# ON MEROMORPHIC FUNCTIONS TAKING THE SAME
# VALUES AT THE SAME POINTS

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**Introduction.** It was shown in [1] that if two non-constant polynomials $p(z)$ and $q(z)$ in the complex plane have the same set of zeros and the same set of preimages of 1 (without counting the multiplicity), then $p(z)\equiv q(z)$. Clearly it is not true for the class of all meromorphic functions. For instance, functions $e^z$, $e^{-z}$ have the same preimages of values 0, 1, and $-1$. In this note we shall study the forms of two functions in a restricted class of transcendental meromorphic functions when they have the same preimage sets of 1 and 0. The class of meromorphic functions $F$ which we are going to consider are all the transcendental meromorphic functions $f$ of the form:

$$f(z)=\mu_1(z)e^{\alpha(z)}+\mu_2(z)\,,$$

where $\alpha$ is an entire function of finite order, and $\mu_1$ and $\mu_2$ are meromorphic functions with their orders $\rho_{\mu_1}$ and $\rho_{\mu_2}$, respectively, less than that of $e^\alpha$ ($\mu_1(z)\not\equiv 0$, $\mu_2(z)\not\equiv$ constant). More precisely, we assume that $\rho_{\mu_1}$ and $\rho_{\mu_2}$ are less than $k$ if $\alpha(z)$ is a polynomial of degree $k$, and $\rho_{\mu_1}$ and $\rho_{\mu_2}$ can be any finite quantities if $\alpha(z)$ is a transcendental entire function (since in this case the order of $e^\alpha$ is infinite (see [2, p. 53])). We shall call a meromorphic function $\mu(z)$ a deficient function of $f(z)$ if and only if the following two conditions are satisfied:

(A) $$T(r, \mu(z))=o\{T(r, f(z)\}\,, \quad \text{as } r\to\infty$$

where $T(r, g)$ denotes the Nevanlinna characteristic function of $g$

(B) $$\delta(0, f-\mu)>0$$

where $\delta(0, h)$ is the Nevanlinna deficiency of the function $h$ at value 0.

Then we have the following lemma which is a particular case of a result of Nevanlinna (see [4, p. 76]).

LEMMA 1. *Let $h(z)$ be a meromorphic function and $\alpha(z)$ an entire function. Suppose that the following condition is held:*

(C) $$T(r, h)=o(T(r, e^\alpha))\,, \quad \text{as } r\to\infty\,.$$

*Then the function $h(z)e^{\alpha(z)}$ has no other deficient functions than two constant functions $0$ and $\infty$.*

*Remark*: For any $f \in F$ we have $N(r, (1/F)) \sim (1+o(1))T(r, f)$, as $r \to 0$. As an immediate application of this lemma we have the following theorem which will be used repeatedly in the proof of our results.

THEOREM 1. *Let $h_1(z)$ and $h_2(z)$ ($\not\equiv$ constant) be two meromorphic functions of finite order satisfying condition (C) and $\alpha(z)$ be an entire function$\not\equiv$constant. If the exponent of convergence of zeros of $h_1e^{\alpha}+h_2$ is less than the order of $e^{\alpha}$, then we must have $h_1 \equiv 0$.*

We shall also need the following result essentially due to G. Hiromi and M. Ozawa [3].

LEMMA 2. *Let $a_0(z)$, $a_1(z)$, $\cdots$, $a_n(z)$ and $g_1(z)$, $g_2(z)$, $\cdots$, $g_n(z)$ be entire functions. Suppose that*

(1) $$T(r, a_j(z)) = o(\sum_{i=1}^{n} T(r, e^{g_i})), \qquad j=0, 1, 2, \cdots, n.$$

*If the identity*

$$\sum_{i=1}^{n} a_i(z)e^{g_i(z)} = a_0(z)$$

*holds, then there is an identity*

$$\sum_{i=1}^{n} c_i a_i(z)e^{g_i(z)} = 0,$$

*where $c_i$, $i=1, 2, \cdots, n$ are constants that are not all zero.*

We now proceed to state and prove our results.

