Y. WATANABE
KŌDAI MATH. SEM. REP.
28 (1977), 284–299

# ON THE CHARACTERISTIC FUNCTIONS OF QUATERNION KÄHLERIAN SPACES OF CONSTANT Q-SECTIONAL CURVATURE

BY YOSHIYUKI WATANABE

## 1. Introduction.

In Riemannian geometry, one of interest problems is the determination of of all Einstein spaces. But it is difficult to solve this problem, for we cannot even classify the class of harmonic spaces defined by analytic conditions, which is less general than that of Einstein spaces. The problems of classifying harmonic spaces, those of finding canonical forms for their metrics and those of determining its characteristic functions have taken up by E. T. Copson and H. S. Ruse [1], A. G. Walker [9], [10], [11], A. Lichnérowicz [5], T. J. Willmore [13], A. J. Ledger [4], S. Tachibana [7], the author [11] and others.

Its typical examples are the following: (1) Euclidean space $R^n$, (2) sphere $S^n$, (3) real projective space $RP^n$, (4) complex projective space $CP^m$, (5) quaternion projective space $HP^m$ and (6) the Cayley projective plane $\mathfrak{C}P(2)$ ([4], [6]). The characteristic functions of $S^n$, $RP^n$ and $CP^m$ have been already obtained as follows ([6], [7]): An $n$-dimensional space of constant curvature ($k\neq 0$) is characterized as a harmonic Riemannian space with characteristic function

$$f(\Omega)=1+(n-1)\sqrt{2k\Omega}\cot\sqrt{2k\Omega}$$

and a $2m$-dimensional space of constant holomorphic curvature ($k\neq 0$) as a harmonic Kählerian space with characteristic function

$$f(\Omega)=1+(2m-1)(ls)\cot(ls)-(ls)\tan(ls)$$

or

$$f(\Omega)=1+(2m-1)(ls)\coth(ls)+(ls)\tanh(ls),$$

according to $k=4l^2$, or $k=-4l^2$ respectively, where $s$ means the geodesic distance and $\Omega=(1/2)s^2$.

In spite of these facts, the characteristic functions of $HP^m$ and $\mathfrak{C}P(2)$ have

Received Dec. 18, 1975.
*) This work was partially supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, which is gratefully acknowledged.

not been known. In the present paper, we give the characteristic functions of quaternion Kählerian spaces of constant $Q$-sectional curvature. For this purpose, we consider quaternion Fubinian spaces, which are showed to be harmonic and also quaternion Kählerian spaces of constant $Q$-sectional curvature.

The author wishes to express his sincere thanks to Prof. S. Tachibana and Prof. S. Ishihara, who gave him many valuable suggestions and guidances, and prof. H. Mizusawa and prof. S. Sawaki, who gave him guidances.

## 2. Quaternion Kählerian spaces.

Let $M$ be a differentiable manifold of dimension $n$ and assume that there is a 3-dimensional vector bundle $V$ consisting of tensors of type (1,1) over $M$ satisfying the following conditions:

(a) In any coordinate neibourhood $U$ of $M$, there is a local base $\{F, G, H\}$ of $V$ such that

$$F^2=G^2=H^2=-I, \qquad HG=-GH=F,\ FH=-HF=G,\ GF=-FG=H, \tag{2.1}$$

$I$ denoting the identity tensor field of (1,1) in $M$.

Such a local base $\{F, G, H\}$ is called a canonical local base of the bundle $V$ in $U$. Then the bundle $V$ is called an almost quaternion structure in $M$ and the set $(M, V)$ an almost quaternion space. In an almost quaternion space $(M, V)$, we take two intersecting coordinate neighborhoods $U$, $U'$ and local base $\{F, G, H\}$, $\{F', G', H'\}$ satisfying (2.1) in $U$ and $U'$, respectively, then they have relations in $U\cap U'$ as

$$F'=s_{11}F+s_{12}G+s_{13}H, \qquad G'=s_{21}F+s_{22}G+s_{23}H, \qquad H'=s_{31}F+s_{32}G+s_{33}H, \tag{2.2}$$

where $s_{\alpha\beta}(\alpha, \beta=1, 2, 3)$ form an element $s_{U,U'}=(s_{\alpha\beta})$ of the special orthogonal group $SO(3)$ of dimension 3. In any almost quaternion space $(M, V)$, there is a Riemannian metric $g$ such that

$$g(FX, Y)+g(X, FY)=0, \qquad g(GX, Y)+g(X, GY)=0$$
$$g(HX, Y)+g(X, HY)=0,$$

hold for any local base $\{F, G, H\}$ and any vector fields $X, Y$. Now, let us assume that the Riemannain connection $\nabla$ of the Riemannian space $(M, g)$ satisfies the following conditions:

(b) If $\{F, G, H\}$ is a canonical local base of $V$ in $U$, then

$$
\begin{aligned}
&\nabla_X F = \qquad\quad +r(X)G - q(X)H\,, \\
(2.3)\qquad &\nabla_X G = -r(X)F \qquad\quad +p(X)H\,, \\
&\nabla_X H = \ \ q(X)F - p(X)G
\end{aligned}
$$

for any vector fields $X$, where $p$, $q$ and $r$ are certain local 1-forms defined in $U$.

If the set $(M, g, V)$ satisfies the condition (b), then $(M, g, V)$ is called a quaternion Kählerian space and $(g, V)$ a quaternion Kählerian structure.

We take a point $p$ in a quaternion Kählerian space $(M, g, V)$ of dimension $4m$ and a vector $X$ tagent to $M$ at $p$. Putting

$$Q(X)=\{Y \mid Y=aX+bFX+cGX+dHX\}\,,$$

$a$, $b$, $c$ and $d$ being arbitrary real numbers, we call $Q(X)$ the $Q$-section determined by $X$, where $Q(X)$ is a 4-dimensional subspace of the tangent space of $M$ at the point $p$.

We denote by $\sigma(X, Y)$ the sectional curvature of $(M, g)$ with respect to the section spanned by $X$ and $Y$. When $\sigma(Y, Z)$ with respect to the section spanned by any $Y, Z \in Q(X)$ is a constant $\rho(X)$, $\rho(X)$ is called the $Q$-sectional curvature of $(M, g, V)$ with respect to $X$. A quaternian Kählerian space is said to be of constant $Q$-sectional curvature $k$ when any $Q$-section $Q(X)$ has its $Q$-sectional curvature $\rho(X)$ and $\rho(X)$ is a constant $k$ independent of $X$ at each point $p$. By $R=(R^i{}_{jkl})$, we denote the Riemannian curvature tensor of $\nabla$. Then the following propositions are known.

PROPOSITION 2.1 (*c.f.* [3]). *A quaternion Kählerian space of dimension $4m$ $(m>1)$ is of constant $Q$-sectional curvature $k$, if and only if its curvature tensor has components of the form*

$$
\begin{aligned}
&R_{ABCD}=\frac{k}{4}(g_{AD}g_{BC}-g_{BD}g_{AC}+F_{AD}F_{BC}-F_{BD}F_{AC}-2F_{AB}F_{CD} \\
(2.4)\qquad &\qquad +G_{AD}G_{BC}-G_{BD}G_{AC}-2G_{AB}G_{CD}+H_{AD}H_{BC}-H_{BD}H_{AC}-2H_{AB}H_{CD})\,.
\end{aligned}
$$

PROPOSITION 2.2. ([3]) *A quaternion Kählerian space of constant $Q$-sectional curvature is locally symmetric.*

