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ON THE ZERO-ONE SET OF AN ENTIRE FUNCTION

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1. Introduction. Let $\{a_n\}$ and $\{b_n\}$ be two disjoint infinite sequences with no finite limit points. If it is possible to construct an entire function f whose zero sequence is exactly $\{a_n\}$ and whose one sequence is exactly $\{b_n\}$, the pair $(\{a_n\}, \{b_n\})$ is called the zero-one set of f. In general an arbitrary pair of two sequences $\{a_n\}$, $\{b_n\}$ is not a zero-one set of any entire function. This was recently proved by Rubel and Yang [5] explicitly. On giving $\{a_n\}$ they constructed $\{b_n\}$ in a very skillful but artificial manner. It seems to the present author that their $\{b_n\}$ has less arbitrariness in a sense and has too much arbitrariness in the other sense. In this paper we shall discuss the following problem: How can it be arbitrary? Our answer is given in the following.

THEOREM 1. *Let $\{a_n\}$ and $\{b_n\}$ be two arbitrary disjoint infinite sequences with no finite limit points. Let b_1 be different from b_2. Then one of the following three pairs*

$$(\{a_n\}, \{b_n\}_{n=1}^{\infty}), (\{a_n\}, \{b_n\}_{n=2}^{\infty}), (\{a_n\}, \{b_n\}_{n=3}^{\infty} \cup \{b_1\})$$

is not a zero-one set of any entire function.

THEOREM 2. *Suppose that $(\{a_n\}, \{b_n\})$ is the zero-one set of an entire function $N(z)$ of finite non-integral order. Then $(\{a_n\}, \{b_n\}_{n=2}^{\infty})$ is not a zero-one set of any entire function.*

We shall give an example showing that two pairs are really zero-one sets in Theorem 1 and that the finite nonintegrity assumption cannot be omitted in Theorem 2. We shall give other several examples being connected with closely related problems. Our method of proof depends also upon the impossibility of the Borel identity, which had been stated in several ways. See [1], [2], [3], [4]. Theorem 1 corresponds to the so-called three function theorem.

2. Proof of Theorem 1. Suppose that all of the given pairs are zero-one sets. Then there are entire functions N, f and g satisfying

1) $f=Ne^{\alpha}$, $(f-1)(z-b_1)=(N-1)e^{\beta}$,

$g=Ne^{\gamma}$, $(g-1)(z-b_2)=(N-1)e^{\delta}$,

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where α, β, γ and δ are entire functions. Hence

2) $N=\dfrac{z-b_1-e^{\beta}}{e^{\alpha}(z-b_1)-e^{\beta}}$

and

3) $(z-b_1)(z-b_2)(e^{\gamma}-e^{\alpha})+(z-b_2)(e^{\beta}-e^{\beta+\gamma})-(z-b_1)(e^{\delta}-e^{\alpha+\delta})=0$.

Assume that α is a constant c. Since $f(b_n)=1$ for $n\geqq 2$ and $N(b_n)=1$, $e^c=1$, that is, $f=N$, which is absurd by $f(b_1)\neq 1$, $N(b_1)=1$. Hence α is not a constant. Similarly γ is not a constant.

Assume that β is a constant c. Then, by

$$(f(a_n)-1)(a_n-b_1)=(N(a_n)-1)e^{\beta},$$

$$a_n-b_1=e^c.$$

This is impossible. Hence β is not a constant. Similarly neither δ is.

Assume that $\alpha-\gamma$ is a constant c. Then $f=Ne^ce^{\gamma}=e^cg$. Let us put $z=b_n$, $n\geqq 3$. Then $f(b_n)=g(b_n)=1$ and hence $e^c=1$ i. e. $f=g$. Let us put $z=b_1$. Then $f(b_1)\neq 1$, $g(b_1)=1$. This is absurd. Hence $\alpha-\gamma$ is not a constant. Assume that $\beta-\delta$ is a constant c. Then $(f-1)(z-b_1)$ is equal to $e^c(g-1)(z-b_2)$. Let us put $z=a_n$. Then $f(a_n)=g(a_n)=0$ and hence $a_n-b_1=e^c(a_n-b_2)$. If $e^c\neq 1$, then $a_n=(b_1-e^cb_2)/(1-e^c)$, which is impossible. If $e^c=1$, then $b_1=b_2$, which is again absurd. Hence $\beta-\delta$ is not a constant.

Now we recall a form of the impossibility of Borel's identity. Let P_j be polynomials, $P_j\not\equiv 0$ and let g_j be transcendental entire functions satisfying

$$\sum_{j=1}^{p} P_jg_j=P_0.$$

Then

$$\sum_{j=1}^{p}\delta(0,\ g_j)\leqq p-1.$$

This form was already stated in [4] in a simpler form. We shall not any proof of the above result, since its proof is similar as in [4]. Our equation has the following form: By 3)

4) $P_2P_3e^{\gamma-\alpha}+P_2e^{\beta-\alpha}-P_2e^{\beta+\gamma-\alpha}-P_3e^{\delta-\alpha}+P_3e^{\delta}=P_2P_3$,

$P_2=z-b_2,\ P_3=z-b_1$.