THEOREM 2. *Suppose that $f \in F$, $g \in F$, and that $f$ and $g$ have the same preimage sets of the values $c_1$ and $c_2$ with $c_1 \neq c_2$ respectively. Then either $f \equiv g$ or*

$$f(z) = \frac{c_2 - c_1\lambda(z)}{1-\lambda(z)} - \frac{(c_1-c_2)^2\lambda(z)}{1-\lambda(z)} \frac{1}{h(z)e^{\alpha(z)}} \tag{i}$$

and

$$g(z) = \frac{c_1 - c_2\lambda(z)}{1-\lambda(z)} + \frac{h(z)e^{\alpha(z)}}{1-\lambda(z)} \tag{ii},$$

*where $\lambda(z)$ is a meromorphic function of finite order satisfying $T(r, \lambda) = o(T(r, g))$ or $o(T(r, e^{\alpha}))$ and $\alpha$ is an entire function, $h$ is a meromorphic function of finite order satisfying*

$$T\left(r, \frac{1}{h}\right) = o(T(r, g)) \qquad \text{as } r \to \infty.$$

*Proof.* First of all, for each $f \in F$, we have $f(z) = \mu_1(z)e^{\alpha(z)} + \mu_2(z)$,

(2)            $T(r, \mu_1)+T(r, \mu_2)=o(T(r, e^\alpha))$,      as $r \to \infty$.

Next we are going to show that, for any constant $c$ and any $f \in F$, the exponent of convergence of the multiple roots of the equation $f-c=0$ is finite. It is clear that the multiple roots of the equation $f-c=0$ are the common roots of the following system of equations:

(3)
$$\mu_1(z)e^{\alpha(z)}+\mu_2(z)-c=0,$$

$$[\mu_1'(z)+\mu_1(z)\alpha'(z)]e^{\alpha(z)}+\mu_2'(z)=0$$

Here we note that the miltiplicities of these common roots are one less than those of the multiple roots of the equation $f(z)-c=0$. It is easy to see, from the above system equation, that the multiple roots of the equation $f(z)-c=0$ are no other than those of the roots of the following equation:

(4)
$$\frac{\mu_1(z)}{\mu_2(z)-c}-\frac{\mu_1'(z)+\mu_1(z)\alpha'(z)}{\mu_2'(z)}=0.$$

This cannot be an identity. For otherwise, by rewriting the equation we would have

(5)
$$\frac{\mu_2'(z)}{\mu_2(z)-c}=\frac{\mu_1'(z)+\mu_1(z)\alpha'(z)}{\mu_1'(z)}$$

or

(6)
$$\frac{\mu_2'(z)}{\mu_2(z)-c}=\frac{\mu_1'(z)}{\mu_1(z)}+\alpha'(z).$$

After integration we get

(7)
$$e^{\alpha(z)} \equiv a\frac{\mu_2(z)-c}{\mu_1(z)},$$

where $a$ is a constant. This will violate the inequality (2). Therefore, we conclude that equation (4) is not an identity and hence the zeros (counting multiplicity) of equation (4) coincide with the zeros of the following function

(8)        $A(z) \equiv (\mu_2(z)-c)(\mu_1'(z)+\mu_1(z)\alpha'(z))-\mu_1(z)\mu_2'(z)$.

Since all the orders of $\mu_2$ (so is $\mu_2'$), $\mu_1$ (so is $\mu_1'$) and $\alpha$ (so is $\alpha'$) are finite, it follows that the order of $A(z)$ is also finite. This also proves the assertion that the exponents of convergence of the multiple roots of the equation $f-c=0$ is finite. Now let $g(z)=\lambda_1(z)e^{\beta(z)}+\lambda_2(z)$. Then the above analysis indicates that both the exponent of convergence of the multiple roots of the equations $f-c=0$ and $g-c=0$ are finite. From this and noting that the exponents of the poles of $f$ and $g$ are finite, we have

(9)
$$\frac{f(z)-c_1}{g(z)-c_1}=\frac{\mu_1 e^\alpha+\mu_2-c_1}{\lambda_1 e^\beta+\lambda_2-c_1}=h_1 e^{\gamma_1}$$

and

(10) 
$$\frac{f(z)-c_2}{g(z)-c_2}=\frac{\mu_1 e^\alpha+\mu_2-c_2}{\lambda_1 e^\beta+\lambda_2-c_2}=h_2 e^{\gamma_2}$$

where $h_1$ and $h_2$ are meromorphic functions of finite order and $\gamma_1$ and $\gamma_2$ are two entire functions.