Let $(M, g, V)$ and $(M', g', V')$ be two quaternion Kählerian spaces of dimension $4m$ of constant $Q$-sectional curvature $k$. Let $\{F, G, H\}$ be a local base of $V$ in a coordinate neibourhood $U$ in $M$ and $\{F', G', H'\}$ a local base of $V'$ in a coordinate neighbourhood $U'$ in $M'$. Then, we can choose an orthonormal basis $\mathfrak{F}=\{e_1, \cdots, e_m, Fe_1, \cdots, Fe_m, Ge_1, \cdots, Ge_m, He_1, \cdots, He_m\}$ of the tangent space $T_p(M)$ at $p$ and an orthonormal basis $\mathfrak{F}'=\{e_1', \cdots, e_m', F'e_1', \cdots, F'e_m', G'e_1', \cdots, G'e_m', H'e_1', \cdots, H'e_m'\}$ of the tangent space $T_{p'}(M')$ at $p'$ respectively. It easily seen from Proposition 2.1 that all components of the curvature tensor $R$ with respect to $\mathfrak{F}$ are equal to corresponding those of the curvature tensor $R'$ with respect to $\mathfrak{F}'$. Thus, from Proposition 2.2 we can see that $(M, g, V)$ and $(M', g', V')$ are locally isomorphic, that is, there is an local isometry $f$ such that $\{f^*F, f^*G, f^*H\}$ is a canonical local base of $V'$ in $f(U)$, where $f^*F$ denotes the tensor field induced by $f$ from $F$ and so on. Moreover, it is proved that if $(M, g, V)$ and $(M', g', V')$ are complete and simply connected, then $(M, g, V)$ and $(M', g', V')$ isomorphic to each other, that is, there is an isometry $f$ such that $\{f^*F, f^*G, f^*H\}$ is a canonical local base of $V'$ in $f(U)$.

For the reasons stated above, the subsequent sections are devoted to study quaternion Fubinian spaces, which will be showed to be quaternion Kählerian spaces of constant $Q$-sectional curvature.

## 3. Quaternion Fubinian spaces.

Let $H=\{x+y\mathrm{i}+z\mathrm{j}+w\mathrm{k} \mid x, y, z, w\in R^1\}$ be the set of quaternions. For

$$\mathrm{p}=x+y\mathrm{i}+z\mathrm{j}+w\mathrm{k}\in H, \qquad \bar{p}=x-y\mathrm{i}-z\mathrm{j}-w\mathrm{k}$$

is the conjugate of p. Putting $|\mathrm{p}|=\sqrt{\mathrm{p}\bar{\mathrm{p}}}$, we denote $\mathrm{p}^{-1}=(\mathrm{p}/|\mathrm{p}|^2)$ for $\mathrm{p}\neq 0$. Let $H^m$ the $m$-dimensional right hand module over $H$ and

$$|\mathrm{p}|=\sqrt{\sum_{\alpha=1}^{m} \mathrm{q}_\alpha\bar{\mathrm{q}}_\alpha} \qquad \text{for } \mathrm{p}=(\mathrm{q}_1, \cdots, \mathrm{q}_m)\in H^m .$$

$H^m$ will be always identified with $R^{4m}$ as follows: For $\mathrm{q}_\alpha=x_\alpha+y_\alpha\mathrm{i}+z_\alpha\mathrm{j}+w_\alpha\mathrm{k}$,

$$\phi:(q_1, \cdots, q_m) \longrightarrow (x_1, \cdots, x_m, y_1, \cdots, y_m, z_1, \cdots, z_m, w_1, \cdots, w_m).$$

Throughout the paper we shall agree with the following conventions.

( I ) The range of indeces.

$$A, B, C, \cdots=1, \cdots, 4m, \qquad \alpha_{(1)}, \beta_{(1)}, \gamma_{(1)}, \cdots=m+1, \cdots, 2m ,$$

$$a, b, c, \cdots=1, \cdots, m+1 \qquad \alpha_{(2)}, \beta_{(2)}, \gamma_{(2)}, \cdots=2m+1, \cdots, 3m ,$$

$$\alpha, \beta, \gamma, \cdots=1, \cdots, m, \qquad \alpha_{(3)}, \beta_{(3)}, \gamma_{(3)}, \cdots=3m+1, \cdots, 4m .$$

(II) $$(\xi_A)=(\xi_\alpha, \xi_{\alpha(1)}, \xi_{\alpha(2)}, \xi_{\alpha(3)}) =(x_\alpha, y_\alpha, z_\alpha, w_\alpha) =(x_1, \cdots, x_m, y_1, \cdots, y_m, z_1, \cdots, z_m, w_1, \cdots, w_m).$$

(III) The summation convention. For examples,

$$\mathrm{q}_\alpha\bar{\mathrm{q}}_\alpha=\mathrm{q}_1\bar{\mathrm{q}}_1+\cdots+\mathrm{q}_m\bar{\mathrm{q}}_m ,$$

$$x_\alpha dy_\alpha=x_1dy_1+\cdots+x_mdy_m ,$$

$$F_B{}^AX^B=F_1{}^AX^1+\cdots+F_{4m}{}^AX^{4m}.$$

$H^m$ is a real analytic manifold with the coordinate system $(x_\alpha, y_\alpha, z_\alpha, w_\alpha)$.

We introduce real valued functions $u$ and $\theta$ on $H^m$ defined by

$$u=\mathrm{q}_\alpha\bar{\mathrm{q}}_\alpha=x_\alpha x_\alpha+y_\alpha y_\alpha+z_\alpha z_\alpha+w_\alpha w_\alpha,$$

(3.1)

$$\theta=1+\frac{k}{4}u,$$

where $k$ is a non-zero real number. We consider the maximal connected domain $D^m$ such that $\theta>0$, and define a (positive definite) Riemannian metric on it given by

$$\begin{aligned} ds^2=\frac{1}{\theta}\Big\{&\theta(dx_\lambda dx_\lambda+dy_\lambda dy_\lambda+dz_\lambda dz_\lambda+dw_\lambda dw_\lambda\\ &-\frac{k}{4}(a_{\lambda\mu}dx_\lambda dx_\mu+b_{\lambda\mu}dx_\lambda dy_\mu+c_{\lambda\mu}dx_\lambda dz_\mu+d_{\lambda\mu}dx_\lambda dw_\mu\\ &-b_{\lambda\mu}dy_\lambda dx_\mu+a_{\lambda\mu}dy_\lambda dy_\mu+d_{\lambda\mu}dy_\lambda dz_\mu-c_{\lambda\mu}dy_\lambda dw_\mu\\ &-c_{\lambda\mu}dz_\lambda dx_\mu-d_{\lambda\mu}dz_\lambda dy_\mu+a_{\lambda\mu}dz_\lambda dz_\mu+b_{\lambda\mu}dz_\lambda dw_\mu\\ &-d_{\lambda\mu}dw_\lambda dx_\mu+c_{\lambda\mu}dw_\lambda dy_\mu-b_{\lambda\mu}dw_\lambda dz_\mu+a_{\lambda\mu}dw_\lambda dw_\mu)\Big\}, \end{aligned} \tag{3.2}$$

where
$$\begin{aligned} a_{\lambda\mu}&=x_\lambda x_\mu+y_\lambda y_\mu+z_\lambda z_\mu+w_\lambda w_\mu,\\ b_{\lambda\mu}&=x_\lambda y_\mu-x_\lambda y_\mu+z_\lambda w_\mu-z_\lambda w_\mu,\\ c_{\lambda\mu}&=x_\lambda z_\mu-x_\lambda z_\mu-y_\lambda w_\mu+y_\lambda w_\mu,\\ d_{\lambda\mu}&=x_\lambda w_\mu-x_\lambda w_\mu+y_\lambda z_\mu-y_\lambda z_\mu. \end{aligned}$$

The matrix expression of the matric $g$ is given by

$$(g_{AB})=\frac{1}{\theta}\begin{pmatrix} \theta\delta_{\lambda\mu}-\frac{k}{4}a_{\lambda\mu} & -\frac{k}{4}b_{\lambda\mu} & -\frac{k}{4}c_{\lambda\mu} & -\frac{k}{4}d_{\lambda\mu}\\ \frac{k}{4}b_{\lambda\mu} & \theta\delta_{\lambda\mu}-\frac{k}{4}a_{\lambda\mu} & -\frac{k}{4}d_{\lambda\mu} & \frac{k}{4}c_{\lambda\mu}\\ \frac{k}{4}c_{\lambda\mu} & \frac{k}{4}d_{\lambda\mu} & \theta\delta_{\lambda\mu}-\frac{k}{4}a_{\lambda\mu} & -\frac{k}{4}b_{\lambda\mu}\\ \frac{k}{4}d_{\lambda\mu} & -\frac{k}{4}c_{\lambda\mu} & \frac{k}{4}b_{\lambda\mu} & \theta\delta_{\lambda\mu}-\frac{k}{4}a_{\lambda\mu} \end{pmatrix}. \tag{3.2$'$}$$

