(z)=P(g(z))e^{M(z)}.$$

$H(g_1(z))+M(z)-L(g(z))=2m\pi i$, m: an integer. Let $\{z_l\}$ be the set of $g_1(z_l)=0$. Then $P(g(z_l))=0$ and hence $g(z_l)=w_j$ with $P(w_j)=0$. Therefore

$$M(z_l)=L(w_j)-H(0)+2m\pi i,$$

which is bounded for $l\to\infty$, since j runs over a finite number of indices. Thus $M(z)$ is a constant. Hence

$$g_1(z)=CP(g(z)),$$

which contradicts the primeness of $g_1(z)$.

(c). Suppose that $f(w)=0$ has only one root w_1. (There must be at least one zero of $f(w)$.) In this case

$$f(w)=(w-w_1)^n e^{L(w)}$$

with a non-constant entire function $L(w)$. Thus

$$g_1(z)e^{H(g_1(z))}=(g(z)-w_1)^n e^{L(g(z))}.$$

Since $nN(r, w_1, g)=N(r, 0, g_1)\leqq m(r, g_1)$ and $\rho_{g_1}<\infty$, we can construct the canonical product $N(z)$ by the zeros of $g(z)-w_1$. Then $N(z)$ is of finite order. Let us put

$$g(z)-w_1=N(z)e^{Q(z)}$$

with entire $Q(z)$. Hence

$$g_1(z)e^{H(g_1(z))}=N(z)^n e^{nQ(z)+L(g(z))}$$

and

$$g_1(z)=N(z)^n e^{M(z)}$$

with a polynomial $M(z)$. Hence

$$g_1(z)=W^n \circ (N(z)e^{M(z)/n}).$$

Then primeness of $g_1(z)$ implies $n=1$. Thus

$$g_1(z)=N(z)e^{M(z)}.$$

Further

$$M(z)+H(N(z)e^{M(z)})=Q(z)+L(w_1+N(z)e^{Q(z)})+2m\pi i.$$

If $\rho_g=\infty$, then $Q(z)$ is transcendental and then the right hand side is of infinite order but the left hand side is of finite order. This is a contradiction. Hence $\rho_g<\infty$ and then $Q(z)$ is a polynomial. Let $\{z_l\}$ be the set of zeros of $g_1(z)$. Then

$$H(0)+M(z_l)=Q(z_l)+L(w_1)+2m\pi i.$$

Thus $M(z_l)-Q(z_l)$ is bounded and $M(z)-Q(z)$ reduces to a constant. Therefore

$$g_1(z)=c(g(z)-w_1)$$

and

$$H(g_1(z))=A+L(g(z))+2m\pi i, \qquad c=e^{-A}$$

with a constant A. Thus

$$g_1(z)e^{H(g_1(z))}=c(g(z)-w_1)e^{H(c(g(z)-w_1))}.$$

Hence

$$\begin{aligned} f(w)&=c(w-w_1)e^{H(c(w-w_1))} \\ &=\{We^{H(W)}\}\circ\lambda(w), \qquad \lambda(w)=c(w-w_1) \end{aligned}$$

and

$$g(z)=\lambda^{-1}\circ g_1(z).$$

Thus $f(g(z))$ is equivalent to $\{we^{H(w)}\}\circ g_1(z)$.

Case 2). $f(w)$ is a polynomial. If $f(w)=0$ has two different roots, then by Lemma 2 $\rho_g<\infty$ and then by the second fundamental theorem of Nevanlinna $\rho_g \leqq \rho_{g_1}<\infty$. Then

$$g_1(z)e^{H(g_1(z))}=f(g(z))$$

is clearly absurd. Hence $f(w)$ has only one zero w_1 and

$$f(w)=A(w-w_1)^n,$$

$$g_1(z)e^{H(g_1(z))}=A(g(z)-w_1)^n.$$

Let us put $g(z)-w_1=N(z)e^{Q(z)}$ with the canonical product $N(z)$ formed by the zeros of $g(z)-w_1$ and entire $Q(z)$. $N(z)$ is well defined. Then

$$g_1(z)e^{H(g_1(z))}=AN(z)^n e^{nQ(z)}$$

and

$$g_1(z)=BN(z)^n e^{M(z)}$$
$$=\{BW^n\}\circ\{N(z)e^{M(z)/n}\}$$

with a polynomial $M(z)$ and a constant B. The primeness of $g_1(z)$ shows $n=1$. Therefore $f(w)$ is linear. We may omit this case.