Here we note that the functions $\alpha$ and $\beta$ are either polynomials of the same degree or transcendental entire functions of the same order. In fact, they satisfy $T(r, e^\alpha) \sim T(r, e^\beta)$ or equivalently $T(r, f) \sim T(r, g)$.

Equations (9) and (10) can be rewritten as

(11) 
$$(\lambda_1 e^\beta+\lambda_2-c_1)h_1 e^{\gamma_1}=\mu_1 e^\alpha+\mu_2-c_1$$

and

(12) 
$$(\lambda_1 e^\beta+\lambda_2-c_2)h_2 e^{\gamma_2}=\mu_1 e^\alpha+\mu_2-c_2,$$

respectively. Subtracting (12) from (11) we obtain

(13) 
$$c_2-c_1=\lambda_1 e^\beta(h_1 e^{\gamma_1}-h_2 e^{\gamma_2})+h_1(\lambda_2-c_1)e^{\gamma_1}-h_2(\lambda_2-c_2)e^{\gamma_2}$$

or

(14) 
$$c_2-c_1=h_1\lambda_1 c^{\beta+\gamma_1}+h_1(\lambda_2-c_1)e^{\gamma_1}-h_2\lambda_1 e^{\beta+\gamma_2}-h_2(\lambda_2-c_2)e^{\gamma_2}.$$

Since the order of the porduct of two meromorphic functions is less than or equal to the larger order of the two functions, it follows that the orders of the functions $h_1\lambda_1$, $h_1(\lambda_2-c_2)$, $h_1(\lambda_2-c_1)$, and $h_2\lambda_1$ are finite. Going back to identity (14), we shall treat the following two cases separately:

Case (i)   $\beta$ is transcendent entire.

Case (ii)   $\beta$ is a polynomial.

Suppose that Case (i) occurs. In this case the following two subcases may arise:

Subcase (i1) all the exponentials of equation (14) are polynomials.

Subcase (i2) not all the exponentials of equation (14) are polynomials.

Clearly, subcase (i1) is ruled out right away, because it means that $\beta+\gamma_1$, $\gamma_1$, $\beta+\gamma_2$, and $\gamma_2$ are all polynomials. It follows that $\beta$ itself must be a polynomial, giving a contradiction. Suppose that subcase (i2) holds. Then Lemma 2 is applicable, therefore there exist constants $a_1$, $a_2$, $a_3$, and $a_4$ that are not all zero such that

(15) 
$$a_1 h_1\lambda_1 e^{\beta+\gamma_1}+a_2 h_1(\lambda_2-c_1)e^{\gamma_1}+a_3 h_2\lambda_1 e^{\beta+\gamma_2}+a_4 h_2(\lambda_2-c_2)e^{\gamma_2}\equiv 0.$$

Next we assume that one of $\{a_i\}^4_{i=1}$, is not zero. (The case $a_1=0$ will be treated later.) Then from above we have

(16) 
$$a_1\equiv -\frac{a_2(\lambda_2-c_1)}{\lambda_1}e^{-\beta}-\frac{a_3 h_2}{h_1}e^{\gamma_2-\gamma_1}-\frac{a_4 h_2(\lambda_2-c_2)}{h_1\lambda_1}e^{\gamma_2-\gamma_1-\beta}.$$

We shall consider subcase (i) $a_2=0$ and subcase (ii) $a_2\neq 0$ separately. Suppose subcase (i) holds, that is $a_1\neq 0$ and $a_2=0$. Then identity (16) becomes

(17) $$a_1 \equiv e^{\gamma_2 - \gamma_1} \left[ -e^{-\beta} \frac{a_4 h_2 (\lambda_2 - c_2)}{h_1 \lambda_1} - \frac{a_3 h_2}{h_1} \right].$$

Since $e^{-\beta}$ is of infinite order while $(a_4 h_2 (\lambda_2 - c_2)/h_1 \lambda_1)$ and $a_3 (h_2 / h_1)$ are meromorphic functions of finite order by virtue of Theorem 1, we have to conclude that $a_4 = 0$. (Hence $a_3 \neq 0$, otherwise it would lead to the conclusion that $a_1 = 0$, a contradiction). Thus we have