Such an $(D^m, g)$ will be called a quaternion Fubinian space of real dimension $4m$ and denoted by $HF^m$. Then $(g^{AB})$ is given by

$$(3.3)\qquad (g^{AB})=\theta\left(\begin{array}{c|c|c|c}
\delta_{\lambda\mu}+\frac{k}{4}a_{\lambda\mu} & \frac{k}{4}b_{\lambda\mu} & \frac{k}{4}c_{\lambda\mu} & \frac{k}{4}d_{\lambda\mu} \\ \hline
-\frac{k}{4}b_{\lambda\mu} & \delta_{\lambda\mu}+\frac{k}{4}a_{\lambda\mu} & \frac{k}{4}d_{\lambda\mu} & -\frac{k}{4}c_{\lambda\mu} \\ \hline
-\frac{k}{4}c_{\lambda\mu} & -\frac{k}{4}d_{\lambda\mu} & \delta_{\lambda\mu}+\frac{k}{4}a_{\lambda\mu} & \frac{k}{4}b_{\lambda\mu} \\ \hline
-\frac{k}{4}d_{\lambda\mu} & \frac{k}{4}c_{\lambda\mu} & -\frac{k}{4}b_{\lambda\mu} & \delta_{\lambda\mu}+\frac{k}{4}a_{\lambda\mu}
\end{array}\right).$$

Putting $(\xi_A)=(x_\alpha, y_\alpha, z_\alpha, w_\alpha)$, the Christoffel symbols $\left\{ {A \atop B\ C} \right\}$ are given by

$$\left\{ {A \atop B\ C} \right\}=\frac{1}{2}g^{AD}\Big(\frac{\partial g_{BD}}{\partial \xi_C}+\frac{\partial g_{CD}}{\partial \xi_B}-\frac{\partial g_{BC}}{\partial \xi_D}\Big).$$

Then (3.2)′ and (3.3) give the following results:

(3.4) (a) $\left\{ {\kappa \atop A\ B} \right\}$

$$=\frac{k}{4\theta}\left(\begin{array}{c|c|c|c}
-x_\lambda\delta_{\mu\kappa}-x_\mu\delta_{\lambda k} & -y_\lambda\delta_{\mu\kappa}-y_\mu\delta_{\lambda\kappa} & -z_\lambda\delta_{\mu\kappa}-z_\mu\delta_{\lambda\kappa} & -w_\lambda\delta_{\mu\kappa}-w_\mu\delta_{\lambda\kappa} \\ \hline
-y_\lambda\delta_{\mu\kappa}-y_\mu\delta_{\lambda\kappa} & x_\lambda\delta_{\mu\kappa}+x_\mu\delta_{\lambda\kappa} & -w_\lambda\delta_{\mu\kappa}+w_\mu\delta_{\lambda k} & z_\lambda\delta_{\mu\kappa}-z_\mu\delta_{\lambda\kappa} \\ \hline
-z_\lambda\delta_{\mu\kappa}-z_\mu\delta_{\lambda\kappa} & w_\lambda\delta_{\mu\kappa}-w_\mu\delta_{\lambda\kappa} & x_\lambda\delta_{\mu\kappa}+x_\mu\delta_{\lambda\kappa} & -y_\lambda\delta_{\mu\kappa}+y_\mu\delta_{\lambda\kappa} \\ \hline
-w_\lambda\delta_{\mu\kappa}-w_\mu\delta_{\lambda\kappa} & -z_\lambda\delta_{\mu\kappa}+z_\mu\delta_{\lambda\kappa} & y_\lambda\delta_{\mu\kappa}-y_\mu\delta_{\lambda\kappa} & x_\lambda\delta_{\mu\kappa}+x_\mu\delta_{\lambda\kappa}
\end{array}\right)$$

for fixed $\kappa$;

(b) $\left\{ {\kappa_{(1)} \atop A\ B} \right\}$

$$=\frac{k}{4\theta}\left(\begin{array}{c|c|c|c}
y_\lambda\delta_{\mu\kappa}+y_\mu\delta_{\lambda\kappa} & -x_\lambda\delta_{\mu\lambda}-x_\mu\delta_{\lambda\kappa} & w_\lambda\delta_{\mu\kappa}-w_\mu\delta_{\lambda\kappa} & -z_\lambda\delta_{\mu\kappa}+z_\mu\delta_{\lambda\kappa} \\ \hline
-x_\lambda\delta_{\mu\kappa}-x_\mu\delta_{\lambda\kappa} & -y_\lambda\delta_{\mu\kappa}-y_\mu\delta_{\lambda\kappa} & -z_\lambda\delta_{\mu\kappa}-z_\mu\delta_{\lambda\kappa} & -w_\lambda\delta_{\mu\kappa}+z_\mu\delta_{\lambda\kappa} \\ \hline
-w_\lambda\delta_{\mu\kappa}+w_\mu\delta_{\lambda\kappa} & -z_\lambda\delta_{\mu\kappa}-z_\mu\delta_{\lambda\kappa} & y_\lambda\delta_{\mu\kappa}+y_\mu\delta_{\lambda\kappa} & x_\lambda\delta_{\mu\kappa}-x_\mu\delta_{\lambda\kappa} \\ \hline
-z_\lambda\delta_{\mu\kappa}-z_\mu\delta_{\lambda\kappa} & -w_\lambda\delta_{\mu\kappa}-w_\mu\delta_{\lambda\kappa} & -x_\lambda\delta_{\mu\kappa}+x_u\delta_{\lambda\kappa} & y_\lambda\delta_{\mu\kappa}+y_\mu\delta_{\lambda\kappa}
\end{array}\right)$$

for fixed $\kappa_{(1)}$;

(c) $\left\{ {\kappa_{(2)} \atop A\ B} \right\}$

$$=\frac{k}{4\theta}\left(\begin{array}{c|c|c|c}
z_\lambda\delta_{\mu\kappa}+z_\mu\delta_{\lambda\kappa} & -w_\lambda\delta_{\mu\kappa}+w_\mu\delta_{\lambda\kappa} & -x_\lambda\delta_{\mu\kappa}-x_\mu\delta_{\lambda\kappa} & y_\lambda\delta_{\mu\kappa}-y_\mu\delta_{\lambda\kappa} \\ \hline
w_\lambda\delta_{\mu\kappa}-w_\mu\delta_{\lambda\kappa} & z_\lambda\delta_{\mu\kappa}+z_\mu\delta_{\lambda\kappa} & -y_\lambda\delta_{\mu\kappa}-y_\mu\delta_{\lambda\kappa} & -x_\lambda\delta_{\mu\kappa}+x_\mu\delta_{\lambda\kappa} \\ \hline
-x_\lambda\delta_{\mu\kappa}-x_\mu\delta_{\lambda\kappa} & -y_\lambda\delta_{\mu\kappa}-y_\mu\delta_{\lambda\kappa} & -z_\lambda\delta_{\mu\kappa}-z_\mu\delta_{\lambda\kappa} & -w_\lambda\delta_{\mu\kappa}-w_\mu\delta_{\lambda\kappa} \\ \hline
-y\delta_{\mu\kappa}+y_\mu\delta_{\lambda\kappa} & x_\lambda\delta_{\mu\kappa}-x_\mu\delta_{\lambda\kappa} & -w_\lambda\delta_{\mu\kappa}-w_\mu\delta_{\lambda\kappa} & z_\lambda\delta_{\mu\kappa}+z_\mu\delta_{\lambda\kappa}
\end{array}\right)$$

for fixed $\kappa_{(2)}$;

