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ON HARMONIC MAJORATION

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It is easily verified that if $u(z)$ is subharmonic in the unit disk Δ and has a harmonic majorant on some annulus $[\rho<|z|<1]$, then on Δ so does u. Royden [6] showed that if u is harmonic in a finite Riemann surface W and has a positive harmonic majorant on some boundary neighborhood, then on W so does u. On the other hand Gauthier and Hengartner [1] has recently shown that if u is subharmonic in the unit disk Δ and has harmonic majorants on sufficiently small (relative) neighborhoods of each point of the frontier $\partial\Delta$, then on Δ so does u.

We are concerned with partitionality of harmonic majoration. In the present paper we give some extensions of the above statements.

1. Sectional Majoration Theorem.

THEOREM 1. *On an open Riemann surface W, let A be a relatively compact ring domain with frontier $\partial A=\gamma_1\cup\gamma_2$, where γ_1 and γ_2 are mutually disjoint simple closed curves. Let W_1 and W_2 be regions on W satisfying that:* i) $W_1\cap W_2=A$, ii) *W_k contains $\gamma_k(k=1, 2)$,* iii) *W_1 has positive ideal boundary. Suppose that u is subharmonic in $W_1\cup W_2$ and has harmonic majorants on each of W_1 and W_2. Then u has a harmonic majorant on $W_1\cup W_2$.*

Proof. Let us take an analytic simple closed curve γ(in A) separating ∂A. Let $W_1'(W_2')$ denote the subregion of $W_1(W_2$, resp.) obtained by removing the closed ring domain bounded by γ and γ_2 (γ_1, resp.).

Let b be a bounded harmonic function on W_1' with the boundary values h_2-h_1 on γ, where h_1 and h_2 are harmonic majorants of u on W_1 and W_2, respectively. Let $\omega\not\equiv 0$ be a nonnegative bounded harmonic function on W_1' continuously vanishing toward γ. We define a function w on $W_1\cup W_2$ as follows:

$$w=\begin{cases} h_2+M & \text{on } W_2' \\ h_1-K\omega+b+M & \text{on } W_1', \end{cases}$$

where K and M are positive constants satisfying that

$$-K\omega+b+M>0 \quad \text{in } W_1.$$

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Here we may assume that both ω and b are harmonically extended over the closed annulus A_1 bounded by γ and γ_2, and that $\omega\leqq 0$ on A_1. Hence, for sufficiently large K,

$$h_2+M\leqq h_1-K\omega+b+M \qquad \text{on } A_1 .$$

This inequality implies that w is superharmonic in $W_1\cup W_2$ for such K, which proves the theorem.

A meromorphic function f on a Riemann surface W is called Lindelöfian on W if $\log^+|f(z)|$ has a superharmonic majorant on W. (See Heins [2].) As an application of the theorem we give a decomposition formula for Lindelöfian meromorphic functions on a plane region.

THEOREM 2. *Let Ω be a plane region whose complement in the extended plane $\bar{C}$ consists of mutually disjoint n continua $E_1, E_2, \cdots, E_n$. Then every Lindelöfian meromorphic function f on Ω can be represented in the form*

$$f=f_1+f_2+\cdots+f_n ,$$

where $f_k(k=1, 2, \cdots, n)$ are Lindelöfian meromorphic functions respectively on the simply-connected regions $\bar{C}\smallsetminus E_k$.

Proof. Without loss of generality we may assume that Ω contains the point at infinity at which f is analytic and vanishes. Let P be the set of poles of f. Since P clusters only on $\partial\Omega$, we can divide P into mutually disjoint n subsets $P_k(k=1, 2, \cdots, n)$ each of which is either a countable set clustering only on E_k or the empty set. Then the usual Aronszajn decomposition

$$f=f_0+f_1+f_2+\cdots+f_n ,$$

where

$$f_k(z)=\frac{1}{2\pi i}\int_{c_k}\frac{f(\zeta)}{\zeta-z}d\zeta \qquad (k=0, 1, 2, \cdots, n) ,$$

c_0 being a circle centered at the origin with sufficiently large radius and $c_k(k=1, 2, \cdots, n)$ being a cycle bounding $E_k\cup P_k$, enjoys the following properties: i) $f_0\equiv 0$, ii) f_k is meromorphic in $\bar{C}\smallsetminus E_k$ and vanishes at the point at infinity ($k=1, 2, \cdots, n$), iii) the set of poles of f_k is precisely $P_k(k=1, 2, \cdots n)$. We must verify that $f_k(k=1, 2, \cdots, n)$ are Lindelöfian on their respective domains. To do this for f_1 (for other f_k, the reasoning is the same), let us take a closed ring domain A off $E_1\cup E_2\cup\cdots\cup E_n$ satisfying that the bounded component of the complement of A contains P_1 and the unbounded component contains $P_2, P_3, \cdots, P_n$. The previous theorem is available to the case: W_1=the interior of the set obtained by removing $E_1\cup P_1$ from the union of A and the bounded component of the complement of A, W_2=the interior of the union of A and the unbounded component, $u(z)=\log^+|f(z)|$, $h_1(z)=h(z)+M$, $h_2(z)=M$, where $h(z)$ is a harmonic majorant of $u(z)$ on Ω less P and M is a sufficiently large constant.

