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ON INVARIANT CLOSED GEODESICS UNDER ISOMETRIES

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§0. Introduction

It is an interesting problem to estimate the number of distinct closed geodesics on a compact Riemannian manifold. In [2] Gromoll and Meyer proved the existence of infinitely many geometrically distinct closed geodesics on a compact Riemannian manifold satisfying a certain topological condition. Recently Grove [5] extended their result by means of invariant closed geodesics under involutive isometries. In this paper we will prove a more general theorem than their results. Let M be a connected Riemannian manifold and h an isometry on the manifold M. A geodesic $\gamma:\boldsymbol{R}\rightarrow M$ is called an invariant geodesic under h if there exists some nonnegative constant θ such that $h(\gamma(t))=\gamma(t+\theta)$ for all $t\in\boldsymbol{R}$. Two such geodesics γ_1, γ_2 are said to be geometrically distinct if $\gamma_1(\boldsymbol{R})\neq\gamma_2(\boldsymbol{R})$. Let $C^0(M, h)$ be the topological space of all continuous curves $\sigma:[0,1]\rightarrow M$ satisfying $h(\sigma(0))=\sigma(1)$ with the compact open topology. Now we will state our main theorem.

MAIN THEOREM. *Let M be a compact simply connected Riemannian manifold and f an isometry satisfying $f^s=id.$ for some prime integer s. Then there exist infinitely many geometrically distinct invariant closed geodesics under f if the sequence of the Betti numbers for the space $C^0(M, f)$ is not bounded.*

Note. If $s=1$, i.e., $f=id.$, (resp. $s=2$) in our main theorem then we obtain the result of Gromoll and Meyer (resp. Grove).

§1. Preliminaries

Let $(M, \langle , \rangle)$ be an $n+1$ $(\geqq 2)$ dimensional compact Riemannian manifold, and h an isometry on the manifold M. A continuous curve $\gamma:[0,1]\rightarrow M$ will be called an H^1-curve when it is absolutely continuous and $\int_0^1\langle\dot{\gamma},\dot{\gamma}\rangle dt<\infty$, where $\dot{\gamma}$ denotes the velocity vector of γ. For each H^1-curve γ, a continuous vector field X along the curve γ will be called an H^1-vector field along γ when it is absolutely continuous and $\int_0^1\langle X', X'\rangle dt<\infty$, where X' denotes the covariant de-

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rivative of X along γ. Let $\Omega(M, h)$ be the set of H^1-curves σ from the unit interval I into M satisfying $h(\sigma(0))=\sigma(1)$. For each $\sigma\in\Omega(M, h)$, let $T_\sigma\Omega(M, h)$ be the set of H^1-vector fields X along the curve σ satisfying $h_*(X(0))=X(1)$, where h_* denotes the differential of the map h. The inner product on $T_\sigma\Omega(M, h)$ is defined by

$$(1) \qquad \ll X, Y\gg=\int_0^1(\langle X, Y\rangle+\langle X', Y'\rangle)dt \quad \text{for } X, Y\in T_\sigma\Omega(M, h).$$

By this inner product $T_\sigma\Omega(M, h)$ becomes a Hilbert space. $\Omega(M, h)$ has a structure of Riemannian Hilbert manifold [3]. The model spaces of $\Omega(M, h)$ are given by $\{T_\sigma\Omega(M, h); \sigma\in\Omega(M, h)\}$ and the Riemannian structure is given by (1). For each $\sigma\in\Omega(M, h)$ we can regard the model space $T_\sigma\Omega(M, h)$ as the tangent space of $\Omega(M, h)$ at σ. On $\Omega(M, h)$ we have the energy function E^h: $\Omega(M, h)\to\mathbf{R}$ defined by

$$E^h(\sigma)=1/2\int_0^1\langle\dot\sigma, \dot\sigma\rangle dt \quad \text{for } \sigma\in\Omega(M, h).$$

The following are well known facts.

(a) $E^h: \Omega(M, h)\to\mathbf{R}$ is a smooth function and satisfies condition (C) of Palais and Smale (see [3]).
(b) $\sigma\in\Omega(M, h)$ is a critical point for E^h if and only if σ is a geodesic on M satisfying $h_*\dot\sigma(0)=\dot\sigma(1)$ (see [3]). Particularly $\sigma\in\Omega(M, id.)$ is a critical point for $E^{id}: \Omega(M, id.)\to\mathbf{R}$ if and only if σ is a closed geodesic in M.
(c) The Hessian H_c of E^h at a critical point c is given by

$$H_c(X, Y)=\int_0^1(\langle X', Y'\rangle-\langle R(X, \dot c), \dot c Y\rangle)dt,$$

where R denotes the curvature tensor of M.

For each $\sigma\in\Omega(M, h)$ we always assume that σ is naturally defined on $\mathbf{R}$, i.e.,

$$(2) \qquad \sigma(t)=h^{[t]}(\sigma(t-[t])) \quad \text{for } t\in\mathbf{R},$$

where $[t]$ denotes the greatest integer $\leqq t$.

Let g be an isometry on M such that $g^s=id.$ for some positive integer s, and $SO(2)$ the parameter circle $[0, s]/\{0, s\}$. We may regard $SO(2)$ as an operation on $\Omega(M, g)$ as follows;

$$SO(2)\times\Omega(M, g)\longrightarrow\Omega(M, g),$$

$$(\alpha, \sigma)\longmapsto\alpha(\sigma), \text{ where } \alpha(\sigma)(t)=\sigma(t+\alpha).$$

Note that $\sigma(t+s)=\sigma(t)$ for all $t\in\mathbf{R}$ and $\sigma\in\Omega(M, g)$. This action is continuous and for each $\alpha\in SO(2)$, $\alpha: \Omega(M, g)\to\Omega(M, g)$ is an isometry [4]. A critical point c for E^g in $\Omega(M, g)$ lies always on a critical submanifold of $\Omega(M, g)$,

$SO(2)c$ when c is non constant, i.e., $E^g(c)\neq 0$. Now we shall construct a tubular neighborhood $\mathcal{D}$ of $SO(2)c$. We can take for $\mathcal{D}$ the diffeomorphic image of a sufficiently small tubular neighborhood of the zero section in the normal bundle $\mathcal{N}$ of $SO(2)c$ by the induced map from the exponential map exp of M, i.e., the map $\overline{\exp}: \mathcal{N}\to\Omega(M, g)$ with $Y\mapsto \exp\circ Y$ is a local diffeomorphism along the zero section of $\mathcal{N}$. So the normal space $\mathcal{N}_c$ over c is the tangent space of the fiber $\mathcal{D}_c$ at c and $\alpha(\mathcal{D}_c)=\mathcal{D}_{\alpha(c)}$ for $\alpha\in SO(2)$. Let $E_c{}^g$ be the restriction of the energy E^g to $\mathcal{D}_c$. For the Hessian $\tilde{H}_c$ of $E_c{}^g$ at c we obtain immediately $\tilde{H}_c=H_c|\mathcal{N}_c\oplus\mathcal{N}_c$.