Case 3). $g(z)$ is a polynomial. In this case $f(w)=0$ has infinitely many zeros $\{w_n\}_{n=1}^{\infty}$. Suppose that the degree of $g(z)$ is not less than 3. Then $g(z)=w_n$ has roots not lying on the real axis when n is sufficiently large. Suppose that $g(z)$ is quadratic, that is, $a(z-\alpha)^2+b$, $a\neq 0$. Let x be $z-\alpha$. Then

$$g_1(x+\alpha)e^{H(g_1(x+\alpha))}=f(ax^2+b).$$

Hence $g_1(x+\alpha)$ and $g_1(-x+\alpha)$ have the same zeros and hence with a polynomial $M(x)$

$$g_1(x+\alpha)=g_1(-x+\alpha)e^{M(x)}.$$

Therefore by

$$g_1(x+\alpha)e^{H(g_1(x+\alpha))}=g_1(-x+\alpha)e^{H(g_1(-x+\alpha))},$$
$$M(x)+H(g_1(x+\alpha))=H(g_1(-x+\alpha))+2p\pi i$$

with an integer p. Let $\{z_n\}=\{x_n+\alpha\}$ be the set of zeros of $g_1(z)$. Then

$$M(x_n)+H(0)=H(0)+2p\pi i.$$

Hence $\{M(x_n)\}$ is bounded and hence $M(x)$ is a constant. So we have $M(x)=2p\pi i$. Therefore $g_1(x+\alpha)=g_1(-x+\alpha)$, that is, with entire $G(w)$ $g_1(x+\alpha)=G(x^2)$, $g_1(z)=G((z-\alpha)^2)$. This contradicts the primeness of $g_1(z)$.

Case 4). $f(w)$ is meromorphic (not entire) and transcendental. In this case g is entire and with entire $f^*(w)$, $M(z)$

$$f(w)=\frac{f^*(w)}{(w-w_0)^n},\qquad f^*(w_0)\neq 0,$$
$$g(z)=w_0+e^{M(z)}.$$

Therefore

$$F(z)=e^{-nM(z)}f^*(w_0+e^{M(z)}).$$

If $f^*(w)=0$ has infinitely many zeros $\{w_l\}_{l=1}^{\infty}$, then $w_0+e^{M(z)}=w_l$ has only real roots for all l. Thus by Lemma 1 $w_0+e^{M(z)}$ must be a polynomial, which is evidently a contradiction. Hence $f^*(w)$ has only a finite number of zeros. Hence with non-constant entire $N(w)$

$$f^*(w)=P(w)e^{N(w)}, \qquad P(w_0)\neq 0,$$

$$F(z)=e^{-nM(z)}P(w_0+e^{M(z)})e^{N(w_0+e^{M(z)})}.$$

Let us consider

$$P(w_0+e^{M(z)})=\sum_{j=0}^{m}A_j e^{jM(z)}\equiv Q(w)\circ e^{M(z)}.$$

If $Q(w)$ has two different zeros, then by Lemma 2 $\rho_{\exp M}\leqq\rho_{g_1}<\infty$ and hence M is a polynomial. Thus with a polynomial $R(z)$

$$g_1(z)=P(w_0+e^{M(z)})e^{R(z)},$$

$$H(g_1(z))+R(z)=-nM(z)+N(w_0+e^{M(z)})+2s\pi\imath$$

with an integer s. Let $\{z_l\}$ be the set of zeros of $g_1(z)$. Then $P(w_0+e^{M(z_l)})=0$ and $w_0+e^{M(z_l)}=w_j$, $P(w_j)=0$. Thus

$$R(z_l)+nM(z_l)=N(w_j)-H(0)+2s\pi i.$$

Here j can move onto at most finitely many integers. Hence $R(z)+nM(z)$ is a constant d. Hence

$$g_1(z)=AP(w_0+e^{M(z)})e^{-nM(z)}$$

$$\equiv\left\{\frac{AP(w_0+W)}{W^n}\right\}\circ e^{M(z)}.$$

This gives a contradiction by the primeness of $g_1(z)$. Hence $Q(w)$ has only one zero. Hence

$$P(w_0+e^{M(z)})=A(w-w_1)^m\circ e^{M(z)}.$$

Here $w_1\neq 0$, since $P(w_0)\neq 0$ and $(-1)^m A w_1^m=P(w_0)$. In this case

$$\rho_{\exp M}=\rho_{N(r,w_1,e^M)}\leqq\rho_{N(r,0,g_1)}\leqq\rho_{g_1}<\infty.$$