(18) $$-\frac{a_1}{a_3} = e^{\gamma_2 - \gamma_1} \frac{h_2}{h_1}.$$

From this and taking the quotient of two expressions (9) and (10) we get

(19) $$\frac{f - c_2}{g - c_2} = a \frac{f - c_1}{g - c_1},$$

where $a = -(a_1/a_3)$. Thus we have the following identity:

(20) $$fg - c_2 g - c_1 f + c_1 c_2 \equiv afg - ac_1 g - ac_2 f + ac_1 c_2$$

or

(21) $$(1 - a)fg - (c_1 + ac_2)f = (c_2 - ac_1)g + (a - 1)c_1 c_2.$$

Hence

(22) $$f[(1 - a)g - (c_1 - ac_2)] = (c_2 - ac_1)g + (a - 1)c_1 c_2.$$

We claim $a = 1$. If $a \neq 1$, then from the above identity we have

(23) $$N\left(r, \frac{1}{f[(1-a)g-(c_1-ac_2)]}\right) = N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{(1-a)g-(c_1-ac_2)}\right)$$
$$\sim (1 - o(1))T(r, f) + (1 - o(1))T(r, g) + o(1),$$

according to the remark after Lemma 1 and Nevanlinna's first fundamental theorem. On the other hand,

(24) $$N\left(r, \frac{1}{(c_2 - ac_1)g + (a-1)c_1 c_2}\right) \sim (1 - o(1))T(r, g) + o(1).$$

This, inequality (23), and identity (22) lead to a contradiction. Therefore we must have $a = 1$. Identity (19) becomes

(25) $$\frac{f - c_1}{g - c_1} = \frac{f - c_2}{g - c_2}.$$

It follows that $f \equiv g$. Now suppose subcase (ii) holds, that is, $a_1 \neq 0$ and $a_2 \neq 0$. Then Lemma 2 is applicable to identity (16). Therefore there exist three constants $b_1$, $b_2$, and $b_3$ that are not all zero such that

(26) $$\frac{b_1(\lambda_2 - c_1)}{\lambda_1} e^{-\beta} + \frac{b_2 h_2}{h_1} e^{\gamma_2 - \gamma_1} + \frac{b_3 h_2(\lambda_2 - c_2)}{h_1 \lambda_1} e^{\gamma_2 - \gamma_1 - \beta} \equiv 0.$$

This yields

(27)
$$\frac{b_1(\lambda_2-c_1)}{\lambda_1}e^{-\beta}=e^{\gamma_2-\gamma_1}\left[-\frac{b_3h_2(\lambda_2-c_2)}{h_1\lambda_1}e^{-\beta}-\frac{b_2h_2}{h_1}\right].$$

In the first place, we note that $b_1$ cannot be zero because if it were so, then it would yield $b_2=0$ and $b_3=0$, giving a contradiction. Now the exponent of convergence of zeros of the left hand side of equation (27) is finite while on the right hand side it is infinite unless $b_2=0$ or $b_3=0$. Suppose that $b_2=0$. Then identity (27) becomes

$$\frac{b_1(\lambda_2-c_1)}{\lambda_1}e^{-\beta}\equiv e^{\gamma_2-\gamma_1}\frac{-b_3h_2(\lambda_2-c_2)}{h_1\lambda_1}e^{-\beta}$$

or

(28)
$$e^{\gamma_2-\gamma_1}\equiv\frac{-b_1h_1(\lambda_2-c_1)}{b_3h_2(\lambda_2-c_2)}.$$

Hence $e^{\gamma_2-\gamma_1}$ is a function of finite order. Substituting identity (28) into identity (16) we get

(29)
$$a_1=e^{-\beta}\left(-\frac{a_2(\lambda_2-c_1)}{\lambda_1}-\frac{a_4h_2(\lambda_2-c_2)}{h_1\lambda_1}e^{\gamma_2-\gamma_1}\right)-\frac{a_3h_2}{h_1}e^{\gamma_2-\gamma_1}.$$

By virtue of Theorem 1 and noting that

(30)
$$H(z)\equiv-\frac{a_2(\lambda_2-c_1)}{\lambda_1}-\frac{a_4h_2(\lambda_2-c_2)}{h_1\lambda_1}e^{\gamma_2-\gamma_1}$$

is a function of finite order, we conclude that $H(z)\equiv0$. Hence

$$a_1=-\frac{a_3h_2}{h_1}e^{\gamma_2-\gamma_1}$$

or

(31)
$$e^{\gamma_2-\gamma_1}\frac{h_2}{h_1}=-\frac{a_1}{a_3}.$$

We thus come back to the previously proved case (i.e., subcase (i)).