The next lemma is essentially proved by Gromoll and Meyer [2].

LEMMA 1. *Let $c\in\Omega(M, g)$ be a non constant critical point. Then the operator $A_c: T_c\Omega(M, g)\to T_c\Omega(M, g)$ defined by*

$$\langle\!\langle A_cX, Y\rangle\!\rangle=H_c(X, Y)$$

admits a decomposition $A_c=id+k$ with a compact operator k. Clearly the corresponding operator $\tilde{A}_c$ for $\tilde{H}_c$ is also of the form $\tilde{A}_c=id+\tilde{k}$, where $\tilde{k}$ is compact.

In general let j be a smooth (C^∞) function defined on some open neighborhood of the origin in a Hilbert space $(H, \langle, \rangle)$ such that the origin 0 is an isolated critical point of j, and $j(0)=0$. Let d^2j_0 be the Hessian for j at the origin, and we assume that the operator $A: H\to H$ defined by $\langle Ax, y\rangle=d^2j_0(x, y)$ admits a decomposition $A=id+K$, where K is a compact operator. We put $N=\ker A$ and $E=N^\perp$, the orthogonal complement in H, so that $H=E\oplus N$. The next "splitting lemma" is due to Gromoll and Meyer [1].

LEMMA 2. *(Splitting lemma) Let j satisfy the assumptions as above. Then there exist an origin preserving diffeomorphism Φ of some neighborhood of 0 in H into H and an origin preserving smooth map h defined in some neighborhood of 0 in N into E such that $j\circ\Phi(x, y)=\langle Px, Px\rangle-\langle(I-P)x, (I-P)x\rangle+j(h(y), y)$ with an orthogonal projection $P: E\to E$.*

COROLLARY 3. *The function j satisfies condition (C) of Palais and Smale in some neighborhood of the origin.*

Proof. Let $\{\sigma_n\}$ be any sequence such that the gradient vector of j at σ_n, ∇j_{σ_n}, tends to zero as $n\to\infty$. We set $(x_n, y_n)=\Phi^{-1}(\sigma_n)$. If the points σ_n are in a sufficiently small neighborhood of the origin, the points y_n are in a bounded set. Since N is a finite dimensional linear subspace, $\{y_n\}$ has a convergent subsequence. On the other hand, by the splitting lemma

$$P_E(\nabla(j\circ\Phi)_{(x,y)})=2(2P-I)x\,,$$

where P_E denotes the orthogonal projection to E in H. Hence

$$2\|x\|=2\|(2P-I)x\|\leqq\|\nabla(j\circ\Phi)_{(x,y)}\|\leqq\|\Phi_{*(x,y)}\|\cdot\|\nabla j_{\Phi(x,y)}\|\,,$$

where $\|\cdot\|$ denotes the norm induced by the inner product $\langle, \rangle$. So if $\nabla j_{\sigma_n}\to 0$, then x_n tends to zero. Therefore the sequence $\{\sigma_n\}$ has a convergent subsequence. (q. e. d.)

Using Lemma 1 and Corollary 3, we have

PROPOSITION 4. *If c is an isolated critical point of E_c^g and $\mathcal{D}_c$ is sufficiently small, then condition (C) holds for E_c^g.*

Now we will define a local homological invariant $\mathcal{H}(E^g, SO(2)c)$ of the energy E^g at the isolated critical orbit $SO(2)c$ by using the construction and the notation of [1]. Choose a sufficiently small tubular neighborhood $\mathcal{D}$ such that E_c^g satisfies condition (C) and such that c is an isolated critical point of E_c^g (see p. 502 in [2]). Thus we can define a local homological invariant of E_c^g at c;

$$\mathcal{H}(E_c^g, c)=H_*(W_c, W_c^-),$$

where W_c and W_c^- are admissible regions for the function E_c^g on the fiber $\mathcal{D}_c$ at c (see [1]). For convenience we use singular homology with a field of characteristic zero. We define a local homological invariant $\mathcal{H}(E^g, SO(2)c)$ of the energy E^g at the isolated critical orbit $SO(2)c$ by

$$\mathcal{H}(E^g, SO(2)c)=H_*(W, W^-) \text{ where } W=SO(2)W_c \text{ and } W^-=SO(2)W_c^-.$$

It does not depend on the choice of the $\mathcal{D}$ and admissible regions W_c, W_c^-.

The next lemma is proved by Gromoll and Meyer [2].

LEMMA 5. *Let b be the only critical value of the energy $E^g: \Omega(M, g)\to \mathbf{R}$ in $[b-\varepsilon, b+\varepsilon]$ for some $\varepsilon>0$. Assume that the critical set in $(E^g)^{-1}(b)$ consists of finitely many critical orbits $SO(2)c^1, \cdots, SO(2)c^r$. Then*

$$H_*(\Omega^{b+\varepsilon}(M, g), \Omega^{b-\varepsilon}(M, g))=\sum_{i=1}^{r}\mathcal{H}(E^g, SO(2)c^i),$$

where $\Omega^{b\pm\varepsilon}(M, g)=(E^g)^{-1}[0, b\pm\varepsilon]$.

Let $a<b$ be regular values of the energy E^g such that the critical orbits in $(E^g)^{-1}[a, b]$ consist of finitely many critical orbits $SO(2)c^1, \cdots, SO(2)c^r$. Then we have the Morse inequalities from Lemma 5;

$$b_k(\Omega^b(M, g), \Omega^a(M, g))\leqq\sum_{i=1}^{r}B_k(c^i, g), \tag{3}$$

where $b_k(\Omega^b(M, g), \Omega^a(M, g))=\dim H_k(\Omega^b, \Omega^a)$

and

$$B_k(c^i, g)=\dim \mathcal{H}_k(E^g, SO(2)c^i).$$

If we define a map π of $(SO(2)\times W_c, SO(2)\times W_c^-)$ onto (W, W^-) by $(\alpha, e)\mapsto \alpha(e)$, then the map π is a covering map. Put $\Gamma=\{\alpha\in SO(2); \alpha(c)=c\}$, which is called the isotropy group at c. We can regard Γ as covering transformations on $(SO(2)\times W_c, SO(2)\times W_c^-)$ by $(\alpha, e)\mapsto(\alpha\beta^{-1}, \beta(e))$ for each $\beta\in\Gamma$. Since

$$(W, W^-)=(SO(2)\times W_c, SO(2)\times W_c^-)/\Gamma,$$

we have

(4) $$H_*(W, W^-)\subset H_*(SO(2)\times W_c, SO(2)\times W_c^-).$$

By the künneth formula

(5) $$\mathcal{H}(E^g, SO(2)c)\subset H_*(SO(2))\otimes\mathcal{H}(E_c^g, c).$$

Let λ be the index of c in $\Omega(M, g)$. Using the shifting theorem [1]

$$\mathcal{H}_{k+\lambda}(E_c^g, c)=\mathcal{H}_k^0(E_c^g, c),$$

where $\mathcal{H}_k^0$ denotes the characteristic invariant.