Hence $M(z)$ is a polynomial. If M is not linear, then

$$M(z)=\log w_1+2t\pi i$$

has zeros not lying on the real axis if t moves all integers. But they must be a subset of zeros of $g_1(z)$. This is absurd. If $M(z)$ is linear, then $M(z)=az+b$. Again we put

$$g_1(z)=P(w_0+Be^{az})e^{R(z)}.$$

Then $R(z)$ is a polynomial and

$$R(z)+nM(z)=N(w_0+Be^{az})-H(g_1(z))+2p\pi\imath$$

with an integer p. Again we put $z=z_l$, $g_1(z_l)=0$. Then

$$R(z_l)+nM(z_l)$$

is bounded. This is a constant d. Hence

$$R(z)=d-naz-nb\,.$$

Thus

$$g_1(z)=\{A(w-w_1)^m\circ e^{az+b}\}e^{-naz-nb}$$
$$=A\frac{(W-w_1)^m}{W^n}\circ e^{az+b}\,.$$

Since $W^p\circ e^{(az+b)/p}=e^{az+b}$, the above $g_1(z)$ is not prime. This is a contradiction.

Case 5). $f(w)$ is rational (not polynomial).

(a). Suppose that $g(z)$ is entire. In this case

$$f(w)=\frac{P(w)}{(w-w_0)^n}\,,\qquad g(z)=w_0+e^{M(z)}\,,$$
$$P(w_0)\neq 0\,.$$

Here $P(w)$ is a polynomial and $M(z)$ is entire. Hence

$$g_1(z)e^{H(g_1(z))}=e^{-nM(z)}P(w_0+e^{M(z)})\,.$$

In this case we can arrive at a contradiction quite similarly as in Case 4).

(b). Suppose that $g(z)$ is really meromorphic. In this case $f(w)$ should have at most two different poles by Picard's theorem. If

$$f(w)=\frac{Q(w)}{(w-w_1)^{n_1}(w-w_2)^{n_2}}$$

with a polynomial $Q(w)$ and two positive integers n_1, n_2, then

$$g(z)=w_1+\frac{Be^M}{1-Be^M}(w_1-w_2)\qquad (B\neq 0)$$
$$=w_2+\frac{w_1-w_2}{1-Be^M}\,.$$

The degree of Q should not be greater than n_1+n_2. Let us consider

$$g^*(z)=\lambda(g(z))=\frac{g(z)-w_1}{g(z)-w_2}\equiv Be^M$$

and

$$f^*(w)=f(\lambda^{-1}(w))\,.$$

Then

$$f^*(g^*(z))=f(g(z))\,.$$

$g^*(z)$ is entire. Hence this case reduces to Case 5), (a). If

$$f(w)=\frac{Q(w)}{(w-w_1)^{n_1}}, \qquad \deg Q(w)\leqq n_1$$

with a polynomial $Q(w)$ and a positive integer n_1, then

$$g(z)=w_1+\frac{Be^M}{h(z)}$$

with entire $h(z)\neq$ const. Then by putting

$$g^*(z)\equiv \lambda(g(z))\equiv \frac{1}{g(z)-w_1}=B^{-1}h(z)e^{-M(z)},$$

$$f^*(w)=f(\lambda^{-1}(w))$$

we have the equivalence of

$$f^*(g^*(z))=f(g(z)).$$

Here $g^*(z)$ is entire. Hence this case reduces to 5), (a).

This completes our proof of Theorem 1.

5. Proof of Theorem 2. Let $F(z)\equiv(g_1(z))^2$ be $f(g(z))$.

Case 1). f and g are transcendental entire. In this case the order ρ_f of f is equal to 0 by Pólya's theorem [8]. Hence $f(w)=0$ has infinitely many roots. Assume that $f(w)$ has three zeros w_1, w_2, w_3 of odd orders. Then at least one of $g(z)=w_j$, $j=1$, 2, 3 has at least one simple zero z^*. Then $F(z^*)=f(g(z^*))=0$ and $F(z)$ has the zero z^* of odd order. However the order of z^* should be even by the form of $F(z)$. This is a contradiction. Therefore $f(w)=0$ has at most two zeros of odd order. Hence three cases may occur:

i). $f(w)=(w-w_1)^{\lambda_1}(w-w_2)^{\lambda_2}h(w)^2$,

ii). $f(w)=(w-w_1)^{\lambda_1}h(w)^2$,

iii). $f(w)=h(w)^2$,

where $h(w)$ is entire and λ_1, λ_2 are odd integers.

Case iii). Then

$$(g_1(z))^2=(h(g(z)))^2,$$

$$g_1(z)=\pm h(g(z)).$$

This contradicts the primeness of $g_1(z)$.

Case ii). Then $g(z)=w_1+X(z)^2$ and hence

$$(g_1(z))^2=X(z)^{2\lambda_1}h(w_1+X(z)^2)^2.$$

Therefore

$$g_1(z)=\pm X(z)^{\lambda_1}h(w_1+X(z)^2),$$

which contradicts the primeness of $g_1(z)$.

