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ANALYTIC SELF-MAPPING INDUCING THE IDENTITY ON $H_1(W, Z/mZ)$

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1. Introduction

Let W be a compact Riemann surface of genus greater than one, and let f be a conformal automorphism of W. If f induces the identity on $H_1(W, Z)$, the first homology group on W, then f is the identity mapping. This was proved by Hurwitz [6]. Grothendieck and Serre [3] showed that if f induces the identity on $H_1(W, Z/mZ)$ for some integer $m \geqq 3$, then f is the identity mapping. Recently, Earle [2] has given a simple alternative proof of their result. In this paper, we shall extend it to open Riemann surfaces and we shall consider the case $m=2$. In addition, we discuss weak homological conditions briefly. The author is grateful to Professor Lipman Bers who suggested this problem to him and gave him various advice in preparing this paper, and also to Professor Jane Gilman who read the first draft and gave him available suggestions.

2. Main theorem

THEOREM 1. *Let W be a Riemann surface whose fundamental group is nonabelian. Let f be an analytic self-mapping of W. For each cycle γ on W, suppose that there is a cycle δ_γ and there is an integer $m_\gamma \geqq 2$, so that $f(\gamma)$ is homologous to $\gamma + m_\gamma \delta_\gamma$. Then, either f is the identity, or W can be represented as a two-sheeted covering of a simply connected Riemann surface and f as the interchanging of sheets. In the latter case f induces the identity on $H_1(W, Z/2Z)$.*

As corollaries to Theorem 1, we have

COROLLARY 1. *Let W be a Riemann surface whose fundamental group is nonabelian. Let f be an analytic self-mapping of W. If f induces the identity on $H_1(W, Z/mZ)$ for some integer $m \geqq 3$, then f is the identity mapping.*

COROLLARY 2. *Let W be a compact Riemann surface of genus greater than one. If f induces the identity on $H_1(W, Z/2Z)$ and f is not the identity, then W is hyperelliptic.*

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3. Preparatory lemmas.

To prove Theorem 1 we prepare some lemmas. Let W be a compact Riemann surface of genus greater than one. If f is an analytic self-mapping of W, then f is a conformal automorphism of finite period. Let $\langle f\rangle$ denote the cyclic group generated by f, and W_0 the quotient space $W/\langle f\rangle$. We can consider W_0 as a Riemann surface which has the natural conformal structure induced from W. If the period of f is prime, the points over the branch points of the projection π of W into W_0 are exactly the fixed points of f.

First we shall consider open Riemann surfaces.

LEMMA 1 (Heins [5]). *Let W be an open Riemann surface whose fundamental group is nonabelian, f an analytic self-mapping of W. If f is not an automorphism of finite period, then for every compact subset K of W, $f^n(K)$ tends to a point in W or an ideal boundary component uniformly as n tends to infinity. Here f^n denotes the n-th iteration of f.*

LEMMA 2. *Let W be an open Riemann surface of positive genus, f an analytic self-mapping of W. Suppose there are a nondividing cycle γ which is a simple closed curve on W, and a cycle δ and an integer $m\geqq 2$ so that $f(\gamma)$ is homologous to $\gamma+m\delta$. Then f is an automorphism of finite period.*

Proof. Obviously $f^n(\gamma)$ is homologous to $\gamma+m(\delta+f(\delta)+\cdots+f^{n-1}(\delta))$. Let γ' be a closed curve such that the intersection number of γ to γ' is one. Therefore, the intersection number of $f^n(\gamma)$ to γ' is not zero for each n. Hence, by Lemma 1, f is an automorphism of finite period.

LEMMA 3 (Maskit [8], Oikawa [9]). *Let W be a compact bordered Riemann surface, and f an automorphism of W. Then there is a compact Riemann surface $\hat{W}$ satisfying the following properties. (i) the genus of $\hat{W}$ is equal to that of W, (ii) W is conformally equivalent to a subregion of $\hat{W}$, and (iii) f induces the restriction of an automorphism f' of $\hat{W}$ to the subregion.*

LEMMA 4. *Let W, $\hat{W}$, f and f' be as in Lemma 3. Assume that the genus of W is positive and f' is the hyperelliptic involution so that $\hat{W}_0=\hat{W}/\langle f'\rangle$ is the sphere. Assume further that f induces the identity on $H_1(W, Z/2Z)$. Then W_0 is simply connected.*