The last equality and (5) give

(6) $$\mathcal{H}_k(E^g, SO(2)c)\subset\mathcal{H}_{k-\lambda}^0(E_c^g, c)\oplus\mathcal{H}^0_{k-\lambda-1}(E_c^g, c).$$

Hence

(7) $$B_k(c, g)\leqq B_{k-\lambda}^0(c, g)+B^0_{k-\lambda-1}(c, g),$$

where $B_k^0(c, g)=\dim\mathcal{H}_k^0(E_c^g, c)$.

§2. Estimations of the indexes and nullities of all the critical orbits

For each $\sigma\in\Omega(M, g)$ and non zero integer m, we define a curve

$$\sigma_m\in\Omega(M, g^m) \quad \text{by} \quad \sigma_m(t)=\sigma(mt).$$

Note that each element in $\Omega(M, g)$ is assumed to be a map from $\boldsymbol{R}$ into M by (2). Then we can define the interation map $m: \Omega(M, g)\to\Omega(M, g^m)$ by $\sigma\mapsto\sigma_m$ for each non zero integer m. The next theorem is important for us. It is essentially proved by Gromoll and Meyer [2].

THEOREM 6. *Let $SO(2)c$ be a non constant critical orbit in $\Omega(M, g)$ such that $SO(2)c_m$ is an isolated critical orbit in $\Omega(M, g^m)$ and $\nu(c, g)=\nu(c_m, g^m)$ for some non zero integer m. Then $\mathcal{H}_k^0(E_c^g, c)=\mathcal{H}_k^0(E_{c_m}^{g^m}, c_m)$ for all all k. Here $\nu(c, g)$ (resp. $\nu(c_m, g^m)$) denotes the nullity of the critical submanifold $SO(2)c$ (resp. $SO(2)c_m$) in $\Omega(M, g)$ (resp. $\Omega(M, g^m)$).*

Let f be an isometry on M with an order s, and we assume that s is prime. Now we will study the indexes and nullities of all the critical orbits in $\Omega(M, f)$ generated by the iteration of a critical point. Let σ be a non constant critical point. Since $f(\sigma(t))=\sigma(t+1)$ and $f^s=id$, then $\sigma(t+s)=\sigma(t)$, that is, c is a closed geodesic with the fundamental period s/m, where m is some positive integer. For a critical point $\gamma\in\Omega(M, f)$ there are the following possibilities.

1) $\gamma(t)=p$ for all $t\in[0, 1]$ where the point p is a fixed point of f.
2) The fundamental period of a critical point is $1/m_0$ for some positive integer m_0.
3) The fundamental period of a critical point is s/m_0 for some positive integer m_0 with $(m_0, s)=1$.

A critical point of type 1) is constant. The other critical points are non constant. At first we will study a critical point c of type 3). Since m_0 and s are relatively prime, there exist some integers n_0 and k_0 satisfying $m_0 n_0 = 1 + s k_0$, hence $n_0 = 1/m_0 + (s/m_0)k_0$. If we set $\bar{c}(t) = c(t/m_0)$ for $t \in [0, 1]$ and $g = f^{n_0}$, then $\bar{c}$ is a critical point for E^g and the fundamental period of $\bar{c}$ is s. Clearly for each integer m and r with $ms + rm_0 \neq 0$, $\bar{c}_{ms+rm_0}$ is a critical point for E^{f^r}. The critical orbits $SO(2)\bar{c}_{ms+m_0}$, $m \in \boldsymbol{Z}$, are all the orbits in $\Omega(M, f)$ generated by the closed geodesic c. We may assume $1 \leqq m_0 < s$ without loss of generality.

Let $V_{\bar{c}}$ be the vector space of smooth vector fields along $\bar{c}$ orthogonal to $\bar{c}$. A linear map $L_{\bar{c}}: V_{\bar{c}} \to V_{\bar{c}}$ is defined by $L_{\bar{c}} X = -X'' - R(X, \dot{\bar{c}})\dot{\bar{c}}$. Let $\lambda(\bar{c}_{ms+rm_0}, f^r)$ and $\nu(\bar{c}_{ms+rm_0}, f^r)$ be the index and the nullity of the submanifold

$$SO(2)\bar{c}_{ms+rm_0} \quad \text{in} \quad \Omega(M, f^r)$$

respectively. We have

$$\lambda(\bar{c}_{ms+m_0}, f) = \sum_{\mu<0} \dim\{X \in V_{\bar{c}};\ L_{\bar{c}} X = \mu X,\ X(t+ms+m_0) = f_*(X(t)) \quad \text{for all } t \in \boldsymbol{R}\},$$

$$\nu(\bar{c}_{ms+rm_0}, f^r) = \dim\{X \in V_{\bar{c}};\ L_{\bar{c}} X = 0,\ X(t+ms+rm_0) = f_*{}^r(X(t)) \quad \text{for all } t \in \boldsymbol{R}\}$$

(See Theorem 2.3 in [6, p. 45].)

Let us complexify $V_{\bar{c}}$ and write it as $V_{\bar{c}}$ again. We also extend f_*, g and $L_{\bar{c}}$ to $\boldsymbol{C}$-linear maps and write them as f_*, g_*, $L_{\bar{c}}$ again respectively. For a complex number $\omega \in S^1 \subset \boldsymbol{C}$, a real number μ and a non zero integer m, let $S_{\bar{c}}[\mu, m, \omega g_*{}^m]$ denote the vector space of complex vector fields Y in $V_{\bar{c}}$ satisfying $L_{\bar{c}} Y = \mu Y$ and $Y(t+m) = \omega g_*{}^m(Y(t))$.

LEMMA 7. $S_{\bar{c}}[\mu, m, g_*{}^m] = \bigoplus_{\omega^m=1} S_{\bar{c}}[\mu, 1, \omega g_*]$.

Proof. It is trivial that $S_{\bar{c}}[\mu, m, g_*{}^m] \supset \bigoplus_{\omega^m=1} S_{\bar{c}}[\mu, 1, \omega g_*]$. We assume that m is positive. We can prove the lemma analogously for negative intgers. For any $Y \in S_{\bar{c}}[\mu, m, g_*{}^m]$ and ω with $\omega^m = 1$, we set

$$Y_\omega(t) = 1/m \sum_{l=0}^{m-1} \omega^{-l} g_*{}^{-l+1}(Y(t+l-1)).$$

Clearly, $L_{\bar{c}} Y_\omega = \mu Y_\omega$ and $Y = \sum_{\omega^m=1} \omega Y_\omega$. From the definition of Y_ω,

$$Y_\omega(t+1) = 1/m\Big[\sum_{l=0}^{m-1} \omega^{-l} g_*{}^{-l+1}(Y(t+l))\Big]$$

$$= \omega/m\Big[g_*\Big(\sum_{l=0}^{m-1} \omega^{-l-1} g_*{}^{-l}(Y(t+l))\Big)\Big]$$

$$=\omega/m[g_*\{\sum_{l=1}^{m-1}\omega^{-l}g_*{}^{-l+1}(Y(t+l-1))$$
$$+\omega^{-m}g_*{}^{-m+1}(Y(t+m-1))\}]$$
$$=\omega g_*(Y_\omega(t))\,.$$

Hence $Y_\omega\in S_{\bar c}[\mu, 1, \omega g_*]$. (q. e. d.)