(d) $\left\{ {\kappa_{(3)} \atop A\quad B} \right\}$

$$
= \frac{k}{4\theta}\left(\begin{array}{c|c|c|c}
w_\lambda\delta_{\mu\kappa}+w_\mu\delta_{\lambda\kappa} & z_\lambda\delta_{\mu\kappa}-z_\mu\delta_{\lambda\kappa} & -y_\lambda\delta_{\mu\kappa}+y_\mu\delta_{\lambda\kappa} & -x_\lambda\delta_{\mu\kappa}-x_\mu\delta_{\lambda\kappa} \\
\hline
-z_\lambda\delta_{\mu\kappa}+z_\mu\delta_{\lambda\mu} & w_\lambda\delta_{\mu\kappa}+w_\mu\delta_{\lambda k} & x_\lambda\delta_{\mu\kappa}-x_\mu\delta_{\lambda\kappa} & -y_\lambda\delta_{\mu\kappa}-y_\mu\delta_{\lambda\kappa} \\
\hline
y_\lambda\delta_{\mu\kappa}-y_\mu\delta_{\lambda\kappa} & -x_\lambda\delta_{\mu\kappa}+x_\mu\delta_{\lambda\kappa} & w_\lambda\delta_{\mu\kappa}+w_\mu\delta_{\lambda\kappa} & -z_\lambda\delta_{\mu\kappa}-z_\mu\delta_{\lambda\kappa} \\
\hline
-x_\lambda\delta_{\mu\kappa}-x_\mu\delta_{\lambda\kappa} & -y_\lambda\delta_{\mu\kappa}-y_\mu\delta_{\lambda\kappa} & -z_\lambda\delta_{\mu\kappa}-z_\mu\delta_{\lambda\kappa} & -w_\lambda\delta_{\mu\kappa}-w_\mu\delta_{\lambda\kappa}
\end{array}\right)
$$

for fixed $\kappa_{(3)}$.

## 4. Geodesics in $HF^m$.

Consider a quaternion Fubinian space $HF^m$. We shall find the equation of geodesics passing through a point $p_0 \in HF^m$.

The differential equation of geodesic is given by

$$\xi_A'' + \left\{ {A \atop B\quad C} \right\} \xi_B' \xi_C' = 0, \tag{4.1}$$

where throughout the paper we denote by ($'$) the derivative with respect to the arc length $s$. Since our space is a quaternion Fubinian space, substituting (3.4) into (4.1) implies

(4.2)

(a)
$$
\begin{aligned}
x_\lambda'' = -\frac{k}{2\theta}\{ & -x_\alpha x_\alpha' x_\lambda' + x_\alpha y_\alpha' y_\lambda' + x_\alpha z_\alpha' z_\lambda' + x_\alpha w_\alpha' w_\lambda' \\
& - y_\alpha y_\alpha' x_\lambda' - y_\alpha x_\alpha' y_\lambda' - z_\alpha z_\alpha' x_\lambda' - z_\alpha x_\alpha' z_\lambda' - w_\alpha w_\alpha' x_\lambda' - w_\alpha x_\alpha' w_\lambda' \\
& - w_\alpha y_\alpha' z_\lambda' + w_\alpha z_\alpha' y_\lambda' + z_\alpha y_\alpha' w_\lambda' - z_\alpha w_\alpha' y_\lambda' - y_\alpha z_\alpha' w_\lambda' + y_\alpha w_\alpha' z_\lambda' \},
\end{aligned}
$$

(b)
$$
\begin{aligned}
y_\lambda'' = -\frac{k}{2\theta}\{ & -y_\alpha y_\alpha' y_\lambda' + y_\alpha x_\alpha' x_\lambda' + y_\alpha z_\alpha' z_\lambda' + y_\alpha w_\alpha' x_\lambda' \\
& - x_\alpha x_\alpha' y_\lambda' - x_\alpha y_\alpha' x_\lambda' + w_\alpha x_\alpha' z_\lambda' - w_\alpha z_\alpha' x_\lambda' - z_\alpha x_\alpha' w_\lambda' + z_\alpha w_\alpha' x_\lambda' \\
& - z_\alpha z_\alpha' y_\lambda' - z_\alpha y_\alpha' z_\lambda' - w_\alpha w_\alpha' y_\lambda' - w_\alpha y_\alpha' w_\lambda' + x_\alpha z_\alpha' w_\lambda' - x_\alpha w_\alpha' z_\lambda' \},
\end{aligned}
$$

(c)
$$
\begin{aligned}
z_\lambda'' = -\frac{k}{2\theta}\{ & -z_\alpha z_\alpha' z_\lambda' + z_\alpha x_\alpha' x_\lambda' + z_\alpha y_\alpha' y_\lambda' + z_\alpha w_\alpha' w_\lambda' \\
& - w_\alpha x_\alpha' y_\lambda' + w_\alpha y_\alpha' x_\lambda' - x_\alpha x_\alpha' z_\lambda' - x_\alpha z_\alpha' x_\lambda' - y_\alpha x_\alpha' w_\lambda' - y_\alpha w_\alpha' x_\lambda' \\
& - y_\alpha y_\alpha' z_\lambda' - y_\alpha z_\alpha' y_\lambda' - x_\alpha y_\alpha' w_\lambda' + x_\alpha w_\alpha' y_\lambda' - w_\alpha z_\alpha' w_\lambda' - w_\alpha w_\alpha' z_\lambda' \},
\end{aligned}
$$

d) $$w_\lambda''=-\frac{k}{2\theta}\{-w_\alpha w_\alpha' w_\lambda'+w_\alpha x_\alpha' x_\lambda'+w_\alpha y_\alpha' y_\lambda'+w_\alpha z_\alpha' z_\lambda'$$
$$+z_\alpha x_\alpha' y_\lambda'-z_\alpha y_\alpha' x_\lambda'-y_\alpha x_\alpha' z_\lambda'+y_\alpha z_\alpha' x_\lambda'-x_\alpha x_\alpha' w_\lambda'-x_\alpha w_\alpha' x_\lambda'$$
$$+x_\alpha y_\alpha' z_\lambda'-x_\alpha z_\alpha' y_\lambda'-y_\alpha y_\alpha' w_\lambda'-y_\alpha w_\alpha' y_\lambda'-z_\alpha z_\alpha' w_\lambda'-z_\alpha w_\alpha' z_\lambda'\}\,.$$

Now, we shall find a geodesic passing through a point $p_0=(x^0_\alpha, y^0_\alpha, z^0_\alpha, w^0_\alpha)$. Let $t(s)$ be a real valued function of $s$ satisfying $t(0)=0$ and $t'(0)>0$. If for any constant $A=(a_\alpha, b_\alpha, c_\alpha, d_\alpha)$, substituting $x_\alpha=a_\alpha t(s)+x^0_\alpha$, $y_\alpha=b_\alpha t(0)+y^0_\alpha$, $z_\alpha=c_\alpha t(s)+z^0_\alpha$ and $w=d_\alpha t(s)+w^0_\alpha$ into (4.2), then we get

(4.3) (a) $$a_\lambda t''(s)=\frac{k(t'(s))^2}{2\theta}\{|A|^2 a_\lambda t(s)+va_\lambda-mb_\lambda-nc_\lambda-rd_\lambda\}\,,$$

(b) $$b_\lambda t''(s)=\frac{k(t'(s))^2}{2\theta}\{|A|^2 b_\lambda t(s)+ma_\lambda+vb_\lambda+rc_\lambda-nd_\lambda\}\,,$$

(c) $$c_\lambda t''(s)=\frac{k(t'(s))^2}{2\theta}\{|A|^2 c_\lambda t(s)+na_\lambda-rb_\lambda+vc_\lambda+md_\lambda\}\,,$$