If $e^{\beta-\alpha}$, $e^{\beta+\gamma-\alpha}$ and $e^{\delta-\alpha}$ are transcendental, we have

$$\delta(0,\ e^{\gamma-\alpha})+\delta(0,\ e^{\beta-\alpha})+\delta(0,\ e^{\beta+\gamma-\alpha})+\delta(0,\ e^{\delta-\alpha})+\delta(0,\ e^{\delta})\leqq 4.$$

But the left-hand side is equal to 5, which is clearly impossible.

If $e^{\beta-\alpha}$ is a constant $c\neq 0$, we have

5) $P_2P_3e^{\gamma-\alpha}-cP_2e^{\gamma}-P_3e^{\delta-\alpha}+P_3e^{\delta}=P_2P_3-cP_2\not\equiv 0$.

Since $\beta-\delta\neq$constant, $\delta-\alpha$ is not a constant in this case. Hence

$$4=\delta(0,\ e^{\gamma-\alpha})+\delta(0,\ e^{\gamma})+\delta(0,\ e^{\delta-\alpha})+\delta(0,\ e^{\delta})\leqq 3,$$

which is a contradiction. If $e^{\beta+\gamma-\alpha}$ is a constant $c\neq 0$, we have

6) $P_2P_3e^{\gamma-\alpha}+cP_2e^{-\gamma}-P_3e^{\delta-\alpha}+P_3e^{\delta}=P_2P_3+cP_2\not\equiv 0$.

If further $e^{\delta-\alpha}$ is a constant $d\neq 0$, we have

7) $P_2P_3e^{\gamma-\alpha}+cP_2e^{-\gamma}+dP_3e^{\alpha}=P_2P_3+cP_2+dP_3\not\equiv 0$.

This is again impossible. Hence $e^{\delta-\alpha}$ is not a constant. Then by 6) we have again a contradiction. If $e^{\delta-\alpha}$ is a constant $c\neq 0$, we have

8) $P_2P_3e^{\gamma-\alpha}+P_2e^{\beta-\alpha}-P_2e^{\beta+\gamma-\alpha}+cP_3e^{\alpha}=P_2P_3+cP_3\not\equiv 0$,

which gives a contradiction similarly.

In the above proof the fact that there is no linear relation among P_2P_3, P_2, P_3 is very important. If we only make use of the classical form of the impossibility of Borel's identity, we need a very lengthy but almost trivial proof.

3. Proof of Theorem 2. By the well-known Borel theorem [3] it is easy to show that the order ρ_N of N satisfies

$$\rho_N > \rho_{exp\beta},$$

when we put

$$f=Ne^{\alpha},\ (f-1)(z-b_1)=(N-1)e^{\beta}.$$

By 2) in the proof of Theorem 1

$$N(r,\ 0,\ N)=N(r,\ 0,\ z-b_1-e^{\beta})-N(r,\ 0,\ z-b_1-e^{\beta-\alpha})$$
$$\leqq N(r,\ 0,\ z-b_1-e^{\beta})\leqq m(r,\ e^{\beta})(1+o(1)).$$

Hence

$$\rho_N=\rho_{N(r,0,N)}\leqq \rho_{exp\beta}.$$

This is absurd.

4. Examples and remarks.

(a) Let us consider

$$N(z)=\frac{z+c-ce^z}{ze^z},\ c\neq 0$$

and

$$f(z)=\frac{z+c-ce^z}{z}.$$

Then $f=Ne^z$, $(f-1)(z+c)=c(N-1)e^z$. This example and an example in (c) show that two pairs are really zero-one sets in Theorem 1 and that the non-integrity assumption in Theorem 2 cannot be omitted.

(b) $(\{a_n\},\ \{b_n\}_{n=n_0}^{\infty})$ may be a zero-one set if $(\{a_n\},\ \{b_n\}_{n=1}^{\infty})$ is. This is shown by

$$N(z)=\frac{z^2-2\pi iz+c-ce^z}{e^z(z^2-2\pi iz)}, \quad c\neq 0$$

and $f(z)=N(z)e^z$. Indeed

$$(f-1)(z^2-2\pi iz+c)=c(N-1)e^z .$$

It is always possible to construct an example having the above property, although we only have showed an example in the case $n_0=3$. Further infinitely many $\{b_{\nu_j}\}$ may be omitted from $\{b_n\}$. This is shown by

$$N=\frac{e^c e^{3z}+e^c e^{2z}+e^c e^z+e^c-1}{(e^c-1)e^{3z}}, \quad e^c\neq 1$$

and $f=Ne^{3z}$. Then

$$(f-1)(e^z-1+e^c)=(N-1)e^c e^{4z} .$$

If we do not persist in the case of order one, then we can construct examples by composition such as

$$N(P(z))=\frac{P(z)+c-ce^{P(z)}}{P(z)e^{P(z)}}, \quad c\neq 0 ,$$

$$f(P(z))=N(P(z))e^{P(z)} .$$

We may adopt $P(z)$ as a polynomial or an entire function.