Since $f^r=g^{ms+rm_0}$, $S_{\bar c}[\mu, ms+rm_0, f_*{}^r]=\bigoplus_{\omega^{ms+rm_0}=1} S_{\bar c}[\mu, 1, \omega g_*]$.

Putting $\Lambda_{\bar c}(\omega)=\sum_{\mu<0}\dim_C S_{\bar c}[\mu, 1, \omega g_*]$ and $N_{\bar c}(\omega)=\dim_C S_{\bar c}[0, 1, \omega g_*]$, we obtain

$$\lambda(\bar c_{ms+m_0}, f)=\sum_{\omega^{ms+m_0}=1}\Lambda_{\bar c}(\omega)\,,$$
(8)
$$\nu(\bar c_{ms+rm_0}, f^r)=\sum_{\omega^{ms+rm_0}=1}N_{\bar c}(\omega)\,.$$

It follows that $\lambda(\bar c_{ms+m_0}, f)$ and $\nu(\bar c_{ms+rm_0}, f^r)$ are completely determined by the nonnegative integer valued functions $\Lambda_{\bar c}(\cdot)$ and $N_{\bar c}(\cdot)$ on the unit circle respectively.

Let E denote the complexification of the orthogonal complement of $\dot{\bar c}(0)$ in the tangent space $M_{\bar c(0)}$ at $\bar c(0)$. Then so called Poincaré map P is defined in the following;

$$P: E\oplus E\longrightarrow E\oplus E,\quad (u, v)\longmapsto (g_*{}^{-1}(Y(1)), g_*{}^{-1}(Y'(1)))\,,$$

where Y is the unique complex Jacobi field (i.e. $L_{\bar c}Y=0$) satisfying $Y(0)=u$ and $Y'(0)=v$. Since $N_{\bar c}(z)=\dim_C\ker(P-z)$ and $\dim_C(E\oplus E)=2n$, we obtain

LEMMA 8. *$N_{\bar c}(z)=0$ except for at most $2n$ points which will be called Poincaré points.*

The next theorem is contained in Theorem 3.1 and 3.2 of M. Morse [6, p. 91].

THEOREM 9. *Let J be a bounded interval such that the end points are not in the eigenvalues of $L_{\bar c}$ subject to the boundary condition $Y(t+1)=zg_*(Y(t))$. Then there is a neighborhood U of z in S^1 such that the end points of J are not in the eigenvalues of $L_{\bar c}$ subject to $Y(t+1)=\omega g_*(Y(t))$ for $\omega\in U$ and*

$$\sum_{\mu\in J}\dim_C S_{\bar c}[\mu, 1, \omega g_*]=\sum_{\mu\in J}\dim_C S_{\bar c}[\mu, 1, zg_*]\,.$$

It follows from Theorem 9 that

$\Lambda_{\bar c}(\cdot)$ is locally constant except possibly at Poincaré points,
(9)
and $\lim_{z\to z_0}\Lambda_{\bar c}(z)\geqq\Lambda_{\bar c}(z_0)$.

By using Lemma 8, (8) and (9) we obtain the following two lemmas.

LEMMA 10. *Either* $\lambda(\bar{c}_{ms+m_0}, f)=0$ *for all* m *or there are positive numbers* a *and* ε *such that for any integers* $m_1 \geqq m_2 \geqq 0$

$$\lambda(\bar{c}_{m_1 s+m_0}, f)-\lambda(\bar{c}_{m_2 s+m_0}, f) \geqq (m_1-m_2)\varepsilon-a$$

and such that for any negative integers $m_1 \leqq m_2$

$$\lambda(\bar{c}_{m_1 s+m_0}, f)-\lambda(\bar{c}_{m_2 s+m_0}, f) \geqq (m_2-m_1)\varepsilon-a\,.$$

The proof of Lemma 10 is analogous to that of Lemma 1 in [2].

LEMMA 11. *There exist positive integers* $k_1, \cdots, k_q$ *and sequences* $m_j{}^i \in \boldsymbol{Z}$, $i>0$, $j=1, \cdots, q$, *such that the numbers* $m_j{}^i k_j$ *are mutually distinct,* $\{m_j{}^i k_j\}=\{ms+m_0\,;\, m\in \boldsymbol{Z}\}$ *and*

$$\nu(\bar{c}_{m_j{}^i k_j}, f)=\nu(\bar{c}_{k_j}, f^r) \text{ where } r\cdot m_j{}^i \equiv 1 \bmod s\,.$$

Outline of proof. We can prove analogously to Lemma 2 in [2] that there exist positive integers $\bar{k}_1, \cdots, \bar{k}_l$ and sequences $\bar{m}_j{}^i \in \boldsymbol{Z}$, $i>0$, $j=1, \cdots, l$, such that the numbers $\bar{m}_j{}^i \bar{k}_j$ are mutually distinct, $\{\bar{m}_j{}^i \bar{k}_j\}=\boldsymbol{Z}-\{0\}$ and

$$\sum_{\omega^{\bar{m}_j^i \bar{k}_j}=1} N_{\bar{c}}(\omega)=\sum_{\omega^{\bar{k}_j}=1} N_{\bar{c}}(\omega)\,.$$

Choose some elements $k_1, \cdots k_q$ (resp. $m_j{}^i$) from the set $\{\bar{k}_1, \cdots \bar{k}_l\}$ (resp. $\{\bar{m}_j{}^i\,;\, i>0, j=1, \cdots, l\}$) to satisfy $\{m_j{}^i k_j\}=\{ms+m_0\,;\, m\in \boldsymbol{Z}\}$. We can checkeasily by using (8) that $\nu(\bar{c}_{m_j{}^i k_j}, f)=\nu(\bar{c}_{k_j}, f^r)$ holds. (q. e. d.)

Combining Theorem 6 and Lemma 11 we obtain

COROLLARY 12. *Let* c *be a critical point in* $\Omega(M, f)$ *of type* 3) *and we assume that all the critical orbits* $SO(2)\bar{c}_{ms+m_0}$, $m\in \boldsymbol{Z}$, *are isolated in* $\Omega(M, f)$. *Then there exists some constant* B *such that* $B_k{}^0(\bar{c}_{ms+m_0}, f) \leqq B$ *for all* k *and* m. *Furthermore there exists a number* k_0 *such that* $B_k{}^0(\bar{c}_{ms+m_0}, f)=0$ *for* $k>k_0$ *and all* m.