(d) $$d_\lambda t''(s)=\frac{k(t'(s))^2}{2\theta}\{|A|^2 d_\lambda t(s)+ra_\lambda+nb_\lambda-mc_\lambda+vd_\lambda\}\,,$$

where
$$v=x^0_\alpha a_\alpha+y^0_\alpha b_\alpha+z^0_\alpha c_\alpha+w^0_\alpha d_\alpha\,,$$
$$m=x^0_\alpha b_\alpha-y^0_\alpha a_\alpha-z^0_\alpha d_\alpha+w^0_\alpha c_\alpha\,,$$
$$n=x^0_\alpha c_\alpha+y^0_\alpha d_\alpha-z^0_\alpha a_\alpha-w^0_\alpha b_\alpha\,,$$
$$r=x^0_\alpha d_\alpha-y^0_\alpha c_\alpha+z^0_\alpha b_\alpha-w^0_\alpha a_\alpha\,.$$

From (4.3) and the definition of $\theta$, we can get

(4.4) $$t''(s)=\frac{k(t'(s))^2\{|A|^2 t(s)+v\}}{2+(k/2)\{|A|^2(t(s))^2+2vt(s)+u_0\}}\,,$$

where $u_0=x^0_\alpha x^0_\alpha+y^0_\alpha y^0_\alpha+z^0_\alpha z^0_\alpha+w^0_\alpha w^0_\alpha$. Hence, taking account of $g_{AB}\xi_A\xi_B=1$ and $t(0)=0$, we get

LEMMA 3.1. *A geodesic with direction $A=(a_\alpha, b_\alpha, c_\alpha, d_\alpha)$ passing a point $p_0=(x^0_\alpha, y^0_\alpha, z^0_\alpha, w^0_\alpha)$ is given by*
(i) *for the case of $k>0$,*

(4.5)
$$x_\alpha=a_\alpha\Big\{\frac{h}{\sqrt{k}}\tan\Big(\frac{\sqrt{k}}{2}s+\frac{eh}{\sqrt{k}}\Big)-\frac{v}{|A|^2}\Big\}+x^0_\alpha\,,$$
$$y_\alpha=b_\alpha\Big\{\frac{h}{\sqrt{k}}\tan\Big(\frac{\sqrt{k}}{2}s+\frac{eh}{\sqrt{k}}\Big)-\frac{v}{|A|^2}\Big\}+y^0_\alpha\,,$$
$$z_\alpha=c_\alpha\Big\{\frac{h}{\sqrt{k}}\tan\Big(\frac{\sqrt{k}}{2}s+\frac{eh}{\sqrt{k}}\Big)-\frac{v}{|A|^2}\Big\}+z^0_\alpha\,,$$
$$w_\alpha=d_\alpha\Big\{\frac{h}{\sqrt{k}}\tan\Big(\frac{\sqrt{k}}{2}s+\frac{eh}{\sqrt{k}}\Big)-\frac{v}{|A|^2}\Big\}+w^0_\alpha\,,$$

*where*

$$h=\frac{1}{|A|^2}\sqrt{ku_0|A|^2+4|A|^2-kv}$$

*and*

$$e=\frac{\sqrt{k}}{h}\tan^{-1}\frac{\sqrt{k}\,v}{h|A|^2}.$$

(ii) *for the case of* $k<0$,

$$(4.5)' \quad \begin{aligned} x_\alpha&=a_\alpha\Big\{\frac{h}{\sqrt{|k|}}\tanh\Big(\frac{\sqrt{|k|}}{2}s+\frac{\tilde{e}h}{\sqrt{|k|}}\Big)-\frac{v}{|A|^2}\Big\}+x^0_\alpha. \\ y_\alpha&=b_\alpha\Big\{\frac{h}{\sqrt{|k|}}\tanh\Big(\frac{\sqrt{|k|}}{2}s+\frac{\tilde{e}h}{\sqrt{|k|}}\Big)-\frac{v}{|A|^2}\Big\}+y^0_\alpha, \\ z_\alpha&=c_\alpha\Big\{\frac{h}{\sqrt{|k|}}\tanh\Big(\frac{\sqrt{|k|}}{2}s+\frac{\tilde{e}h}{\sqrt{|k|}}\Big)-\frac{v}{|A|^2}\Big\}+z^0_\alpha, \\ w_\alpha&=d_\alpha\Big\{\frac{h}{\sqrt{|k|}}\tanh\Big(\frac{\sqrt{|k|}}{2}s+\frac{\tilde{e}h}{\sqrt{|k|}}\Big)-\frac{v}{|A|^2}\Big\}+w^0_\alpha. \end{aligned}$$

*where*

$$e=\frac{\sqrt{|k|}}{h}\tanh^{-1}\frac{\sqrt{|k|}\,v}{h|A|^2}.$$

## 5. $\Delta_2 s$ in $HF^m$.

In the next section, getting its characteristic function, we shall show that any quaternion Fubinian space is harmonic. For this purpose, we shall calculate the Laplacian $\Delta_2 s$ in $HF^m$. Putting

$$k=\begin{cases} 4l^2 & \text{if } k>0, \\ -4l^2 & \text{if } k<0, \end{cases}$$

we give an outline of the calculations for the case of $k>0$, where $l$ is a positive constant.

Since each point $p_0\in HF^m$ has a normal neighbourhood $U$ such that arbitrary points in $U$ can be joined with $p_0$ by exactly one geodesic segment contained in $U$, any point of $U$ is represented in the form of (i) or (ii) according to $k>0$, or $k<0$ respectively. Therefore, it follows that

$$(5.1) \quad \begin{aligned} u&=x_\alpha x_\alpha+y_\alpha y_\alpha+z_\alpha z_\alpha+w_\alpha w_\alpha \\ &=\frac{h^2|A|^2}{k}\Big\{\tan\Big(\frac{\sqrt{k}}{2}s+\frac{eh}{\sqrt{k}}\Big)\Big\}^2+u_0-\frac{v^2}{|A|^2}, \end{aligned}$$

from which

$$(5.2)\quad \begin{aligned} u' &= \frac{du}{ds} = \frac{h^2|A|^2}{\sqrt{k}} \tan\Big(\frac{\sqrt{k}}{2}s + \frac{eh}{\sqrt{k}}\Big)\Big\{\sec\Big(\frac{\sqrt{k}}{2}s + \frac{eh}{\sqrt{k}}\Big)\Big\}^2, \\ u'' &= \frac{d^2u}{ds^2} = \frac{h^2|A|^2}{2}\Big\{\sec\Big(\frac{\sqrt{k}}{2}s + \frac{eh}{\sqrt{k}}\Big)\Big\}^2\Big\{1 + 3\Big(\tan\Big(\frac{\sqrt{k}}{2}s + \frac{eh}{\sqrt{k}}\Big)\Big)^2\Big\}. \end{aligned}$$

Differentiating (5.1) by $x_\lambda$, $y_\lambda$, $z_\lambda$ and $w_\lambda$ respectively, we have

$$(5.3)\quad \begin{aligned} &\text{(a)}\quad 2x_\lambda = u'\frac{\partial s}{\partial x_\lambda}, &\qquad &\text{(c)}\quad 2z_\lambda = u'\frac{\partial s}{\partial z_\lambda}, \\ &\text{(b)}\quad 2y_\lambda = u'\frac{\partial s}{\partial y_\lambda}, &\qquad &\text{(d)}\quad 2w_\lambda = u'\frac{\partial s}{\partial w_\lambda}. \end{aligned}$$

Moreover, differentiating (5.3.a) by $x_\mu$, we have

$$2\delta_{\lambda\mu} = u''\frac{\partial s}{\partial x_\mu}\frac{\partial s}{\partial x_\lambda} + u'\frac{\partial^2 s}{\partial x_\lambda \partial x_\mu}.$$

Multiplying this equation by $g^{\lambda\mu}$, then we obtain

$$(5.4)\quad 2g^{\lambda\mu}\delta_{\lambda\mu} = u'' g^{\lambda\mu}\frac{\partial s}{\partial x_\lambda}\frac{\partial s}{\partial x_\mu} + u' g^{\lambda\mu}\frac{\partial^2 s}{\partial x_\lambda \partial x_\mu}.$$