Case i). In this case with entire $L(z)$

$$g(z)=\frac{1}{2}(w_1+w_2)+\frac{1}{2}(w_1-w_2)\cos L(z)$$

by Lemma 4. Then

$$g_1(z)=\pm A\left(\sin\frac{L}{2}\right)^{\lambda_1}\left(\cos\frac{L}{2}\right)^{\lambda_2}h(B+D\cos L).$$

L should be linear by the primeness of $g_1(z)$. Then this function $g_1(z)$ has infinitely many zeros in $\mathcal{R}z\leqq x_0$ for some x_0. This is a contradiction.

Case 2). f is transcendental entire and g is a polynomial. In this case f should have infinitely many zeros $\{w_j\}_{j=1}^{\infty}$. Then $g(z)=w_j$ for $j\geqq j_0$ have infinitely many roots in $\mathcal{R}z\leqq x_0$ for some x_0 if $g(z)$ is not linear. This is a contradiction.

Case 3). $f(w)$ is a polynomial. Similarly as in Case 1) three cases occur:

i). $f(w)=(w-w_1)^{\lambda_1}(w-w_2)^{\lambda_2}h(w)^2$,

ii). $f(w)=(w-w_1)^{\lambda_1}h(w)^2$,

iii). $f(w)=h(w)^2$,

where h is a polynomial and λ_1, λ_2 are odd integers.

Case iii). Then $g_1(z)=\pm h(g(z))$. Hence h should be linear. Let us put $\pm h(w)=aw+b$. Therefore

$$g_1(z)=(aw+b)\circ g(z),$$

$$w^2=f(w)\circ\frac{w-b}{a}.$$

Hence two factorizations are equivalent.

Case ii). Then $g(z)-w_1=X(z)^2$. Hence

$$g_1(z)=\pm X(z)^{\lambda_1}h(w_1+X(z)^2).$$

By the primeness of $g_1(z)$ we have that either $X(z)$ is linear or $h(w)$ is a constant and $\lambda_1=1$. This is absurd.

Case i). Then the same reasoning as in Case 1), i) leads us to a contradiction.

Case 4). $f(w)$ is meromorphic (not entire). Then g is transcendental entire and

$$f(w)=\frac{f^*(w)}{(w-w_0)^n},\ f^*(w_0)\neq 0,$$

$$g(z)=w_0+e^M$$

with entire f^*, a polynomial M and a positive integer n. Hence

$$g_1(z)^2 = F(z) = e^{-nM} f^*(w_0 + e^M).$$

In considering on the growth of both sides $\rho_{f^*}=0$. Hence $f^*(w)$ has infinitely many zeros $\{w_j\}_{j=1}^{\infty}$. Then

$$e^M = w_j - w_0$$

has roots satisfying

$$M = \log(w_j - w_0) + 2t\pi i, \qquad t: \text{an integer},$$

and hence there are infinitely many roots of $g_1(z)=0$ in $\Re z \leqq z_0$ for some x_0.

Case 5). f is rational (not a polynomial) but g is entire. In this case

$$f(w) = \frac{Q(w)}{(w-w_0)^n}, \qquad Q(w_0) \neq 0,$$

$$g(z) = w_0 + e^M,$$

where Q, M are polynomials. Hence

$$g_1(z)^2 = e^{-nM} Q(w_0 + e^M).$$

Similarly as in Case 4) we have a contradiction.

Case 6). f is really rational and g is really meromorphic. This case can be reduced to Case 5) by constructing a linear transformation λ so that

$$g^* = \lambda(g), \qquad f^* = f(\lambda^{-1})$$

and that g^* is entire as in Case 5), (b) in the proof of Theorem 1.

The proof of Theorem 2 has been completed.

6. Proof of Theorem 3.

In order to prove Theorem 3 we need to verify the primeness of $g_1(z)$ defined in Theorem 3.

LEMMA 7. *$g_1(z)$ defined in Theorem 3 is prime.*

Proof. Suppose that $g_1(z)=f(g(z))$ with transcendental entire f and g. In this case $\rho_f=0$ by Pólya's theorem. Hence $f(w)=0$ has infinitely many roots $\{w_n\}_{n=1}^{\infty}$ denoting their orders by $\{\mu_n\}_{n=1}^{\infty}$ respectively. Let us consider $g(z)=w_n$, $n=1, 2, \cdots$. Then there is at least one index, say n_0, for which $g(z)=w_{n_0}$ has infinitely many simple zeros by Lemma 3. Hence there must be infinitely many zeros of $g_1(z)$ having the same multiplicity. This is absurd.