Proof. The period of f is two and W_0 is of genus zero. Assume that W_0 has more than one boundary component. Choose two boundary components C and D. If $\pi^{-1}(C)$ has two components C_1 and C_2, then $f(C_1)=C_2$. But C_1-C_2 is neither homologous to zero nor homologous to twice a cycle. Assume that both $\pi^{-1}(C)$ and $\pi^{-1}(D)$ consist of one component. Take a simple closed curve E on W_0 which encircles C and another boundary component or a branch point, but does not encircle D such that $\pi^{-1}(E)$ consists of two components E_1 and E_2. Then $f(E_1)=E_2$. But E_1-E_2 is neither homologous to zero nor homologous to twice a cycle. This contradicts the fact that f is not the identity and f induces the identity on $H_1(W, Z/2Z)$. Thus W_0 is simply connected.

LEMMA 5. *Let W be a compact surface of genus greater than one. If f is an involution on W which induces the identity on $H_1(W, Z/2Z)$ and f is not the identity, then f is the hyperelliptic involution.*

Proof. This follows from the following fact due to Gilman [3]. There is a homology basis for W on which the action of f is given by the matrix

$$\begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix}$$

where G is the $g \times g$ matrix

$$\begin{pmatrix} 0 & I_{g_0} & 0 \\ I_{g_0} & 0 & 0 \\ 0 & 0 & -I_k \end{pmatrix}$$

Here I_t is the $t \times t$ identity matrix. If g is the genus of W, $g=2g_0+k$, f has $2(k+1)$ fixed points, and g_0 is the genus of W_0. If f induces the identity on $H_1(W, Z/2Z)$, $g_0=0$ and f has $2g+2$ fixed points.

LEMMA 6. *Let W be a compact surface of positive genus. If f is an involution on W and $W/\langle f\rangle$ is the sphere, then f does not induces the identity on $H_1(W, Z/mZ)$ for each $m \geqq 3$.*

Proof. Since W is two-sheeted covering of $W/\langle f\rangle$, there is a cycle γ such that $f(\gamma)$ is homologous to $-\gamma$. This implies the conclusion.

4. The proof of Theorem 1.

The proof of Theorem 1 consists of four steps.

Step one: We may assume that f is an automorphism of finite period.

Proof. This follows for the compact case from the remark at the beginning of section three and in the other case from lemma 2.

Step two: We can reduce the open case to the compact or compact bordered case.

Proof. The reduction procedure depends upon the following lemma which needs no proof.

LEMMA 7. *Let W_0 be $W/\langle f\rangle$ and π be the projection from W to W_0. Then W_0 can be exhausted by compact bordered subregions $W_{0,i}^*$ so that*

i) *$W_i^* = \pi^{-1}(W_{0,i}^*)$ is connected*

ii) *no component of* $W\text{-}W_i^*$ *or* $W_0\text{-}W_{0,i}^*$ *has compact closure*
iii) f *restricted to* W_i^* *satisfies the hypothesis of Theorem* 1.

If W is any open surface, let $\{W_{0,i}^*\}$ be an exhaustion for W_0 as described in Lemma 7. Let f_i be the restriction of f to W_i^*. We assume that Theorem 1 is true for compact bordered surfaces, so that for each i, either f_i is the identity or W_i^* is a two-sheeted covering of $W_{0,i}^*$ which is simply connected and f_i is the sheet interchange. In the latter case f_i induces the identity on $H_1(W_i^*, Z/2Z)$. Since $\{W_{0,i}^*\}$ is an exhaustion for W_0, $W_{0,i}^* \subset W_{0,(i+1)}^*$ and $W_0 = \cup W_{0,i}^*$. If f_i is the identity for all i, then f is the identity and we are done. Let j be the smallest integer for which f_i is not the identity. Then f_i induces the identity on $H_1(W_i^*, Z/2Z)$ for all i greater than or equal to j. Thus f induces the identity on $H_1(W, Z/2Z)$ since $\{W_i^*\}$ is an exhaustion for W. If f_i is not the identity, then f_i is the sheet interchange and $W_{0,i}^*$ is simply connected. Thus $W_{0,i}^*$ is simply connected for all i greater than or equal to j. Therefore, $W_0 = \cup W_{0,i}^*$ is simply connected, and the theorem is proved.

From now on it is assumed that W is compact or compact bordered, that $\pi_1(W)$ is non-abelian, and that f is an automorphism of finite period satisfying the hypothesis of Theorem 1.

Step three: We may assume that there is an integer $m \geqq 2$ such that f induces the identity on $H_1(W, Z/mZ)$.