Note that $\nu(\bar{c}_{ms+m_0}, f) \leqq 2n$ for all m. Hence we can take the number k_0 to be not greater than $2n$.

Combining (7), Lemma 10 and Corollary 12 we obtain

COROLLARY 13. *Under the hypotheses of Corollary* 12, *for the resulting constants* B *and* k_0, $B_k(\bar{c}_{ms+m_0}, f)$ *are uniformly bounded by* $2B$. *Moreover, given* $k>k_0+1$, *the number of orbits* $SO(2)\bar{c}_{ms+m_0}$ *such that* $B_k(\bar{c}_{ms+m_0}, f)\neq 0$ *is bounded by a constant* C *which does not depend on* k.

The proof of the above corollary is the same as that of Corollary 2 in [2].

Next we will prove analogous corollaries to Corollary 12 and Corollary 13 for a critical point c of type 2). If we set $\bar{c}(t)=c(t/m_0)$, then $\bar{c}$ is critical for E^f. The fundamental period of $\bar{c}$ is 1, and the orbits $SO(2)\bar{c}_m$, $m\in \boldsymbol{Z}-\{0\}$, are all the critical orbits in $\Omega(M, f)$ generated by c. Therefore we may assume that the critical point $\bar{c}$ is c, that is, $m_0=1$. Let V_c be a vector space of smooth vector fields along c which are orthogonal to c. A linear map $L_c: V_c \to V_c$ is defined by

$$L_c X = -X'' - R(X', \dot{c})\dot{c}\,.$$

Complexify V_c and write it as V_c again. We also extend f_* and L_c to $\boldsymbol{C}$-linear maps, and write them as f_*, L_c again respectively. For each non zero integer m, real number μ and $\omega \in S^1 \subset \boldsymbol{C}$, let $S_c[\mu, m, \omega f_*]$ be the set of complex vector fields $X \in V_c$ satisfying $L_c X = \mu X$ and $X(t+m) = \omega f_*(X(t))$.

LEMMA 14. *The next equalities hold for any integer r, $m(\neq 0)$ and real μ.*

1) $$S_c[\mu, m, f_*^r] = \bigoplus_{\omega^{ms}=1} S_c[\mu, 1, \omega f_*] \cap S_c[\mu, m, f_*^r]\,,$$

2) $$S_c[\mu, 1, \omega f_*] \cap S_c[\mu, m, f_*^r] = S_c[\mu, 1, \omega f_*] \cap \ker(f_*^{m-r} - \omega^{-m})\,,$$

where the linear map $f_*: V_c \to V_c$ is defined by $(f_* X)(t) = f_*(X(t))$.

3) $$S_c[\mu, 1, \omega f_*] \cap \ker(f_*^{m-r} - \alpha^{-1}) = \bigoplus_{z^{m-r}=\alpha^{-1}} S_c[\mu, 1, \omega f_*] \cap \ker(f_* - z)\,,$$

where we set $\omega^m = \alpha$.

Proof. If $|ms|=1$, then $s=1$ and $m=\pm 1$. Since $S_c[\mu, 1, id.] = S_c[\mu, -1, id.]$, the first equality is trivial. If $|ms| \geqq 2$, for any $Y \in S_c[\mu, m, f_*^r]$ and ω with $\omega^{ms}=1$, we set $Y_\omega(t) = 1/|ms| \sum_{q=0}^{|ms|-1} \omega^{-q} f_*^{-q+1}(Y(t+q-1))$. It is easy to check that $Y_\omega \in S_c[\mu, 1, \omega f_*] \cap S_c[\mu, m, f_*^r]$ and that $Y = \sum_{\omega^{ms}=1} \omega Y_\omega$ (see Lemma 7). Thus the first equality holds since it is trivial that

$$S_c[\mu, m, f_*^r] \supset \bigoplus_{\omega^{ms}=1} S_c[\mu, 1, \omega f_*] \cap S_c[\mu, m, f_*^r]\,.$$

We derive the second equality from a direct computation.

It is trivial that the third equality holds for $m-r=1$ and that

$$S_c[\mu, 1, \omega f_*] \cap \ker(f_*^{m-r} - \alpha^{-1}) \supset \bigoplus_{z^{m-r}=\alpha^{-1}} S_c[\mu, 1, \omega f_*] \cap \ker(f_* - z)\,.$$

If $m-r \geqq 2$, for any $Y \in S_c[\mu, 1, \omega f_*] \cap \ker(f_*^{m-r} - \alpha^{-1})$ and z with $z^{m-r} = \alpha^{-1}$, we set

$$Y_z = 1/(m-r) \sum_{l=1}^{m-r-1} z^{-l} f_*^{l-1}(Y)\,.$$

We can check easily that $Y_z \in S_c[\mu, 1, \omega f_*] \cap \ker(f_* - z)$ and $Y = \sum_{z^{m-r}=\alpha^{-1}} z Y_z$. Hence the equality holds for $m-r \geqq 1$. If $m-r=0$.

$$S_c[\mu, 1, \omega f_*]\cap\ker(f_*^0-\alpha^{-1})=S_c[\mu, 1, \omega f_*]\cap\ker(f_*^s-\alpha^{-1})$$
$$=\bigoplus_{z^s=\alpha^{-1}} S_c[\mu, 1, \omega f_*]\cap\ker(f_*-z)$$
$$=\bigoplus_{z^0=\alpha^{-1}} S_c[\mu, 1, \omega f_*]\cap\ker(f_*-z),$$

because $z^s=1$ for any z with $\ker(f_*-z)\neq\{0\}$.

If $m-r<0$,

$$S_c[\mu, 1, \omega f_*]\cap\ker(f_*^{m-r}-\alpha^{-1})=S_c[\mu, 1, \omega f_*]\cap\ker(f_*^{r-m}-\alpha)$$
$$=\bigoplus_{z^{r-m}=\alpha} S_c[\mu, 1, \omega f_*]\cap\ker(f_*-z)$$
$$=\bigoplus_{z^{m-r}=\alpha^{-1}} S_c[\mu, 1, \omega f_*]\cap\ker(f_*-z).$$

(q. e. d.)