Suppose that f is transcendental entire and g is a polynomial. If f has infinitely many zeros, then g should be linear by the same reasoning as in the above. Since g_1 is of order less than one, $\rho_f<1$. Hence f has infinitely many zeros.

Suppose that f is a polynomial. Then by Lemma 3 f has only one zero w_1. Hence

$$f(w)=A(w-w_1)^{\mu_1}.$$

If $\mu_1 \geqq 2$, then every zero of g_1 has its order $\mu_1 x_n$ with a positive integer x_n depending on the zero. The coprimeness of ν_1 and ν_2 implies a contradiction. Hence $\mu_1=1$ and f is linear.

g_1 is not periodic. Hence by Gross' theorem [3] g_1 is prime

Proof of Theorem 3. The case $g_1(z)^2$. In this case all zeros have orders $2\nu_n$. Hence by a small modification of the proof of Lemma 5 it is possible to conclude that

$$f(w)=A^2(w-w_1)^2,$$

$$g(z)=w_1 \pm A g_1(z).$$

This gives an equivalent factorization to $W^2 \circ g_1(z)$. It still remains to consider the case that f is meromorphic or rational. Then it is sufficient to consider g as entire. Thus g has the form

$$w_0+e^{L(z)}.$$

Since then $\rho_g \geqq 1 > \rho_{g_1^2}$, we arrive at a contradication.

The case $g_1(z) \exp H(g_1(z))$. Let $F(z) \equiv g_1(z) \exp H(g_1(z))$ be $f(g(z))$. If f is transcendental (entire or meromorphic), then f has only one zero by Lemma 3. Hence

$$f(w)=(w-w_1)^p e^{L(w)}$$

or

$$f(w)=\frac{(w-w_1)^p}{(w-w_0)^n} e^{L(w)}, \qquad g(z)=w_0+e^{M(z)}.$$

If the latter case occurs, then

$$g_1 e^{H(g_1)}=e^{-nM}(w_0-w_1+e^M)^p e^{L(w_0+e^M)}.$$

On compareing the growth of both sides we have a contradiction. Hence the former case occurs. By the coprimeness of ν_1 and ν_2 p should be equal to one. Thus

$$f(w)=(w-w_1)e^{L(w)},$$

$$F(z)=(g(z)-w_1)e^{L(g(z))}.$$

We now put $g(z)-w_1=g_1(z)e^{M(z)}$. Then

$$H(g_1(z))=M(z)+L(w_1+g_1(z)e^{M(z)})+2m\pi\imath.$$

Let us consider the growth of both sides. Then $M(z)$ should be a constant. Hence $g(z)-w_1=cg_1(z)$ and

$$H(g_1(z)) = \log c + L(w_1 + cg_1(z)) + 2m\pi i\,.$$

Therefore

$$\lambda(g) = g_1\,, \qquad \lambda(x) = \frac{x - w_1}{c}\,,$$

$$f(\lambda^{-1}) = We^{H(W)}\,.$$

If $f(w)$ is a polynomial, then it should be linear by Lemma 3. If $f(w)$ is rational and g is entire, then

$$f(w) = \frac{w - w_1}{(w - w_0)^n}\,, \qquad g(z) = w_0 + e^{M(z)}\,.$$

This is again a contradition, since $g(z) = w_1$ has infinitely many simple zeros. If $f(w)$ is rational and g is meromorphic, then the same process as in the proof of Theorem 1 does work.

7. Proof of Theorem 4.

In the proof we shall describe only parts for which we need a different method from the one in the proof of Theorem 1. Let $F(z) \equiv g_1(z)e^{H(g_1(z))}$ be $f(g(z))$.

Case 1). f and g are transcendental entire.

a). Suppose that $f(w) = 0$ has infinitely many roots $\{w_n\}_{n=1}^{\infty}$. Let $\{z_{nl}\}$ be the set of roots of $g(z) = w_n$. Then

$$|\pi - \arg z_{nl}| \leqq \begin{cases} \dfrac{\pi}{2(\rho_{g_1} + \varepsilon)} & (\rho_{g_1} \geqq 1)\,, \\ \omega < \dfrac{\pi}{2} & (\rho_{g_1} < 1)\,. \end{cases}$$

Evidently $\rho_g \leqq \rho_{g_1} < \infty$. If $\rho_g \geqq 1$, then

$$|\pi - \arg z_{nl}| \leqq \frac{\pi}{2(\rho_{g_1} + \varepsilon)} \leqq \frac{\pi}{2(\rho_g + \varepsilon)}\,.$$

The complementary sector has its aperture

$$2\pi - \frac{\pi}{\rho_g + \varepsilon} > 2\pi - \frac{\pi}{\rho_g}\,.$$

Hence by Lemma 5 there must be infinitely many roots of $g(z) = w_n (n \geqq n_0)$ in this complementary sector. This is a contradiction.