Proof. By assumption on W, $H_1(W, Z) = Z^n$ for some $n \geqq 2$. Let $\phi(\gamma) = f(\gamma) - \gamma$. Then $\phi(Z^n)$ is a subgroup of Z^n, so Z^n has a basis $x_1, x_2, \cdots, x_n$ such that $\phi(Z^n)$ is generated by $d_1x_1, \cdots, d_nx_n$ where $d_j \geqq 0$ and d_j divides d_{j+1} for all j. It suffices to show that $d_1 \neq 1$. If $d_1 = 1$, then $x_1 = \phi(\gamma) = f(\gamma) - \gamma$ for some cycle γ, so by hypothesis $x_1 = m'w$ for some $w = \sum C_j x_j$ in Z^n and some $m' \geqq 2$. This is a contradiction.

Step four: The proof has been reduced to the following cases which we treat separately.

1. W is a compact surface of genus at least two.
2. W is a compact bordered surface of genus at least two.
3. W is a compact bordered surface of genus one and W_0 is of genus one.
4. W is a compact bordered surface of genus one and W_0 is of genus zero.
5. W is the sphere with n open discs removed.

Case 1. If f induces the identity on $H_1(W, Z/mZ)$ and $m \geqq 3$, then f is the identity by the results of [2] or [4]. Thus we may assume that f induces the identity on $H_1(W, Z/2Z)$. It follows that f^2 induces the identity on $H_1(W, Z/4Z)$. Thus f^2 is the identity. By Lemma 5 f is the hyperelliptic involution and W_0 is the sphere.

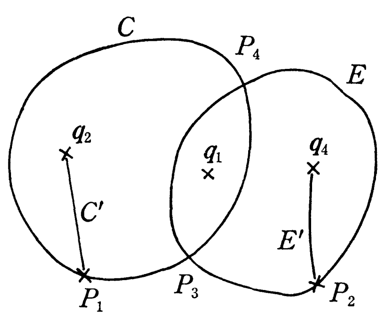
Case 2. Let $\hat{W}$ and f' be as in Lemma 3. Then either f' is the identity so that f is also the identity or f' is the hyperelliptic involution by Case 1.

In the latter case, we may assume that $m=2$ and apply Lemma 4 to obtain the theorem.

Case 3. Let $\hat{W}$ and f' be as in Lemma 3. To simplify notation we assume $W\subset\hat{W}$. Since W_0 is of genus one, $\hat{W}_0=\hat{W}/\langle f'\rangle$ is also. Let $\hat{\pi}$ be the projection. From the Riemann-Hurwitz relation we can conclude that $\hat{W}$ is a smooth n-sheeted covering of $\hat{W}_0$. It is not difficult to show that under these circumstances there is a simple closed curve C on W_0 such that $\hat{\pi}^{-1}(C)$ is n distinct curves which divide $\hat{W}$ into n disjoint copies of $\hat{W}_0$-C. (This is analogous to a lemma of Accola [1] for the case $g\geqq 2$.) Since we may assume that C is contained in W_0, we can consider that $\pi^{-1}(C)$ divides W into n copies of W_0-C. Let γ_1 and γ_2 be two components of $\pi^{-1}(C)$ so that $f(\gamma_1)=\gamma_2$. Then $\gamma_1-\gamma_2$ is homologous to lifting of the sum of the boundary curves of W_0 to the component of $W-\pi^{-1}(C)$ between γ_1 and γ_2. Each boundary curve occurs exactly once in this sum. Thus $\gamma_1-\gamma_2$ cannot be homologous to m times a cycle for $m\geqq 2$. Thus, f is the identity.

Case 4. We let $\hat{W}$ and f' be as in Lemma 3 and we assume that $W\subset\hat{W}$. We will show that under this assumption f' is the hyperelliptic involution and then apply Lemma 4 to complete the proof for this case.

Firstly, assume that the period of f' is prime. Since $\hat{W}$ is a surface of genus one and $\hat{W}_0=\hat{W}/\langle f'\rangle$ has genus zero, the Riemann-Hurwitz relation tell us that the covering $\hat{\pi}:\hat{W}\to\hat{W}_0$ has at least three branch points. Let q_1, q_2, q_3 be three arbitrary branch points. Take two points p_1, p_2 which are not branch points of $\hat{\pi}$. Let C be a simple closed curve which starts and ends at p_1, and encircles no branch point but q_1 and q_2. Let E be a simple closed curve which starts and ends at p_2, and encircles no branch point but q_1 and q_3. Assume that no branch point is on C and E and that C and E have only two points p_3, p_4 common. Let C' be a path which connects from p_1 to q_2 and E' a path which connects from p_2 to q_3. C' is disjoint from E and E'. E' is disjoint from C and C' (figure 1).