Now suppose that $b_3=0$. Then identity (27) becomes

(32)
$$\frac{\lambda_2-c_1}{\lambda_1}e^{-\beta}=-e^{\gamma_2-\gamma_1}\left(\frac{b_2h_2}{b_1h_1}\right).$$

Substituting this into identity (16) we get

$$a_1\equiv\left\{a_2\frac{b_2h_2}{b_1h_1}-\frac{a_3h_2}{h_1}\right\}e^{\gamma_2-\gamma_1}-\frac{a_4h_2(\lambda_2-c_2)}{h_1\lambda_1}e^{\gamma_2-\gamma_1-\beta}$$

or

(33)
$$a_1\equiv e^{\gamma_2-\gamma_1}\left\{\left(\frac{a_2b_2}{b_1}-a_3\right)\frac{h_2}{h_1}-\frac{a_4h_2(\lambda_2-c_2)}{h_1\lambda_1}e^{-\beta}\right\}.$$

According to Theorem 1 we have to conclude that $a_4=0$ and hence

(34)
$$e^{\gamma_2-\gamma_1}\frac{h_2}{h_1}=\frac{a_1}{d}=\text{constant},$$

where

$$d=\frac{a_2b_2}{b_1}-a_3\neq 0.$$

From this and by exactly the same argument used in the subcase (i), we will arrive at the same conclusion that $f\equiv g$. This also completes the proof of the case $a_1\neq 0$. Now we consider the case that $a_1=0$ in equation (16). We shall treat case (a) $a_1=0$ and $a_4=0$, and case (b) $a_1=0$ and $a_4\neq 0$, separately. We shall first show that the former case will lead to a contradiction. Suppose that case (a) holds. In this case we have from equation (16) that

$$\frac{h_2}{h_1}e^{\beta}e^{\gamma_2-\gamma_1}=-\frac{a_2(\lambda_2-c_1)}{a_3\lambda_1}.$$

From this it follows that $\beta+\gamma_2-\gamma_1\equiv$ polynomial. Now, rewriting equation (9) as

(35)
$$\mu_1e^{\alpha}+\mu_2-c_1=h_1\lambda_1e^{\beta+\gamma_1}+h_1(\lambda_2-c_1)e^{\gamma_1},$$

multiplying $e^{\gamma_2-2\gamma_1}$ on both sides of (35) we obtain

(36)
$$\mu_1e^{\alpha+\gamma_2-2\gamma_1}+(\mu_2-c_1)e^{\gamma_2-2\gamma_1}=h_1\lambda_1e^{\beta+\gamma_2-\gamma_1}+h_1(\lambda_2-c_1)e^{\gamma_2-\gamma_1}.$$

Clearly $\gamma_2-\gamma_1$ can't be a polynomial, since $\beta+\gamma_2-\gamma_1$ is a polynomial and $\beta$ is assumed to be transcendental. Suppose neither $\alpha+\gamma_2-2\gamma_1$ nor $\gamma_2-2\gamma_1$ is a polynomial. Then, according to lemma 2, there exist constants $d_1$, $d_2$, and $d_3$, not all zero such that

$$d_1\mu_1e^{\alpha+\gamma_2-2\gamma_1}+d_2(\mu_2-c_1)e^{\gamma_2-2\gamma_1}+d_3h_1(\lambda_2-c_1)e^{\gamma_2-\gamma_1}\equiv 0$$

or

(37)
$$d_1\mu_1e^{\alpha}+d_2(\mu_2-c_1)+d_3h_1(\lambda_2-c_1)e^{\gamma_1}\equiv 0.$$