If $\rho_g < 1$ but $\rho_{g_1} \geqq 1$, then

$$|\pi - \arg z_{nl}| \leqq \frac{\pi}{2(\rho_{g_1} + \varepsilon)} = \omega < \frac{\pi}{2}\,.$$

If $\rho_g < 1$ and $\rho_{g_1} < 1$, then

$$|\pi - \arg z_{nl}| \leqq \omega < \frac{\pi}{2}.$$

In these cases $g(z)$ reduces to a linear polynomial by Lemma 6.

Case 3) and Case 4). We need different processes from those in the proof of Theorem 1. However it becomes easier.

8. Proof of Theorem 5. Let $F(z)=f_1(g_1(z))$.

Case 1). f_1 and g_1 and transcendental entire. In this case the orders of f_1 and g_1 are less than one. Hence there are infinitely many zeros $\{w_n\}$ of $f_1(w)$ and so for some w_n there are infinitely many simple roots of $g_1(z)=w_n$. Therefore there are infinitely many roots of $F(z)=0$ having the same multiplicity. This is absurd.

Case 2). f_1 is transcendental entire and g_1 is a polynomial. Let us put

$$f_1 = A_w{}^{\varepsilon\mu_0} \prod_{l=1}^{\infty} \left(1-\frac{w}{w_l}\right)^{\mu_l},$$

where $\{\mu_j\}$ are positive integers and ε is equal to either 1 if $w=0$ is a zero of f_1 or 0 if $w=0$ is not a zero of f_1 and A is a constant. Since $g_1(z)=w$ has only simple roots if $|w|$ is sufficiently large, $g_1(z)=w_l$, $f_1(w_l)=0$ have only roots having the same order μ_l if l is sufficiently large. Hence μ_l is equal to a ν_m. Let us list up all the μ_l which are equal to ν_m. There are only finitely many such μ_{l_j}, $j=1, \cdots, s$. Here we may assume that $l_j \geqq l_0$. Let us put

$$G(z) = \prod_{t=1}^{s} \left(1-\frac{g_1(z)}{w_{l_t}}\right).$$

Then

$$G(z) = B\left(1-\frac{g(z)}{a_m}\right)$$

by $\rho_G<1$, $\rho_g<1$. Hence if $s\geqq 2$ g is not prime. This is absurd. Thus only one μ_l coincides with ν_m for any sufficiently large $l \geqq l_0$. We now put $l=l(m)$. Further

$$g(z)-a_m = C(g_1(z)-w_{l(m)})$$

with a constant $C=C(m)$ for every $m \geqq m_0$. Let z tend to ∞. Then

$$\frac{g(z)-a_m}{g_1(z)-w_{l(m)}} = C(m)$$

tends to a constant α_0/β_0, where

$$g = \alpha_0 z^p + \cdots + \alpha_p,$$

$$g_1 = \beta_0 z^p + \cdots + \beta_p.$$

Hence $C(m)$ does not depend on m. And

$$C(m)=C=\alpha_0/\beta_0\,, \qquad m\geqq m_0\,.$$

We may assume that the set of integers $\{l\geqq l_0\}$ and the set of integers $\{m\geqq m_0\}$ have one-to-one correspondence so that l_0 corresponds to m_0 by rerabelling of $\{l\}$ and $\{m\}$ if necessary. Under this assumption

$$1-\frac{g_1}{w_{l(m_0)}}=\frac{a_{m_0}}{w_{l(m_0)}}\frac{1}{C}\Big(1-\frac{g}{a_{m_0}}\Big), \qquad l(m_0)=l_0\,.$$

Hence

$$F=f_1(g_1)=A\Big(\frac{Cw_{l_0}-a_{m_0}}{C}\Big)^{\varepsilon\mu_0}\Big(1-\frac{g}{a_{m_0}-Cw_{l_0}}\Big)^{\varepsilon\mu_0}$$

$$\cdot\prod_{l=1}^{\infty}\Big(1+\frac{a_{m_0}-Cw_{l_0}}{w_lC}\Big)^{\mu_l}$$

$$\cdot\prod_{l=1}^{\infty}\Big(1-\frac{g}{w_lC+a_{m_0}-Cw_{l_0}}\Big)^{\mu_l}.$$