It follows from the above lemma that

$$S_c[\mu, m, f_*^r]=\bigoplus_{\omega^{sm}=1}\ \bigoplus_{z^{m-r}=\omega^{-m}} S_c[\mu, 1, \omega f_*]\cap\ker(f_*-z)$$
$$=\bigoplus_{\alpha^s=1}\ \bigoplus_{\omega^m=\alpha}\ \bigoplus_{z^{m-r}=\alpha^{-1}} S_c[\mu, 1, \omega f_*]\cap\ker(f_*-z).$$

Therefore we have

$$\lambda(c_m, f)=\sum_{\alpha^s=1}\ \sum_{\omega^m=\alpha}\ \sum_{z^{m-1}=\alpha^{-1}} \Lambda_c^z(\omega),$$
(10)
$$\nu(c_m, f^r)=\sum_{\alpha^s=1}\ \sum_{\omega^m=\alpha}\ \sum_{z^{m-r}=\alpha^{-1}} N_c^z(\omega),$$

where we put

$$\Lambda_c^z(\omega)=\sum_{\mu<0}\dim_{\boldsymbol{C}}\{S_c[\mu, 1, \omega f_*]\cap\ker(f_*-z)\}$$

and

$$N_c^z(\omega)=\dim_{\boldsymbol{C}}\{S_c[0, 1, \omega f_*]\cap\ker(f_*-z)\}.$$

If follows that $\lambda(c_m, f)$ and $\nu(c_m, f^r)$ are completely determined by the non-negative integer valued functions $\Lambda_c^z(\cdot)$ and $N_c^z(\cdot)$ on the unit circle. We obtain (see Lemma 8 and Theorem 9) the following lemma.

LEMMA 15.
i) $N_c^z(\omega)=0$ *except for at most* $2n$ *points which will be called Poincaré points with respect to* z.
ii) $\Lambda_c^z(\omega)$ *is locally constant except possibly at Poincaré points with respect to* z.
iii) $\lim_{\omega\to\omega_0}\Lambda_c^z(\omega)\geqq\Lambda_c^z(\omega_0)$.
iv) *For any* z *with* $\ker(f_*-z)=\{0\}$, $\Lambda_c^z\equiv 0$ *and* $N_c^z\equiv 0$.

Let $\boldsymbol{Z}^+$ and $\boldsymbol{Z}^-$ denote the set of all positive integers and the set of all negative integers respectively. For each integer l, we put

$$D_l^+=\{m\in \boldsymbol{Z}^+\,;\, m-1\equiv l \bmod s\},\ D_l^-=\{m\in \boldsymbol{Z}^-\,;\, m-1\equiv l \bmod s\}$$

and $D_l=D_l^+\cup D_l^-$.

LEMMA 16. *For each $0\leqq l<s$, either $\lambda(c_m, f)=0$ for all $m\in D_l$ or there exist positive numbers ε_l and a_l such that for any $m_i\in D_l^+$, $i=1, 2$ with $m_1\geqq m_2$,*

$$\lambda(c_{m_1}, f)-\lambda(c_{m_2}, f)\geqq(m_1-m_2)\varepsilon_l-a_l$$

and such that for any $m_i\in D_l^-$, $i=1, 2$ with $m_2\geqq m_1$,

$$\lambda(c_{m_1}, f)-\lambda(c_{m_2}, f)\geqq(m_2-m_1)\varepsilon_l-a_l\,.$$

Proof. It follows from (10) and Lemma 15 that for each $m\in D_l$,

$$\lambda(c_m, f)=\sum_{\alpha^s=1}\ \sum_{\omega^m=\alpha} F_\alpha^{\,l}(\omega)\,,$$

where

$$F_\alpha^{\,l}(\omega)=\sum_{z^l=\alpha^{-1}} \Lambda_c^{\,z}(\omega)\,.$$

If $F_\alpha^{\,l}\not\equiv 0$, then there exist some positive numbers ε_l^α and a_l^α such that

$$\sum_{\omega^{m_1}=\alpha} F_\alpha^{\,l}(\omega)-\sum_{\omega^{m_2}=\alpha} F_\alpha^{\,l}(\omega)\geqq(|m_1|-|m_2|)\varepsilon_l^\alpha-a_l^\alpha$$

for any $m_i\in D_l$, $i=1, 2$ with $|m_1|\geqq|m_2|$. We can prove the existence of such numbers ε_l^α and a_l^α analogously to Lemma 1 in [2]. Therefore if $\lambda(c_{m_0}, f)\neq 0$ for some $m_0\in D_l$, then $F_\alpha^{\,l}\not\equiv 0$ for some α. Set $\varepsilon_l=\sum_\alpha' \varepsilon_l^\alpha$ and $a_l=\sum_\alpha' a_l^\alpha$, where $\sum_\alpha'$ denotes the sum of all α, $\alpha^s=1$, satisfying $F_\alpha^{\,l}\not\equiv 0$. For any $m_i\in D_l$, $i=1, 2$ with $|m_1|\geqq|m_2|$

$$\begin{aligned}\lambda(c_{m_1}, f)-\lambda(c_{m_2}, f)&=\sum_\alpha'\Big(\sum_{\omega^{m_1}=\alpha} F_\alpha^{\,l}(\omega)-\sum_{\omega^{m_2}=\alpha} F_\alpha^{\,l}(\omega)\Big)\\&=\sum_\alpha'\,[(|m_1|-|m_2|)\varepsilon_l^\alpha-a_l^\alpha]\\&=(|m_1|-|m_2|)\varepsilon_l-a_l\,.\end{aligned}$$

(q. e. d.)

LEMMA 17. *For each $0\leqq l<s$, there exist positive integers $k_1, \cdots, k_q$ and sequences $m_j^{\,i}$, $j=1, \cdots, q$, $i>0$ such that the numbers $m_j^{\,i}k_j$ are mutually distinct, $\{m_j^{\,i}k_j\}=D_l$, and for $m_j^{\,i}$ with $(m_j^{\,i}, s)=1$*

$$\nu(c_{m_j^ik_j}, f)=\nu(c_{k_j}, f^r)\qquad \textit{where } r\cdot m_j^{\,i}\equiv 1 \bmod s\,,$$

and for $m_j^{\,i}$ with $(m_j^{\,i}, s)\neq 1$,

$$\nu(c_{m_j^ik_j}, f)=\nu^T(c_{m_j^ik_j})=\nu^T(c_{k_j})\,.$$

Here $\nu^T(\bar{c})$ denotes the nullity of the critical submanifold $SO(2)\bar{c}$ in $\Omega(\mathrm{Fix}(f)$, *id.*) where Fix(f) is the set of the fixed points of f. In general for any isometry h, Fix(h) is a totally geodesic submanifold of M.