On the other hand, it follows from (3.2)′ and (5.3.a) that

$$g^{\lambda\mu}\delta_{\lambda\mu} = \theta\Big(\delta_{\lambda\mu} + \frac{k}{4}a_{\lambda\mu}\Big)\delta_{\lambda\mu} = \theta\Big(m + \frac{k}{4}u\Big),$$

and

$$g^{\lambda\mu}\frac{\partial s}{\partial x_\lambda}\frac{\partial s}{\partial x_\mu} = \frac{4\theta}{(u')^2}\Big(x_\lambda x_\lambda + \frac{k}{4}a_{\lambda\mu}x_\lambda x_\mu\Big).$$

Substituting these into (5.4) gives

$$g^{\lambda\mu}\frac{\partial^2 s}{\partial x_\lambda \partial x_\mu} = \frac{2\theta}{u'}\Big(m + \frac{k}{4}u\Big) - \frac{4u''\theta}{(u')^3}\Big(x_\lambda x_\lambda + \frac{k}{4}a_{\lambda\mu}x_\lambda x_\mu\Big).$$

Similarly we get

$$g^{\lambda\mu(1)}\frac{\partial^2 s}{\partial x_\lambda \partial y_\mu} = -\frac{ku''\theta}{(u')^3}b_{\lambda\mu}x_\lambda y_\mu,$$

$$g^{\lambda\mu(2)}\frac{\partial^2 s}{\partial x_\lambda \partial z_\mu} = -\frac{ku''\theta}{(u')^3}c_{\lambda\mu}x_\lambda z_\mu,$$

$$g^{\lambda\mu(3)}\frac{\partial^2 s}{\partial x_\lambda \partial w_\mu} = -\frac{ku''\theta}{(u')^3}d_{\lambda\mu}x_\lambda w_\mu,$$

$$g^{\lambda(1)\mu(1)}\frac{\partial^2 s}{\partial y_\lambda \partial y_\mu} = \frac{2\theta}{u'}\Big(m + \frac{k}{4}u\Big) - \frac{4u''\theta}{(u')^3}\Big(y_\lambda y_\lambda + \frac{k}{4}a_{\lambda\mu}y_\lambda y_\mu\Big),$$

$$g^{\lambda(1)\mu(2)}\frac{\partial^2 s}{\partial y_\lambda \partial z_\mu} = -\frac{ku''\theta}{(u')^3}d_{\lambda\mu}y_\lambda z_\mu,$$

$$g^{\lambda(1)\mu(3)}\frac{\partial^2 s}{\partial y_\lambda \partial w_\mu}=\frac{ku''\theta}{(u')^3}c_{\lambda\mu}y_\lambda w_\mu ,$$

$$g^{\lambda(2)\mu(2)}\frac{\partial^2 s}{\partial z_\lambda \partial z_\mu}=\frac{2\theta}{u'}\Big(m+\frac{k}{4}u\Big)-\frac{4u''\theta}{(u')^3}\Big(z_\lambda z_\lambda+\frac{k}{4}a_{\lambda\mu}z_\lambda z_\mu\Big),$$

$$g^{\lambda(2)\mu(3)}\frac{\partial^2 s}{\partial z_\lambda \partial w_\mu}=-\frac{ku''\theta}{(u')^3}b_{\lambda\mu}z_\lambda w_\mu ,$$

$$g^{\lambda(3)\mu(3)}\frac{\partial^2 s}{\partial w_\lambda \partial w_\mu}=\frac{2\theta}{u'}\Big(m+\frac{k}{4}u\Big)-\frac{4u''\theta}{(u')^3}\Big(w_\lambda w_\lambda+\frac{k}{4}a_{\lambda\mu}w_\lambda w_\mu\Big).$$

Hence we get

$$\begin{aligned} g^{AB}\frac{\partial^2 \xi}{\partial s_A \partial \xi_B}&=\frac{8}{u'}\Big(1+\frac{k}{4}u\Big)\Big(m+\frac{k}{4}u\Big)\\ &\quad-\frac{u''}{(u')^3}\Big(1+\frac{k}{4}u\Big)\Big\{4(x_\lambda x_\lambda+y_\lambda y_\lambda+z_\lambda z_\lambda+w_\lambda w_\lambda)\\ &\quad+ka_{\lambda\mu}(x_\lambda x_\mu+y_\lambda y_\mu+z_\lambda z_\mu+w_\lambda w_\mu)\\ &\quad+2k(b_{\lambda\mu}x_\lambda y_\mu+c_{\lambda\mu}x_\lambda z_\mu+d_{\lambda\mu}x_\lambda w_\mu+d_{\lambda\mu}y_\lambda z_\mu-c_{\lambda\mu}y_\lambda w_\mu+b_{\lambda\mu}z_\lambda w_\mu)\Big\}\\ &=\frac{8}{u'}\Big(1+\frac{k}{4}u\Big)\Big(m+\frac{k}{4}u\Big)-\frac{4uu''}{(u')^3}\Big(1+\frac{k}{4}u\Big). \end{aligned} \tag{5.5}$$

First, we are going to obtain $g^{AB}\left\{ {C \atop A\ B} \right\}\frac{\partial s}{\partial \xi_C}$ by using (3.2)′ and (3.4):

$$\begin{aligned} g^{AB}\left\{ {\kappa \atop A\ B} \right\}&=\frac{k}{4}\Big\{2(x_\lambda\delta_{\mu\kappa}+x_\mu\delta_{\lambda\kappa})\Big(\delta_{\lambda\mu}+\frac{k}{4}a_{\lambda\mu}\Big)\\ &\qquad-kw_\lambda d_{\lambda\kappa}-kz_\lambda c_{\lambda\kappa}-ky_\lambda b_{\lambda\kappa}\Big\}\\ &=\frac{k}{4}\{4x_\kappa+\kappa(x_\lambda a_{\lambda\kappa}-w_\lambda d_{\lambda\kappa}-z_\lambda c_{\lambda\kappa}-y_\lambda b_{\lambda\kappa})\}\\ &=k\theta x_\kappa , \end{aligned}$$

and similarly

$$g^{AB}\left\{ {\kappa(1) \atop A\ \ B} \right\}=k\theta y_\kappa,\qquad g^{AB}\left\{ {\kappa(2) \atop A\ \ B} \right\}=k\theta z_\kappa,\qquad g^{AB}\left\{ {\kappa(3) \atop A\ \ B} \right\}=k\theta w_\kappa .$$

Thus, we get

$$g^{AB}\left\{ {C \atop A\ B} \right\}\frac{\partial s}{\partial \xi_C}=\frac{2ku}{u'}\Big(1+\frac{k}{4}u\Big). \tag{5.6}$$

Since the Laplacian $\Delta_2 s$ is given by

$$\Delta_2 s = g^{AB}\frac{\partial^2 s}{\partial\xi_A\partial\xi_B} - g^{AB}\left\{ {C \atop A\ \ B} \right\}\frac{\partial s}{\partial\xi_C},$$

it follows from (5.5) and (5.6) that on $U$,

$$\Delta_2 s = \frac{8m}{u'}\left(1+\frac{k}{4}u\right) - \frac{4uu''}{(u')^3}\left(1+\frac{k}{4}u\right)^2, \tag{5.7}$$

where ( ;) denotes the covariant differentiation with respect to the Christoffel symbols $\left\{ {C \atop A\ \ B} \right\}$.

## 6. The main theorem.

An analytic Riemannian space of dimension $n$ is harmonic if every point $p_0$ is the origin of a nomal neighbourhood $U$ such that, if $\Omega$ is the distance function $\Omega(p_0, p)=(1/2)s^2$, then its Laplacian $\Delta_2\Omega$, calculated for fixed $p_0$ and variable $p$, is a function depending upon $\Omega$ and not otherwise upon $p$, i.e., $\Delta_2\Omega=f(\Omega)$, where $f(\Omega)$ called the characteristic function.