Since $\mu_l=\nu_m$ for $m\geqq m_0(l\geqq l_0)$,

$$a_m-w_lC=a_{m_0}-Cw_{l_0}\,, \qquad m\geqq m_0\,.$$

Thus a_m-w_lC is a constant D for $m\geqq m_0$. Hence $g=Cg_1+D$. Further

$$f(g)=F=f_1(g_1)$$

implies that

$$A\Big(\frac{D}{C}\Big)^{\varepsilon\mu_0}\Big(1-\frac{g}{D}\Big)^{\varepsilon\mu_0}\prod_{l=1}^{\infty}\Big(1+\frac{D}{w_lC}\Big)^{\mu_l}\prod_{l=1}^{l_0-1}\Big(1-\frac{g}{w_lC+D}\Big)^{\mu_l}$$

$$=\prod_{m=1}^{m_0-1}\Big(1-\frac{g}{a_m}\Big)^{\nu_m}.$$

Hence

$$A\Big(\frac{D}{C}\Big)^{\varepsilon\mu_0}\prod_{l=1}^{\infty}\Big(1+\frac{D}{w_lC}\Big)^{\mu_l}=1$$

and

$$\Big(1-\frac{w}{D}\Big)^{\varepsilon\mu_0}\prod_{l=1}^{\infty}\Big(1-\frac{w}{w_lC+D}\Big)^{\mu_l}=\prod_{m=1}^{\infty}\Big(1-\frac{w}{a_m}\Big)^{\nu_m}.$$

This means that

$$f=f_1\Big(\frac{\lambda-D}{C}\Big),$$

$$g=Cg_1+D\,.$$

This gives just two equivalent factorizations of F.

Case 3). f_1 is a polynomial and g_1 is transcendental entire. We assume that deg $f_1 \geqq 2$. If f_1 has two different zeros w_1 and w_2, then both of $g=w_1$ and $g=w_2$ have only roots of order at least $\nu_1 \geqq 3$. This is absurd. Hence $f_1=A(w-w_1)^n$, $n \geqq 2$. In this case $f_1(g_1(z))=0$ has roots of order which is not prime. This is again absurd.

Case 4). f_1 is transcendental meromorphic (not entire). Then

$$f_1=\frac{f_1{}^*(w)}{(w-w_1)^n}, \qquad g_1=w_1+e^{M(z)},$$

since F is not entire if g_1 is a polynomial. In this case $1 \leqq \rho_{g_1} \leqq \rho_F < 1$. This is absurd.

Case 5). f_1 is rational and g_1 is transcendental entire. Then

$$f_1=\frac{P}{\prod_{s=1}^{t}(w-b_s)^{\mu_s}}$$

and $\rho_{g_1}<1$ imply a contradiction.

Case 6). f_1 is rational and g_1 is meromorphic (not entire). Let a_1 be a pole of f_1. Then $g_1-a_1 \neq 0$. Let g_2 be $1/(g_1-a_1)$. Then $F=R(g_2)$. This reduces to the case 5).

9. Proof of Theorem 6. Let $F(z)$ be $f_1(g_1(z))$.

Case 1). f_1 and g_1 are transcendental entire. Evidently f_1 and g_1 are of order zero. Hence

$$f_1(w)=Aw^{\varepsilon\mu_0}\prod_{j=1}^{\infty}\left(1-\frac{w}{c_j}\right)^{\mu_j},$$

$$F(z)=Ag_1{}^{\varepsilon\mu_0}\prod_{j=1}^{\infty}\left(1-\frac{g_1}{c_j}\right)^{\mu_j},$$

where A is a constant and ε is either 1 if $f_1(0)=0$ or 0 if $f_1(0)\neq 0$. We firstly consider the case $\varepsilon=1$. $F(z)$ has only one simple zero b_1. Hence there must be at least one μ_l, which is equal to 1. Further F has only zeros of prime order ν_j or n_l $l \geqq 2$ excepting only one, that is, b_1. Assume that there are two μ_l, being equal to 1, say μ_j and μ_k. Then either $g_1(z)=c_j$ or $g_1(z)=c_k$ has infinitely many simple zeros. Hence F should have infinitely many simple zeros, which is a contradiction. Thus only one μ_l is equal to one. We may put $\mu_0=1$. Then we may put $\mu_l \geqq 2$ for $l \geqq 1$. Since F does not have any zeros of not prime order excepting only b_1, all μ_l $(l \geqq 1)$ should be prime numbers and all the equations $g_1=c_l$ should have only simple zeros. Hence we may assume that $\mu_{lj}=\nu_l$, by relabelling if necessary, for $j=1, 2, \cdots$. Therefore

$$\prod_{j=1}^{\infty}\left(1-\frac{g_1}{c_{lj}}\right)^{\nu_l}\cdot M_l{}^{\nu_l}=\left(1-\frac{g}{a_l}\right)^{\nu_l}$$

with a suitable entire function M_l. If $z=b_l$, then g_1 should be equal to zero whose order is n_l. Hence