Proof. It follows from (10) and Lemma 15 that

$$\nu(c_m, f)=\sum_{\alpha^s=1}\sum_{\omega^m=\alpha}\Big(\sum_{z^l=\alpha^{-1}} N_c^z(\omega)\Big) \qquad \text{for any } m\in D_l .$$

For each $\alpha=\exp(2\pi\imath t/s)$ $(t\not\equiv 0 \bmod s)$, we set

$$Q_l^\alpha=\{q\in \boldsymbol{Z}^+;\ \sum_{z^l=\alpha^{-1}} N_c^z(\exp(2\pi ip/sq))\neq 0,\ 0<{}^\exists p\leqq sq,\ (p, sq)=1\} .$$

And put

$$Q_l^1=\{q\in \boldsymbol{Z}^+;\ \sum_{z^l=1} N_c^z(\exp(2\pi ip/q))=0,\ 0<{}^\exists p\leqq q,\ (p, q)=1\}$$

and $Q_l=\bigcup_{\alpha^s=1} Q_l^\alpha$. Note that if $Q_l=\phi$, then $\nu(c_m, f)=0$ for any $m\in D_l$. In case $Q=\phi$, it is sufficient to prove that for any m with $(m, s)=1$,

$$\nu(c_m, f)\geqq\nu(c, f^r) , \qquad \text{where } r\cdot m\equiv 1 \bmod s ,$$

and for any non zero m,

$$\nu(c_m, f)\geqq\nu^T(c_m)\geqq\nu^T(c) .$$

At first we consider the case where $(m, s)=1$. If we set $\omega=\alpha^r$ for each α with $\alpha^s=1$, then $\omega^m=\alpha$. Hence,

$$\begin{aligned}\nu(c_m, f)&\geqq \sum_{\alpha^s=1}\sum_{z^l=\alpha^{-1}} N_c^z(\alpha^r)\\ &=\sum_{\alpha^s=1}\sum_{z^{lr}=\alpha^{-r}} N_c^z(\alpha^r)\\ &=\sum_{\beta^s=1}\sum_{z^{lr}=\beta^{-1}} N_c^z(\beta) \qquad \text{(Here we set } \beta=\alpha^r.)\\ &=\sum_{\beta^s=1}\sum_{z^{1-r}=\beta^{-1}} N_c^z(\beta)\\ &=\nu(c, f^r)\ \text{(by (10))} .\end{aligned}$$

On the other hand,

$$\begin{aligned}\nu(c_m, f)&=\sum_{\alpha^s=1}\sum_{\omega^m=\alpha}\sum_{z^l=\alpha^{-1}} N_c^z(\omega)\geqq\sum_{\omega^m=1}\sum_{z^l=1} N_c^z(\omega)\\ &\geqq\sum_{\omega^m=1} N_c^1(\omega)\geqq N_c^1(1) .\end{aligned}$$

It follows from (10) that

$$\nu^T(c_m)=\sum_{\omega^m=1} N_c^1(\omega) \qquad \text{and } \nu^T(c)=N_c^1(1) .$$

Combining the above inequality and the last equality, we obtain that $\nu(c_m, f) \geqq \nu^T(c_m) \geqq \nu^T(c)$ for any non zero m.

In case $Q_l \neq \phi$, for each subset $A \subset Q_l$ let $k(A)$ denote the least common multiple of all elements in A. Choose distinct numbers $\bar{k}_1, \cdots, \bar{k}_u$ such that $\{\bar{k}_1, \cdots, \bar{k}_u\} = \{1\} \cup \{k(A) ; A \subset Q_l\}$. Keeping $j \in \{1, \cdots, u\}$ fixed, we select from the sequence $m\bar{k}_j \in \boldsymbol{Z} - \{0\}$, the greatest subsequence $\bar{m}_j{}^i \bar{k}_j$ satisfying $q \nmid \bar{m}_j{}^i \bar{k}_j$ whenever $q \in Q_l$ and $q \nmid \bar{k}_j$. Then the numbers $\bar{m}_j{}^i \bar{k}_j$ are mutually distinct, $\{\bar{m}_j{}^i ; i > 0\}$ contains 1 for each $j \in \{1, \cdots, u\}$ and $\{\bar{m}_j{}^i \bar{k}_j ; i > 0, j = 1, \cdots u\} = \boldsymbol{Z} - \{0\}$. Choose some elements $k_1, \cdots, k_q$ (resp. $m_j{}^i$, $i > 0$, $j = 1, \cdots, q$) from the set $\{\bar{k}_1, \cdots, \bar{k}_u\}$ (resp. $\{\bar{m}_j{}^i ; i > 0, j = 1, \cdots, u\}$) to satisfy $\{m_j{}^i k_j ; i > 0, j = 1, \cdots, q\} = D_l$. If $\sum_{\omega^{m_j{}^i k_j} = \alpha} \sum_{z^l = \alpha^{-1}} N_c{}^z(\omega) \neq 0$ for some $\alpha = \exp(2\pi it/s)$ $(t \not\equiv 0 \bmod s)$, there exist some positive integers $q \in Q_l{}^\alpha$ and p satisfying $(\exp(2\pi ip/sq))^{m_j{}^i k_j} = \exp(2\pi it/s)$. Since $(p/sq) m_j{}^i k_j \equiv t/s \bmod 1$, $(p/q) \cdot m_j{}^i k_j \equiv t \bmod s$. The integer q devides k_j because $q \mid m_j{}^i k_j$ and $q \in Q_l$. Since $((pk_j/q) \cdot m_j{}^i, s) = 1$, $(m_j{}^i, s) = 1$. Therefore if $(m_j{}^i, s) \neq 1$, then

$$\nu(c_{m_j{}^i k_j}, f) = \sum_{\omega^{m_j{}^i k_j} = 1} \sum_{z^l = 1} N_c{}^z(\omega) .$$

If $\omega^{m_j{}^i k_j} = 1$ and $\sum_{z^l = 1} N_c{}^z(\omega) \neq 0$, then $\omega^{k_j} = 1$. Thus

$$\nu(c_{m_j{}^i k_j}, f) = \sum_{\omega^{k_j} = 1} \sum_{z^l = 1} N_c{}^z(\omega) .$$

On the other hand, if we note that $N_c{}^z \equiv 0$ for any z with $z^s \neq 1$, then

$$\sum_{z^l = 1} N_c{}^z(\omega) = N_c{}^1(\omega)$$

for each ω since $l \equiv -1 \bmod s$. We obtain

$$\nu(c_{m_j{}^i k_j}, f) = \sum_{\omega^{m_j{}^i k_j} = 1} N_c{}^1(\omega) = \sum_{\omega^{k_j} = 1} N_c{}^1(\omega) .$$

By using (10)

$$\nu^T(c_{m_j{}^i k_j}, f) = \sum_{\omega^{m_j{}^i k_j} = 1} N_c{}^1(\omega) \quad \text{and} \quad \nu^T(c_{k_j}) = \sum_{\omega^{k_j} = 1} N_c{}^1(\omega) .$$

If $(m_j{}^i, s) = 1$, there exists some integer r with $r \cdot m_j{}^i \equiv 1 \bmod s$. Since

$$\{\omega ; \omega^{m_j{}^i k_j} = \alpha, \sum_{z^l = \alpha^{-1}} N_c{}^z(\omega) \neq 0\} = \{\omega ; \omega^{k_j} = \alpha^r, \sum_{z^l = \alpha^{-1}} N_c{}^z(\omega) \neq 0\}$$

for each α,

$$\nu(c_{m_j{}^i k_j}, f) = \sum_{\alpha^s = 1} \sum_{\omega^{k_j} = \alpha^r} \sum_{z^l = \alpha^{-1}} N_c{}^z(\omega) .$$