It is easy to verify that the above identity is possible only if $d_2=0$ and $\alpha-\gamma_1\equiv$ polynomial. Let $e^{\alpha-\gamma_1}\equiv k$, then (36) becomes

$$\mu_1ke^{\gamma_2-\gamma_1}+(\mu_2-c_1)e^{\gamma_2-2\gamma_1}\equiv\lambda_1h_1e^{\beta+\gamma_2-\gamma_1}+(\lambda_2-c_1)e^{\gamma_2-\gamma_1}$$

or

(38)
$$(\mu_1k-\lambda_2h_1+c_1h_1)e^{\gamma_2-\gamma_1}\equiv-(\mu_2-c_1)e^{\gamma_2-2\gamma_1}+\lambda_1h_1e^{\beta+\gamma_2-\gamma_1}.$$

Again, according to lemma 2, we conclude that a necessary condition for (38) to be held is $\gamma_2-2\gamma_1\equiv$ polynomial, giving a contradiction. Thus we must have either $\alpha+\gamma_2-2\gamma_1\equiv$ polynomial or $\gamma_2-2\gamma_1\equiv$ polynomial. Suppose that $\gamma_2-2\gamma_1\equiv$ polynomial. Then from (36) and again by a Borel type of argument, we have

(39)
$$\mu_1e^{\alpha+\gamma_2-2\gamma_1}\equiv h_1(\lambda_2-c_1)e^{\gamma_2-\gamma_1}$$

and

(40) $$(\mu_2-c_1)e^{\gamma_2-2\gamma_1}\equiv\lambda_1h_1e^{\beta+\gamma_2-\gamma_1}\,.$$

From (39) and (40) we can conclude easily that

$$\alpha-\gamma_1\equiv\text{polynomial}$$

and

$$\beta+\gamma_1\equiv\text{polynomial}\,.$$

Therefore

$$\alpha+\beta\equiv\text{polynomial}\,.$$

Then from equation (9) we have

(41) $$\frac{\mu_1e^{\alpha+\beta}+(\mu_2-c_2)e^{\beta}}{(\lambda_1e^{\beta}+\lambda_2-c_2)e^{\beta}}=\frac{(\mu_2-c_2)(e^{\beta}+(\mu_1e^{\alpha+\beta}/(\mu_2-c_2)))}{\lambda_1(e^{\beta}+(\lambda_2-c_1)/\lambda_1)e^{\beta}}\equiv h_1e^{\gamma_1}\,.$$

Since $\mu_1e^{\alpha+\beta}/(\mu_2-c_2)$ and $(\lambda_2-c_1)/\lambda_1$ both are of finite order and $\beta$ is transcendental, we can conclude from equation (41) that

$$\frac{\mu_1e^{\alpha+\beta}}{\mu_2-c_2}\equiv\frac{\lambda_2-c_2}{\lambda_1}\,,$$

otherwise $h_1(\equiv(\mu_2-c_2)/\lambda_1)$ could not be of finite order. Therefore we have $\gamma_1=-\beta$. Similarly, we can derive $\gamma_2=-\beta$ from equation (10). Then $\gamma_1-\gamma_2\equiv0$ contradicts the fact that $\gamma_1-\gamma_2\not\equiv\text{polynomial}$, which we remarked after equation (36). Now we treat the case that $\alpha+\gamma_2-2\gamma_1\equiv\text{polynomial}$. From (36), by using Borel's argument, we derive

$$\mu_1e^{\alpha+\gamma_2-2\gamma_1}\equiv h_1\lambda_1e^{\beta+\gamma_2-\gamma_1}$$

and

$$(\mu_2-c_1)e^{\gamma_2-2\gamma_1}\equiv h_1(\lambda_2-c_1)e^{\gamma_2-\gamma_1}\,.$$

Hence

(42) $$e^{\alpha-\beta}\equiv\frac{h_1\lambda_1}{\mu_1}e^{\gamma_1}$$

and

(43) $$e^{-\gamma_1}\equiv\frac{h_1(\lambda_2-c_1)}{\mu_2-c_1}\,.$$

It follows from (43) that $\gamma_1$ is a polynomial and hence from (42) that

$$\alpha-\beta\equiv\text{polynomial}\,.$$

This will lead to a contradiction as we did in the case $\alpha+\beta\equiv\text{polynomial}$. Thus we have ruled out the case (a). Now suppose case (b) holds, that is, $a_1=0$, $a_4\neq0$. Then from equation (16), we can conclude that $a_3=0$ (since it can be shown early that $a_2\neq0$). Thus equation (16) becomes

$$\frac{a_4h_2(\lambda_2-c_2)}{h_1\lambda_1}e^{\gamma_2-\gamma_1-\beta}\equiv-\frac{a_2(\lambda_2-c_1)}{\lambda_1}e^{-\beta}$$

or

$$\frac{h_2}{h_1}e^{r_2-r_1} \equiv -\frac{a_3(\lambda_2-c_1)}{a_4(\lambda_2-c_2)} \equiv \lambda(z) \ .$$