2. Local Majoration Theorem.

We turn to another partitionality.

THEOREM 3. *Let W be a finite Riemann surface with border β which consists of a finite number of mutually disjoint smooth simple closed curves. If u is subharmonic in W and has positive harmonic majorants on sufficiently small (relative) neighborhoods of each point of the border β, then u has a positive harmonic majorant on the whole surface W.*

Before proving the theorem, we note

LEMMA. *Let C be a simple closed curve in the plane and $f(\zeta)$ be a continuous function on an open subarc γ of C. Let $\{u_n\}$ be a sequence of harmonic functions converging inside C. Suppose that $u_n (n=1, 2, \cdots)$ have common continuous boundary values $f(\zeta)$ at each point of γ, and that there exists a harmonic function $U(z)$ inside C such that $U(z) \leqq u_n(z)$ $(n=1, 2, \cdots)$ inside C. Then the limit function also has continuous boundary values $f(\zeta)$ at each point of γ.*

The lemma reduces to the case where C is the unit circle, $U(z)\equiv 0$ and $f(\zeta)\equiv 0$. In this case the lemma can be verified by the following fact which is an immediate consequence of Poisson's integral formula for harmonic functions. For $\alpha>0$, let H_α be the family of functions which: i) are nonnegative and continuous on the closed unit disk, ii) are harmonic in the (open) unit disk, iii) are dominated by 1 at the origin, and iv) vanish on the arc $\bar{\gamma}=\{e^{i\theta};\ |\theta|\leqq\alpha\}$. Then H_α is equicontinuous near γ in the sense that for any $\varepsilon>0$ and any positive $\eta(<\alpha)$ there exists a positive $\rho<1$ satisfying that $u(re^{i\theta})<\varepsilon$ for every $u \in H_\alpha$ and r, θ with $\rho<r<1$, $|\theta|\leqq\eta$.

To prove the theorem it suffices to show that u has a harmonic majorant on some neighborhood of the border β, and hence to show the following: Let $u(z)$ be a nonnegative subharmonic function on an annulus $1<|z|<R$ $(<\infty)$. If $u(z)$ has harmonic majorants on sufficiently small (relative) neighborhoods of each point of the circle $|z|=R$, then on some annulus $\rho<|z|<R$ u has a harmonic majorant.

To do this, we have only to show

LEMMA. *Let $\{r_n\}$ be strictly increasing sequence with a finite limit R (>1) and for a fixed positive $\alpha\,(<\pi) f$, $f_n (n=1, 2, \cdots)$ be nonnegative bounded continuous functions on the arcs $\{e^{i\theta};\ |\theta|<\alpha\}$, $\{r_n e^{i\theta};\ |\theta|<\alpha\}$ $(n=1, 2, \cdots)$, respectively. By $h_n\,(n=1, 2, \cdots)$ we denote the bounded harmonic functions on the sets $R_n=\{re^{i\theta};\ 1<r<r_n,\ |\theta|<\beta\}$ having continuous boundary values:*

$$\begin{array}{ll} f(\zeta) & \text{on } \{e^{i\theta};\ |\theta|<\alpha\}, \\ f_n(\zeta) & \text{on } \{r_n e^{i\theta};\ |\theta|<\alpha\}, \\ 0 & \text{otherwise (except at the points with } |\theta|=\alpha). \end{array}$$

By $H_n (n=1, 2, \cdots)$ we denote the bounded harmonic functions on the annuli $A_n=$

$[1<|z|<r_n]$ *having continuous boundary values:*

$$f(\zeta) \quad on\ \{e^{i\theta};\ |\theta|<\alpha\}\,,$$

$$f_n(\zeta) \quad on\ \{r_n e^{i\theta};\ |\theta|<\alpha\}\,,$$

$$0 \quad otherwise\ (except\ at\ the\ points\ with\ |\theta|=\alpha)\,.$$

If for some $\beta(>\alpha)$ $\{h_n\}$ *contains a converging subsequence, so does* $\{H_n\}$.

Proof. Fix a point z_0 with $1<|z_0|<r_1$, $\alpha<\arg z_0<\beta$. Let g_n, G_n be the Green functions with pole z_0 of regions R_n, A_n, respectively $(n=1, 2, \cdots)$.