Since p_k is not a branch point, $\hat{\pi}^{-1}(p_k)$ are n distinct points where n is the period of f'. Let $p_k^1,\cdots,p_k^n$ be these points. Let γ and ε be the analytic continuations of $\hat{\pi}^{-1}$ along C and E from p_1^1 and p_2^1, respectively. Let p_1^i and p_2^j be their terminal points, respectively. Let γ_i and ε_j be the analytic continuations of $\hat{\pi}^{-1}$ along C' and E' from p_1^i and p_2^j, respectively. Then there are i and j such that $\gamma+\gamma_i-\gamma_1$ and $\varepsilon+\varepsilon_j-\varepsilon_1$ are closed curves. We may assume that $\gamma+\gamma_i-\gamma_1$ intersect to $\varepsilon+\varepsilon_j-\varepsilon_1$ at a point over p_3. Since q_1 is a branch point, they do not have any common point over p_4. Therefore the intersection number of $\gamma+\gamma_i-\gamma_1$ to $\varepsilon+\varepsilon_j-\varepsilon_1$ is 1 or -1. Since p_3 is not a branch point, $f'(\gamma+\gamma_i-\gamma_1)$ and $\varepsilon+\varepsilon_j-\varepsilon_1$ do not have any common point over p_3 but may have a common point over p_4. If the period of f' is greater than two and $f'(\gamma+\gamma_i-\gamma_1)$ intersects to $\varepsilon+\varepsilon_j-\varepsilon_1$ at a point over p_4, then $f'^2(\gamma+\gamma_i-\gamma_1)$ and $\varepsilon+\varepsilon_j-\varepsilon_1$ have no common point over p_3 and p_4. Hence, $\varepsilon+\varepsilon_j-\varepsilon_1$ is disjoint from $f'(\gamma+\gamma_i-\gamma_1)$ or $f'^2(\gamma+\gamma_i-\gamma_1)$. This is a contradiction.

Secondly, suppose that the period of f' is not prime, say n. If there is a prime factor $n_1 \geqq 3$, then the period of $f''=f'^{n/n_1}$ is n_1 and f'' induces the identity on $H_1(W, Z/m'Z)$ for some $m' \geqq 3$. This is a contradiction. If $n=2^k$, for some $k \geqq 2$, then $W/\langle f'^{n/2} \rangle$ is of genus zero by Case 3. Since $f'^{n/2}$ induces the identity on $H_1(W, Z/m'Z)$ for some $m' \geqq 3$, the period of $f'^{n/2}$ is not two by Lemma 6. This is a contradiction.

Thus the period of f' is two. Therefore, f' has four fixed points and is the hyperelliptic involution. This says $m=2$. Apply Lemma 4.

Case 5. Since the fundamental group of W is nonabelian, the number of boundary components is more than two. By step one, f is an automorphism of finite period. By virtue of Lemma 3, there are $\hat{W}$ and f' and we may assume that $W \subset \hat{W}$. Since $\hat{W}$ is the sphere, f' is an elliptic linear transformation. Hence, f' has two fixed points. Therefore, the number of invariant boundary components under f is at most two. But W has more than two boundary components. This contradicts the hypothesis of the theorem. Hence, f is the identity.

5. Weak homological conditions.

If we use "weak homology" in stead of "homology", then the following can be said.

THEOREM 2. *Let W be of genus greater than one, f an automorphism of W. For each cycle γ on W, suppose there is a cycle δ_γ and there is an integer $m_\gamma \geqq 2$, so that $f(\gamma)$ is weakly homologous to $\gamma+m_\gamma\delta_\gamma$. Then, either f is the identity, or W can be represented as a two-sheeted covering of a Riemann surface of genus zero (possibly not simply connected) and f as the interchanging of sheets.*

We can prove this theorem by the same method as in the proof of Theorem 1 if we treat step 2 of section 4 carefully.

Weak homological criteria for analytic self-mappings reducing to automorphisms are discussed in Kato [7]. The typical one is: "If an analytic self-mapping f carries dividing cycles to dividing cycles and if there is a nondividing cycle γ such that $f(\gamma)$ is weakly homologous to γ, then f is an automorphism of finite period". Treating the proof of this fact and the proof of Lemma 2, we can replace that "f is an automorphism" in the hypothesis in Theorem 2 by that "f is an analytic self-mapping which carries dividing cycles to dividing cycles".

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