We note here that $\lambda(z)$ is a meromorphic function of finite order. From this and taking the quotient of (9) and (10), we obtain

(44)                        $$\frac{f-c_2}{g-c_2} = \lambda(z)\frac{f-c_1}{g-c_1} \ .$$

Hence

$$fg-c_1f-c_2g+c_1c_2 = \lambda(fg-c_1g-c_2f+c_1c_2)$$

or

(45)                        $$fg(1-\lambda)-f(c_1-c_2\lambda)=g(c_2-c_1\lambda)+c_1c_2(\lambda-1) \ .$$

The case that $\lambda \equiv 1$ will lead to a contradiction by $\lambda_2 \not\equiv$ constant. Thus we consider the case that $\lambda \not\equiv 1$. In this case, we rewrite equation (45) as

(46)                        $$f[g(1-\lambda)-(c_1-c_2\lambda)]=g(c_2-c_1\lambda)+c_1c_2(\lambda-1) \ .$$

We conclude from this that

(47)                        $$g(1-\lambda)-(c_1-c_2\lambda)=he^r \ ,$$

where $\gamma$ is an entire function and $h$ is a meromorphic function satisfying

$$N\left(r, \frac{1}{h}\right)=o(T(r,g)) \qquad \text{as } r \to \infty \ .$$

Otherwise, the left-hand of equation (46) would have many more zeros than that on the right hand side. The fact that $h$ is of finite order comes from the form of functions in $F$. It follows that

$$g=\frac{c_1-c_2\lambda}{1-\lambda}+\frac{he^r}{1-\lambda} \ .$$

Substituting the form of $g$ in (38), we get the form (i) for $f$.

As to the other cases, such as $a_2 \neq 0$ (or $a_1=0$), $a_3 \neq 0$ (or $a_3=0$), or $a_4 \neq 0$ (or $a_4=0$), they can be treated in exactly the same manner as we did in the cases $a_1 \neq 0$ and $a_1=0$, and will lead to similar conclusions.

Thus we have completed the analysis of all the possible situations which may arise in case (i) that is when $\beta$ is transcendental entire.

Finally, we must treat case (ii) that $\beta$ is a polynomial. However, it is clear that whatever situations may arise in this case, they can be handled by exactly the same arguments which we used in the proof of case (i). It is noted that in case (i) we used the fact $e^\beta$ is a function of infinite order and hence cannot have any meromorphic functions of finite order other than the constant functions 0 and $\infty$ as its deficient functions. And in case (ii), by hypothesis, $e^\beta$ has an order greater than that of any of those functions: $h_1\lambda_1$, $h_1(\lambda_2-c_1)$, $h_2\lambda_2$ and $h_2(\lambda_2-c_2)$ which appeared in the identity (16). Therefore, in case (ii), Lemma

2 and Theorem 1 again are applicable and we shall obtain similar conclusions. We leave the details and verifications to the reader. Hence the proof of Theorem 2 is complete.

*Remark.* Professor Ozawa provided the following example:
Let

$$f(z)=\frac{e^z}{e^z-1}e^{z^2}-\frac{1}{e^z-1} \quad \text{and} \quad g=-\frac{e^{-z^2}}{e^z-1}+\frac{e^z}{e^z-1},$$

then $f \in F$, $g \in F$ and they have the same preimage sets of the values 0 and 1 respectively.

It seems plausible that Theorem 2 is still valid for the class of all meromorphic functions $f$ of the form $f=\mu_1 e^\alpha+\mu_2$ satisfying

$$T(r, \mu_1)+T(r, \mu_2)=oT(r, e^\alpha),$$

as $r\to\infty$ with $\mu_2\not\equiv$constant and $\mu_1\not\equiv0$. That is, $\mu_1$ and $\mu_2$ can be meromorphic functions of infinite order.

## *Acknowledgement*

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