We assert that for sufficiently large M

$$G_n(z)\leqq Mg_n(z)$$

in the sets $\{re^{i\theta};\ 1<r<r_n,\ |\theta|<\alpha\}$ $(n=1, 2, \cdots)$. To see this, set

$$M_n=\max G_n(z)/g_n(z)$$

over the segments $\{re^{i\theta};\ 1\leqq r\leqq r_n,\ |\theta|=\alpha\}$, where the values of $G_n(z)/g_n(z)$ at the end points are interpreted as the values of

$$(\partial G_n/\partial r)/(\partial g_n/\partial r) \qquad (n=1, 2, \cdots)\,.$$

Suppose that there exists a subsequence $\{M_{n_k}\}$ with $M_{n_k}\to\infty(k\to\infty)$. We may assume $M_n\to\infty(n\to\infty)$, so that we can take a $\{z_n\}$ such that

$$1<|z_n|<r_n,\ |\arg z_n|=\alpha \qquad (n=1, 2, \cdots)$$

and

$$G_n(z_n)/g_n(z_n)\to\infty \qquad (n\to\infty)\,.$$

By Cauchy's mean value theorem we can find a $\{\rho_n\}$ with $1<\rho_n<r_n$ $(n=1, 2, \cdots)$ satisfying either that $\mu_n\to\infty(n\to\infty)$, where $\mu_n(n=1, 2, \cdots)$ are the values of $(\partial G_n/\partial r)/(\partial g_n/\partial r)$ at $z=\rho_n e^{i\alpha}$ or $\rho_n e^{-i\alpha}$. But this contradicts the uniform convergence in an appropriately extended, or that for iufinite n $\partial G_n/\partial r=\partial g_n/\partial r=0$ at $z=P_n e^{id}$ or $P_n e^{-id}$ region of the sequences of the Green functions and of their derivatives.

3. We finish up with a classification of Riemann surfaces.

By O_{AL}, O_R, O_1 we denote the classes of Riemann surfaces not admitting respectively nonconstant: Lindelöfian analytic functions, analytic functions whose real parts are dominated by positive harmonic functions, H^1-functions (i.e., analytic functions whose moduli are dominated by harmonic functions).

We establish the strict inclusion relations:

$$O_{AL}\subsetneqq O_R\subsetneqq O_1\,.$$

i) $O_{AL} \subsetneqq O_R$. $O_{AL} \subset O_R$ is due to the following theorem (Heins [2]). Every analytic function whose real part is dominated by a positive harmonic function is Lindelöfian. Here we show that Myrberg's example (Myrberg [4]) is an example for the concerned strict inclusion relation. Let F be two-sheeted covering surface of the plane given by the equation

$$w^2 - \sin z = 0 .$$

We remove from F a closed disk K on one sheet, and denote the resulting surface by F_1. The valence of the projection π of F_1 onto the plane is at most two, thereby π is a Lindelöfian analytic function on F_1. (Heins [2].) Hence $F_1 \notin O_{AL}$. Here we need only a weak version of Heins' result. Every meromorphic function such that the set of values taken by the function only a finite number of times has positive (logarithmic) capacity is Lindelöfian. This can be directly proved by Theorem 1 and the following elementary fact: the identity function of the plane is Lindelöfian on regions whose complements have positive capacity.

To show $F_1 \in O_R$ suppose that f is an analytic function whose real part is dominated by a positive harmonic function h on F_1. On the plane less the projection of K, we consider the function

$$\phi(z) = (e^{f(z^+)} - e^{f(z^-)})^2 ,$$

where z^+ and z^- are the points over z. Then ϕ is a single-valued analytic function, and

$$|\phi(z)| \leqq 4e^{2\{h(z^+)+h(z^-)\}} .$$

$H(z) = h(z^+) + h(z^-)$ is a single-valued positive harmonic function. Therefore it follows that ϕ is meromorphic in a neighborhood of the point at infinity. On the other hand, $\phi(z)=0$ at $z=n\pi$ $(n=\pm 1, \pm 2, \cdots)$. Consequently $\phi \equiv 0$, which shows $f(z^+) = f(z^-)$ and that f can be analytically continued onto F. A similar reasoning applied to $e^{f(z)}$ concludes that f is a constant.

ii) $O_R \subsetneqq O_1$. $O_R \subset O_1$ is trivial. Let E_0 be a compact set on the segment $\{-1+iy ; -1/2<y<1/2\}$ of linear measure 0 and positive capacity. Let E be the set $\{-1+iy+im ; -1+iy \in E_0, m \text{ integer}\}$. We show that $R = C \setminus E$ has the desired property. The fact that R belongs to O_1 is seen from the following theorem due to Heins [3]. The sets of linear measure 0 on a finite number of mutually disjoint analytic simple closed curves are null sets for H^1-functions.

To see $R \notin O_R$ let ω be the harmonic measure of the imaginary axis with respect to the left half plane less E. Since $\omega(z+i) = \omega(z)$,

$$\sup_{-\infty<y<\infty} \omega\left(-\frac{1}{2}+iy\right) < 1 .$$

We define a function w on R as follows:

$$w(x+iy)=\begin{cases} x+A & x\geqq 0 \\ A\omega & x<0. \end{cases}$$

For sufficiently large constant A, w is (nonnegative and) superharmonic in R and dominates the real part of the identity function on R. Hence $R\notin O_R$.

The idea used in the present paper of constructing superharmonic functions by two harmonic functions with same boundary values is found in [5].

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