On the other hand, if we note that $k_j - r \equiv lr \bmod s$ since $m_j{}^i k_j - 1 \equiv l \bmod s$, then

$$\begin{aligned}\nu(c_{k_j}, f^r) &= \sum_{\beta^s=1} \sum_{\omega^{k_j}=\beta} \sum_{z^{lr}=\beta^{-1}} N_c^z(\omega) \\ &= \sum_{\alpha^s=1} \sum_{\omega^{k_j}=\alpha^r} \sum_{z^{lr}=\alpha^{-r}} N_c^z(\omega) \text{ (Here we set } \beta^{m_j^i}=\alpha.) \\ &= \sum_{\alpha^s=1} \sum_{\omega^{k_j}=\alpha^r} \sum_{z^l=\alpha^{-1}} N_c^z(\omega) = \nu(c_{m_j^i k_j}, f),\end{aligned}$$

since $\{z\,;\, z^{lr}=\alpha^{-r},\ z^s=1\}=\{z\,;\, z=\alpha^{-1},\ z^s=1\}$. (q. e. d.)

We assume that the critical orbit $SO(2)c_{m_j^i k_j}$ is isolated in $\Omega(M, f)$. If $(m_j^i, s)=1$, it follows from Theorem 6 that

$$\mathcal{H}^0(E^f{}_{c_{m_j^i k_j}}, c_{m_j^i k_j})=\mathcal{H}^0(E_{c_{k_j}}{}^{f^r}, c_{k_j}).$$

If $(m_j^i, s)\neq 1$, then it holds that

$$\mathcal{H}^0(E^f{}_{c_{m_j^i k_j}}, c_{m_j^i k_j})=\mathcal{H}^0(c_{m_j^i k_j})^T.$$

(See the proof of Lemma 3.6 in [5].) Furthermore it follows from Theorem 6 that

$$\mathcal{H}^0(c_{m_j^i k_j})^T=\mathcal{H}^0(c_{k_j})^T.$$

Here $\mathcal{H}^0(c_m)^T$ denotes the characteristic invariant of c_m in the space Ω(Fix (f), *id.*).

COROLLARY 18. *Let c be a critical point of fundamental period* 1. *We assume that all the critical orbits $SO(2)c_m$, $m\in \boldsymbol{Z}-\{0\}$, are isolated in $\Omega(M, f)$. Then there exists some constant B such that $B_k{}^0(c_m, f)\leqq B$ for all $m\in\boldsymbol{Z}-\{0\}$ and k. Furthermore there exists k_0 such that $B_k{}^0(c_m, f)=0$ for $k>k_0$ and all $m(\neq 0)$.*

Combining (7) and Lemma 16 we have

COROLLARY 19. *Under the hypotheses of Corollary* 18, *for the resulting constants B and k_0, $B_k(c_m, f)$ are uniformly bounded by $2B$. Moreover, given $k>k_0+1$, the number of orbits $SO(2)c_m$ such that $B_k(c_m, f)\neq 0$ is bounded by a constant C which does not depend on k.*

The proof of the above corollary is analogous to that of Corollary 2 in [2].

§3. Proof of the main theorem

Let M be a compact simply connected Riemannian manifold. It is known that for any isometry h on M the inclusion of $\Omega(M, h)$ into the space of all continuous maps $\sigma: I\to M$ satisfying $h(\sigma(0))=\sigma(1)$ with the compact open topology is a homotopy equivalence [3]. It is also known that the Betti numbers

$$b_k(\Omega(M, h))=\dim H_k(\Omega(M, h))$$

are finite, when M is simply connected (see [7]).

THEOREM 20. (*Main theorem*) *Let f be an isometry on a simply connected* *compact Riemannian manifold M satisfying $f^s = id.$ for some prime integer s. If the sequence $b_k(\Omega(M, f))$ is not bounded, then there exist infinitely many geometrically distinct invariant closed geodesics under the isometry f on M.*

Proof. If there are only finitely many invariant closed geodesics under f, then we can find some critical points c^i for $E^{f^{n_i}}$ ($1 \leqq i \leqq r$, $n_i \in \boldsymbol{Z}^+$) such that any non constant critical point in $\Omega(M, f)$ lies on some orbit $SO(2)c^i{}_m$, $m \in \boldsymbol{Z}$. It follows from the assumption that all the critical orbits $SO(2)c^i{}_m$ in $\Omega(M, f)$ are isolated. Choose B^i, $k_0{}^i$ and C^i for the critical point c^i according to corollaries 12 and 13 or corollaries 18 and 19, and set $\hat{B} = \max\{B^i ; 1 \leqq i \leqq r\}$, $\hat{k}_0 = \max\{k_0{}^i ; 1 \leqq i \leqq r\}$, and $\hat{C} = \sum_{i=1}^{r} C^i$. Now for any $k > \hat{k}^0 + 1$ the constant $\hat{C}$ is an upper bound for the number of orbits $SO(2)c^i{}_m \in \Omega(M, f)$, $1 \leqq i \leqq r$, with $B_k(c^i{}_m, f) \neq 0$. Hence it follows from the Morse inequalities (3) that we can choose some regular value b satisfying $b_k(\Omega^d(M, f), \Omega^b(M, f)) = 0$ for any fixed $k > \hat{k}_0 + 1$ and any regular value $d \geqq b$. Therefore $b_k(\Omega(M, f)) = b_k(\Omega^b(M, f))$ for $k > \hat{k}_0 + 1$. On the other hand, it follows from (3) that for $k > \hat{k}_0 + 1$ and all regular value $0 < a < b$,

$$b_k(\Omega^b(M, f), \Omega^a(M, f)) \leqq 2\hat{C}\hat{B}.$$

If we choose $0 < a < \min\{E^{f^{n_i}}(c^i) ; 1 \leqq i \leqq r\}$, then $\mathrm{Fix}(f)$ is a strong deformation retract of $\Omega^a(M, f)$ (see [3]). Hence

$$b_k(\Omega^b(M, f), \Omega^a(M, f)) = b_k(\Omega^b(M, f), \mathrm{Fix}(f))$$

holds from the exact sequence of homology. In case $\mathrm{Fix}(f) = \phi$, the last equality is trivial. Since $\mathrm{Fix}(f)$ is a finite dimensional manifold, we derive by using the exact sequence of homology

$$b_k(\Omega^b(M, f), \mathrm{Fix}(f)) = b_k(\Omega^b(M, f)) \qquad \text{for almost all } k.$$

Thus

$$\begin{aligned} b_k(\Omega(M, f)) &= b_k(\Omega^b(M, f)) \\ &= b_k(\Omega^b(M, f), \mathrm{Fix}(f)) \\ &= b_k(\Omega^b(M, f), \Omega^a(M, f)) \leqq 2\hat{C}\hat{B} \qquad \text{for almost all } k. \end{aligned}$$

This contradicts the hypothesis of the theorem. (q. e. d.)

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