$$g_1(z)=g(z)L(z)$$

with a suitable entire function L. Thus

$$N(r, a_l, g)=\sum_{j=1}^{\infty} N(r, c_{lj}, g_1)+N(r, 0, M_l)$$

and

$$N(r, 0, g_1)=N(r, 0, g)+N(r, 0, L).$$

Since the orders of g and g_1 are equal to zero,

$$N(r, 0, g)=(1-o(1))m(r, g),$$

$$N(r, a_l, g)=(1-o(1))m(r, g),$$

$$N(r, 0, g_1)=(1-o(1))m(r, g_1),$$

$$N(r, c_{lj}, g_1)=(1-o(1))m(r, g_1)$$

hold for a set of upper density one. Therefore

$$(1-o(1))m(r, g)\geqq s(1-o(1))m(r, g_1)$$

for an integer s and

$$(1-o(1))m(r, g_1)\geqq(1-o(1))m(r, g).$$

Thus s should be equal to one. This means that $\mu_l=\nu_l$ and

$$1-\frac{g}{a_l}=\left(1-\frac{g_1}{c_l}\right)M_l.$$

Evidently L and M_l are of order zero. Hence

$$(1-o(1))m(r, g)=(1-o(1))m(r, g_1)$$

shows that

$$m(r, L)=o(m(r, g))=o(m(r, g_1)),$$

$$m(r, M_l)=o(m(r, g))=o(m(r, g_1))$$

for a set of upper density one. On the other hand

$$g\left(\frac{L}{c_l}M_l-\frac{1}{a_l}\right)=M_l-1.$$

Hence

$$N(r, 0, g)+N\left(r, \frac{c_l}{a_l}, LM_l\right)=N(r, 1, M_l).$$

Therefore for a set of upper density one

$$(1-o(1))m(r, g)+o(m(r, g))=o(m(r, g)),$$

which gives evidently a contradiction. Hence M_l and L should be constants. Hence

$$g_1=Bg,$$

$$1-\frac{g}{a_l}=C\left(1-\frac{g_1}{c_l}\right);$$

$$\frac{CB}{c_l}=\frac{1}{a_l} \quad \text{and} \quad C=1.$$

Thus $B=c_l/a_l$. This gives

$$f_1(w)=Aw\prod_{m=1}^{\infty}\left(1-\frac{w}{Ba_m}\right)^{\nu_m},$$

$$g_1=Bg.$$

Hence

$$f_1(w)=AB\frac{w}{B}\prod_{m=1}^{\infty}\left(1-\frac{w}{Ba_m}\right)^{\nu_m}$$

$$=\left\{AB\,X\prod_{m=1}^{\infty}\left(1-\frac{X}{a_m}\right)^{\nu_m}\right\}\circ\frac{w}{B}$$

$$=AB\,f\left(\frac{w}{B}\right).$$

Therefore by making use of two representations of F we have $AB=1$. This means that $f_1(g_1(z))$ is equivalent to $f(g(z))$.

If $\mu_1=1$ but $\mu_0\geqq 2$, then we have the equivalence of $f(g(z))$ and $f_1(g(z))$ similarly. For the case of $\varepsilon=0$, we have the same conclusion similarly.

Case 2). f_1 is transcendental but g_1 is not. If the degree of g_1 is larger than 1, then it is impossible to get only one zero b_l of F with a sufficiently large index l. Thus we have a contradiction.

Cases 3), 4), 5), 6). These cases do not occur as in § 8.

REFERENCES

[1] BIEBERBACH, L. Über eine Vertiefung des Picardschen Satzes bei ganzen Funktionen endlicher Ordnung. Math. Zeits. 3 (1919), 175-190.

[2] EDREI, A., Meromorphic functions with three radially distributed values. Trans. Amer. Math. Soc. 78 (1955), 276-293.

[3] GROSS, F., Factorization of entire functions which are periodic mod g. Indian Journ. pure applied Math. 2 (1971), 561-571.

[4] GROSS, F., Factorization of meromorphic functions. Math. Research Center, Washington D.C., 1972.

[5] NEVANLINNA, R., Le Théorème de Picard-Borel et la Théorie de fonctions méromorphes, Gauthier-Villars, Paris, 1929.

[6] OZAWA, M., Sufficient conditions for an entire function to be pseudo-prime. Kōdai Math. Sem. Rep. 27 (1976), 373-378.

[7] OZAWA, M., On a characterization of the exponential function and the cosine function by factorization. to appear.

[8] PÓLYA, G., On an integral function of an integral function. J. London Math. Soc. 1 (1926), 12-15.

[9] TSUZUKI, M., On the value distribution of entire functions of order less than one. J. Saitama Univ. 10 (1974), 1-4.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.