Now, the left hand member of (5.7) does not depend on the choice of coordinate system. So, if we represent the right hand member in terms of the normal coordinate system $(\xi_A)$ on $U$ with origin $p_0$, then by (5.2) we have

$$\Delta_2 s = (4m-1)l\cot(ls) - 3l\tan(ls), \tag{6.1}$$

because of definitions, of $h$, $v$ and $u_0$. Thus, the characteristic function

$$f(\Omega) = 1 + (4m-1)(ls)\cot(ls) - 3(ls)\tan(ls), \tag{6.2}$$

by virtue of the identity $\Delta_2\Omega = 1 + s\Delta_2 s$. Thus we see that $HF^m$ is harmonic.

Similarly, for $k=-4l^2$, we can get

$$f(\Omega) = 1 + (4m-1)(ls)\coth(ls) + 3(ls)\tanh(ls). \tag{6.2$'$}$$

On the other hand, we have

$$x\cot x = 1 - \frac{1}{3}x^2 - \frac{1}{45}x^4 - \frac{2}{945}x^6 - \cdots, \tag{6.3}$$

from which

$$x\tan x = x^2 + \frac{1}{3}x^4 + \frac{126}{945}x^6 + \cdots. \tag{6.4}$$

If we develop $f(\Omega)$ in power series of $\Omega$ and $s$, then taking account of (6.3) and (6.4), we have, respectively

$$f(\Omega) = 4m + \dot{f}(0)\Omega + \frac{1}{2}\ddot{f}(0)\Omega^2 + \frac{1}{3!}\dddot{f}(0)\Omega^3 + \cdots$$

$$= 4m - \frac{4(m+2)}{3}(ls)^2 - \frac{4(m+11)}{45}(ls)^4 - \frac{8(m+47)}{945}(ls)^6 - \cdots.$$

Thus, we have

$$\dot{f}(0)=-\frac{8(m+2)}{3}l^2, \quad \ddot{f}(0)=-\frac{32(m+11)}{45}l^4. \tag{6.5}$$

Similarly, for the $f(\Omega)$ given by (6.2)′, we get

$$\dot{f}(0)=\frac{8(m+2)}{3}l^2, \quad \ddot{f}(0)=-\frac{32(m+11)}{45}l^4. \tag{6.5$'$}$$

Now, we define a base $\{F, G, H\}$ of an almost quaternion structure $V$ in the quaternion Fubinian space $HF^m$ with the coodinate system $(\xi_A)=(x_\alpha, y_\alpha, z_\alpha, w_\alpha)$ as follows:

$$\begin{aligned}
&F\left(\frac{\partial}{\partial x_\lambda}\right)=\frac{\partial}{\partial y_\lambda}, && G\left(\frac{\partial}{\partial x_\lambda}\right)=\frac{\partial}{\partial z_\lambda}, && H\left(\frac{\partial}{\partial x_\lambda}\right)=\frac{\partial}{\partial w_\lambda},\\
&F\left(\frac{\partial}{\partial y_\lambda}\right)=-\frac{\partial}{\partial x_\lambda}, && G\left(\frac{\partial}{\partial y_\lambda}\right)=\frac{\partial}{\partial w_\lambda}, && H\left(\frac{\partial}{\partial y_\lambda}\right)=-\frac{\partial}{\partial z_\lambda},\\
&F\left(\frac{\partial}{\partial z_\lambda}\right)=-\frac{\partial}{\partial w_\lambda}, && G\left(\frac{\partial}{\partial z_\lambda}\right)=-\frac{\partial}{\partial x_\lambda}, && H\left(\frac{\partial}{\partial z_\lambda}\right)=\frac{\partial}{\partial y_\lambda},\\
&F\left(\frac{\partial}{\partial w_\lambda}\right)=\frac{\partial}{\partial z_\lambda}, && G\left(\frac{\partial}{\partial w_\lambda}\right)=-\frac{\partial}{\partial y_\lambda}, && H\left(\frac{\partial}{\partial w_\lambda}\right)=-\frac{\partial}{\partial x_\lambda}.
\end{aligned} \tag{6.6}$$

Then $F$, $G$ and $H$ have numerical components of the form

$$F;\begin{pmatrix} 0 & -E & 0 & 0\\ E & 0 & 0 & 0\\ 0 & 0 & 0 & E\\ 0 & 0 & -E & 0\end{pmatrix}, \quad G;\begin{pmatrix} 0 & 0 & -E & 0\\ 0 & 0 & 0 & -E\\ E & 0 & 0 & 0\\ 0 & E & 0 & 0\end{pmatrix}, \quad H;\begin{pmatrix} 0 & 0 & 0 & -E\\ 0 & 0 & E & 0\\ 0 & -E & 0 & 0\\ E & 0 & 0 & 0\end{pmatrix}, \tag{6.7}$$

where $E$ denotes the identity $(m, m)$-matrix. It is easily seen that $\{F, G, H\}$ satisfies (2.1) and each of $F$, $G$ and $H$ is almost Hermitian with respect to $g$.

We shall calculate $\nabla_X F$ by putting

$$X=s^\lambda\frac{\partial}{\partial x_\lambda}+t^\lambda\frac{\partial}{\partial y_\lambda}+u^\lambda\frac{\partial}{\partial z_\lambda}+v^\lambda\frac{\partial}{\partial w_\lambda}.$$

Since

$$\nabla_C F_B{}^A=\frac{\partial F_B{}^A}{\partial \xi_C}+\left\{{A \atop D\ C}\right\}F_B{}^D-\left\{{D \atop B\ C}\right\}F_D{}^A,$$

it follows from (3.4) and (6.7) that

$$\nabla_{\partial/\partial x_r}F=\frac{k}{2\theta}(-w_rG+z_rH), \quad \nabla_{\partial/\partial y_r}F=\frac{k}{2\theta}(-z_rG-w_rH),$$

$$\nabla_{\partial/\partial z_r}F=\frac{k}{2\theta}(y_rG-x_rH), \quad \nabla_{\partial/\partial w}F=\frac{k}{2\theta}(x_rG+y_rH)$$

from which

$$\nabla_X F=\frac{k}{2\theta}\{(x_\gamma v^\gamma+y_\gamma u^\gamma-z_\gamma t^\gamma-w_\gamma s^\gamma)G-(x_\gamma u^\gamma-y_\gamma v^\gamma-z_\gamma s^\gamma+w_\gamma t^\gamma)H\} .$$

Similarly we can get

$$\nabla_X G=\frac{k}{2\theta}\{-(x_\gamma v^\gamma+y_\gamma u^\gamma-z_\gamma t^\gamma-w_\gamma s^\gamma)F+(x_\gamma t^\gamma-y_\gamma s^\gamma+z_\gamma v^\gamma-w_\gamma u^\gamma)H\}$$

and

$$\nabla_X H=\frac{k}{2\theta}\{(x_\gamma u^\gamma-y_\gamma v^\gamma-z_\gamma s^\gamma+w_\gamma t^\gamma)F-(x_\gamma t^\gamma-y_\gamma s^\gamma+z_\gamma v^\gamma-w_\gamma u^\gamma)G\} .$$

Thus, since $\{F, G, H\}$ satisfies (2.3), we have

PROPOSITION 6.1. *$(HF^m, g, V)$ is harmonic and quaternion Kählerian.*

To prove Proposition 6.3, we need

PROPOSITION 6.2. ([11]) *In any $4m$-dimensional harmonic quaternion Kählerian space $M$, the inequality*

$$\dot{f}^2(0)+\frac{10(m+2)^2}{m+11}\ddot{f}(0)\leqq 0 \tag{6.8}$$

*holds. Equality sign is valid if and only if $M$ is of constant $Q$-sectional curvature.*

It is easily seen from (6.5) and (6.5)′ that the caracteristic function of the triple $(HF^m, g, V)$ satisfies the equality sign in (6.8). Thus we have

PROPOSITION 6.3. *$(HF^m, g, V)$ is a quaternion Kählerian space of constant $Q$-sectional curvature, which is harmonic.*

By Proposition 6.3 and the fact stated in §2, a quaternion Kählerian space of constant $Q$-sectional curvature ($k\neq 0$) is locally regarded as a quaternian Fubinian space. Therefore, such a space is harmonic, and its characteristic function $f(\Omega)$ is given by (6.5) and (6.5)′. Thus we get the following main theorem.