(c) Let $N(z)$ be

$$\frac{z-b_1-(e^c z-b_1)e^z}{(1-e^c)ze^z}, \quad e^c\neq 1,\ b_1\neq 0$$

and $f(z)=N(z)e^z$. Then

$$(f-1)(z-b_1)=(e^c z-b_1)(N-1)e^z .$$

This shows that $(\{a_n\},\ \{b_n\}_{n=2}^{\infty}\cup\{b_1e^{-c}\}$ may be a zero-one set if $(\{a_n\},\ \{b_n\})$ is. This means that the zero-one setness may be preserved even if b_1 moves continuously to b_1e^{-c}.

(d) We can prove several variants of Theorems 1 and 2 based upon the above observations. We shall not discuss them.

5. Two supplements. We shall discuss the following problem: What effects can we expect when $\{b_n\}$ is a nonempty finite sequence in our earlier problem? Let n run from 1 to m. Assume that there exist two entire functions N and f such that

$$f=Ne^{\alpha},\ (f-1)(z-b_1)=(N-1)e^{\beta}$$

and

$$N-1=P(z)e^{\gamma} ,$$

where α, β, γ are entire functions and P is $c(z-b_1)\cdots(z-b_m)$. Then we have

$$N=\frac{z-b_1-e^{\beta}}{e^{\alpha}(z-b_1)-e^{\beta}}, \quad N-1=\frac{(z-b_1)(1-e^{\alpha})}{e^{\alpha}(z-b_1)-e^{\beta}}$$

and

$$P(z)e^{\gamma+\alpha}-\frac{P(z)}{z-b_1}e^{\gamma+\beta}+e^{\alpha}=1\,.$$

It is very easy to prove the non-constancy of α, β and γ. If $\gamma+\alpha$, $\gamma+\beta$, $\beta-\alpha$, $\gamma+\beta-\alpha$ are not constants, we have immediately a contradiction. If some of $\gamma+\alpha$, $\gamma+\beta$, $\beta-\alpha$, $\gamma+\beta-\alpha$ is a constant, we can reduce the above equation to an easier one, which gives immediately a contradiction. Hence we have the following fact: One of the following pairs

$$(\{a_n\},\ \{b_n\}_{n=1}^{m}),\ (\{a_n\},\ \{b_n\}_{n=2}^{m})$$

is not a zero-one set of any entire function.

Next assume that $\{a_n\}$ is a finite set. Then $\{b_n\}$ should be an infinite set if it is the set of one-points of an entire transcendental function. Let N and f be entire functions satisfying

$$f=Ne^{\alpha},\ (f-1)(z-b_1)=(N-1)e^{\beta}$$

and

$$N=Pe^{\gamma}\,,$$

where α, β and γ are entire functions and P is $c(z-a_1)\cdots(z-a_m)$. Similarly we have a contradiction. Hence one of the following pairs

$$(\{a_n\}_{n=1}^{m},\ \{b_n\}_{n=1}^{\infty}),\ (\{a_n\}_{n=1}^{m},\ \{b_n\}_{n=2}^{\infty})$$

is not a zero-one set of any entire function.

The second supplement is the following fact: Let us call a zero-one set $(\{a_n\},\ \{b_n\})$ unigue whenever there is only one entire function whose zero-one set is just the given pair. With this definition of unicity the fact that the given zero-one set is not unique implies that $(\{a_n\},\ \{b_n\}_{n=n_0}^{\infty})(n_0\geqq 2)$ is not a zero-one set.

Assume that $(\{a_n\},\ \{b_n\}_{n=n_0}^{\infty})$ is the zero-one set of an entire function g and that $(\{a_n\},\ \{b_n\}_{n=1}^{\infty})$ is not unique, we have

$$f=Ne^{\alpha},\ f-1=(N-1)e^{\beta}\,,$$

$$g=Ne^{\gamma},\ (g-1)P=(N-1)e^{\delta}$$

with entire α, β, γ, δ and a non-constant polynomial $P=c(z-b_1)\cdots(z-b_{n_0-1})$. It is not difficult to prove the non-constancy of α, β, γ, δ, $\alpha\pm\gamma$, $\beta\pm\delta$. Eliminating f, g and N, we have

9) $P(e^{\gamma}-e^{\beta+\gamma}+e^{\beta}-e^{\alpha})=e^{\delta}-e^{\alpha+\delta}\,.$

This equation is impossible, unless either $e^{\alpha}=e^{\beta}=1$ or $e^{\alpha}=e^{\gamma}=1$. However these cases have already been excluded. In order to conclude something from 9) we need discussions done in the proof of Theorem 1. We shall omit this. Thus we have the desired result.

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