THEOREM. *A $4m$-dimensional space of constant $Q$-sectional curvature ($k\neq 0$) is characterized as a harmonic quaternion Kählerian space with characteristic function given by*

$$f(\Omega)=1+(4m-1)(ls)\cot(ls)-3(ls)\tan(ls)$$

*or*

$$f(\Omega)=1+(4m-1)(ls)\coth(ls)+3(ls)\tanh(ls) ,$$

*according to $k=4l^2$, or $k=-4l^2$ respectirely.*

## 7. The canonical metric of $HP^m$.

Let $H^{m+1}$ be the $(m+1)$-dimensional right module over the quaternions $H$.

$S^{4m+3}$ means the unit hypersphere with center 0 defined by $\mathrm{p}_a\bar{\mathrm{p}}_a=1$. If there is an element x of $H$ such that $\mathrm{q}_a=\mathrm{p}_a\mathrm{x}$, $a=1, \cdots, m+1$, $|x|=1$, then we shall say $(\mathrm{p}_a)$ to be equivalent to $(\mathrm{q}_a)$ and represent this fact by $(\mathrm{p}_a)\sim(\mathrm{q}_a)$. As this relation $(\sim)$ clearly satisfies the three conditions of equivalence relation, $S^{4m+3}$ is classified into the set

$$HP^m=S^{4m+3}/\sim$$

of the equivalence classes. $HP^m$ is called the quaternion projective space. We denote by $[\mathrm{p}_1, \cdots, \mathrm{p}_{m+1}]$ the equivalence class containing $(\mathrm{p}_1, \cdots, \mathrm{p}_{m+1})\in S^{4m+3}$.

The natural local coordinate systems $(U_b, \psi_b)$, $b=1, \cdots, m+1$, of $HP^m$ are introduced as follows: For each $b$, we set

$$U_b=\{[\mathrm{p}_1, \cdots, \mathrm{p}_{m+1}]\in HP^m \mid (\mathrm{p}_1, \cdots, \mathrm{p}_{m+1})\in S^{4m+3}, \mathrm{p}_b\neq 0\} .$$

Then, each $U_b$ is open in $HP^m$ and $\bigcup_{b=1}^{m+1} U_b=HP^m$. Let $\psi_b$ on $U_b$ be

$$\psi_b([\mathrm{p}_1, \cdots, \mathrm{p}_{m+1}])=(\mathrm{p}_1\mathrm{p}_b^{-1}, \cdots, \mathrm{p}_{b-1}\mathrm{p}_b^{-1}, \mathrm{p}_{b+1}\mathrm{p}_b^{-1}, \cdots, \mathrm{p}_{m+1}\mathrm{p}_b^{-1}) .$$

Rewriting $\phi\circ\psi_b$ as $\psi_b$ (see §3), $\psi_b$ is a real coordinate system on $U_b$. Thus, it is easily seen that $HP^m$ is a $4m$-dimensional real analytic manifold. Putting

$$\begin{aligned}\psi_b([p_1, \cdots, p_{m+1}])&=(q_1, \cdots, q_m)\\ &=(x_\alpha, y_\alpha, z_\alpha, w_\alpha) ,\end{aligned}$$

we see that the canonical metric of $HP^m$ is defined by

$$\begin{aligned}ds^2=\frac{1}{\theta^2}\{&\theta(dx_\lambda dx_\lambda+dy_\lambda dy_\lambda+dz_\lambda dz_\lambda+dw_\lambda dw_\lambda)\\ &-(a_{\lambda\mu}dx_\lambda dx_\mu+b_{\lambda\mu}dx_\lambda dy_\mu+c_{\lambda\mu}dx_\lambda dz_\mu+d_{\lambda\mu}dx_\lambda dw_\mu\\ &-b_{\lambda\mu}dy_\lambda dx_\mu+a_{\lambda\mu}dy_\lambda dy_\mu+d_{\lambda\mu}dy_\lambda dz_\mu-c_{\lambda\mu}dy_\lambda dw_\mu\\ &-c_{\lambda\mu}dz_\lambda dx_\mu-d_{\lambda\mu}dz_\lambda dy_\mu+a_{\lambda\mu}dz_\lambda dz_\mu+b_{\lambda\mu}dz_\lambda dw_\mu\\ &-d_{\lambda\mu}dw_\lambda dx_\mu+c_{\lambda\mu}dw_\lambda dy_\mu-b_{\lambda\mu}dw_\lambda dz_\mu+a_{\lambda\mu}dw_\lambda dw_\mu\} ,\end{aligned}$$

where $\theta=1+x_\alpha x_\alpha+y_\alpha y_\alpha+z_\alpha z_\alpha+w_\alpha w_\alpha$. We take any two intersecting coordinate neighborhoods $(U_b, \psi_b)$ and $(U_c, \psi_c)$, where $\psi_b=(\xi_A)$ and $\psi_c=(\widetilde{\xi_A})$ respectively. If we denote by $g_{AB}$ and $\widetilde{g_{AB}}$ the components of $ds^2$ with respect to $(\xi_A)$ and $(\widetilde{\xi_A})$ respectively, then we can see that in $U_b\cap U_c$,

$$g_{AB}=\frac{\partial\widetilde{\xi_A}}{\partial\xi_C}\frac{\partial\widetilde{\xi_B}}{\partial\xi_D}\widetilde{g_{CD}} .$$

This shows that $ds^2$ is (globally) tensor field on $HP^m$. Thus, it is a Riemannian metric on $HP^m$. Since the metric has the form (3.2) with $k=4$, $HP^m$ is naturally a quaternion Kählerian space of constant $Q$-sectional curvature, which is harmonic.

On the other hand, we have already known that the curvature tensor of $HP^m$, which is the base space of the Hopf fibring $S^{4m+3} \to HP^m$ ([2], [3]), is the form (6.6) with $k=4$.

## BIBLIOGRAPHY


[1] E. T. COPSON AND H. S. RUSE, Harmonic Riemannian spaces, Proc. Roy. Soc. Edin. **60** (1939-40) 117-133.

[2] S. ISHIHARA, Quaternion Kählerian manifolds and fibred Riemannian spaces with Sasakian 3-structure, to appear in Kōdai Math. Sem. Rep.

[3] S. ISHIHARA, Notes on quaternion Kählerian manifolds, J. Differential Geometry **9** (1974), 483-500.

[4] A. J. LEDGER, Harmonic homogenous spaces of Lie groups, J. Lond. Math. Soc., **29** (1954), 345-347.

[5] A. LICHNÉROWICZ, Sui les espaces riemanniens complètement harmoiques, Bull. Soc. Math. Fr. **72** (1944) 146-168.

[6] H. S. RUSE, A. G. Walker and T. J. Willmore, Harmonic Spaces, Edizioni Cremonese, Roma, 1961.

[7] S. TAHIBANA, On the characteristic function of spaces of constant holomorphic curvature, Colloq. Math. **26** (1972) 149-155.

[8] A. G. WALKER, On completely harmonic spaces, J. Lond. Math. Soc., **20** (1945) 159-163.

[9] A. G. WALKER, Symmetric harmonic spaces, J. Lond. Math. Soc. **21** (1946) 47-57.

[10] A. G. WALKER, On Lichinérowicz's conjecture for harmonic 4-spaces, J. Lond. Math. Soc. **24** (1948-49) 21-28.

[11] Y. WATANABE, On the characteristic function of harmonic Kählerian spaces, Tôhoku Math. Jour., **27** (1975), 13-24.

[12] Y. WATANABE, On the characteristic function of harmonic quaternion Kählerian spaces, Kōdai Math. Sem. Rep. **27** (1976), 410-420.

[13] T. J. WILLMORE, Some properties of harmonic Riemannian manifolds Convegno di Geometria Differenziale, Venezia (1953), 141-147.

[14] K. YANO AND S. BOCHNER, Curvature and Betti numbers, Ann. of Math. studies, **32**, Princeton, 1953.



FACULTY OF LITERATURE AND SCIENCE
TOYAMA UNIVERSITY
TOYAMA, JAPAN.