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COEFFICIENTS OF MEROMORPHIC UNIVALENT FUNCTIONS

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1. We denote by Σ' the family of functions

$$g(z)=z+\sum_{n=1}^{\infty}\frac{b_n}{z^n}$$

regular and univalent in $1<|z|<\infty$. Let $g(z)$ be a function belonging to Σ' and let

$$G(z)=z+\sum_{n=1}^{\infty}\frac{c_n}{z^n}$$

be the inverse function of $g(z)$. The following results are known:

$$|c_1|=|b_1|\leqq 1,\ |c_2|=|b_2|\leqq\frac{2}{3}.$$

Springer [4] proved that $|c_3|\leqq 1$ and conjectured that

$$|c_{2n-1}|\leqq\frac{(2n-2)!}{n!\,(n-1)!}\qquad(n=1, 2, \cdots).$$

In this paper we shall prove that the conjecture is true for the cases $n=3, 4, 5$.

THEOREM.

$$|c_5|\leqq 2,\ |c_7|\leqq 5,\ |c_9|\leqq 14.$$

In these inequalities equality occurs only for the inverse function of $z+(1/z)$ *and its rotations.*

Ozawa [3] made use of Grunsky's inequality together with Golusin's inequality to prove the Bieberbach conjecture for the sixth coefficient. We apply his method to prove our theorem.

2. Let

$$g(z)=z+\sum_{n=1}^{\infty}\frac{b_n}{z^n}$$

be a function belonging to Σ' and let $F_m(w)$ be the m-th Faber polynomial

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which is defined by

$$g_m(z)=F_m[g(z)]=z^m+\sum_{n=1}^{\infty}\frac{a_{mn}}{z^n}\,.$$

Then Grunsky's inequality has the form

$$\left|\sum_{m,n=1}^{N} n a_{mn} x_m x_n\right| \leqq \sum_{n=1}^{N} n|x_n|^2$$

and Golusin's inequality has the form

$$\sum_{n=1}^{\infty} n\left|\sum_{m=1}^{N} x_m a_{mn}\right|^2 \leqq \sum_{n=1}^{N} n|x_n|^2$$

where N is an arbitrary positive integer and $x_1, x_2, \cdots, x_N$ are arbitrary complex numbers.

By a simple calculation we have

$$a_{1n}=b_n \qquad (n=1, 2, \cdots)\,,$$

$$a_{21}=2b_2\,,$$

$$a_{22}=2b_3+b_1^2\,,$$

$$a_{23}=2b_4+2b_1b_2=\frac{2}{3}a_{32}\,,$$

$$a_{24}=2b_5+2b_1b_3+b_2^2=\frac{1}{2}a_{42}\,,$$

$$a_{25}=2b_6+2b_1b_4+2b_2b_3=\frac{2}{5}a_{52}\,,$$

$$a_{31}=3b_3\,,$$

$$a_{33}=3b_5+3b_1b_3+3b_2^2+b_1^3\,,$$

$$a_{35}=3b_7+3b_1b_5+6b_2b_4+3b_3^2+3b_1^2b_3+3b_1b_2^2=\frac{3}{5}a_{53}\,,$$

$$a_{41}=4b_4\,,$$

$$a_{44}=4b_7+4b_1b_5+8b_2b_4+6b_3^2+4b_1^2b_3+8b_1b_2^2+b_1^4\,,$$

$$a_{55}=5b_9+5b_1b_7+10b_2b_6+15b_3b_5+10b_4^2+5b_1^2b_5+20b_1b_2b_4+15b_1b_3^2+20b_2^2b_3$$
$$+5b_1^3b_3+15b_1^2b_2^2+b_1^5\,.$$

Let

$$g(z)=z+\sum_{n=1}^{\infty}\frac{b_n}{z^n}$$

be a function belonging to Σ' and let

$$G(z)=z+\sum_{n=1}^{\infty}\frac{c_n}{z^n}$$

be the inverse function of $g(z)$, then by a simple calculation we have

$$c_5=-(b_5+4b_1b_3+2b_2^2+2b_1^3)\,,$$

$$c_7=-(b_7+6b_1b_5+6b_2b_4+15b_1^2b_3+3b_3^2+15b_1b_2^2+5b_1^4)\,,$$

$$c_9=-(b_9+8b_1b_7+8b_2b_6+8b_3b_5+4b_4^2+28b_1^2b_5+56b_1b_2b_4+28b_1b_3^2+56b_1^3b_3 +28b_2^2b_3+84b_1^2b_2^2+14b_1^5)\,.$$

In this paper we shall use the following notations:

$$\begin{aligned} &b_1=p+ix'=1-x+ix'\,,\\ &b_2=y+iy'\,,\\ &b_3=\eta+i\eta'\,,\\ &b_4=\xi+i\xi'\,,\\ &b_5=\varphi+i\varphi'\,,\\ &b_6=\phi+i\phi'\,. \end{aligned} \tag{1}$$

3. Firstly we are concerned with the case $n=3$. By Grunsky's inequality with $N=3$, $x_1=x_2=0$, $x_3=1$ we have

$$\left|b_5+b_1b_3+b_2^2+\frac{1}{3}b_1^3\right|\leqq\frac{1}{3}\,.$$

Hence we have

$$|c_5|=|b_5+4b_1b_3+2b_2^2+2b_1^3|\leqq\left|3b_1b_3+b_2^2+\frac{5}{3}b_1^3\right|+\frac{1}{3}\,.$$

We put

$$F=\Re\Big(3b_1b_3+b_2^2+\frac{5}{3}b_1^3\Big)+\frac{1}{3}\,. \tag{2}$$

Since the polynomial $3b_1b_3+b_2^2+(5/3)b_1^3$ is homogeneous, it is sufficient to prove that $F\leqq2$ for $|\arg b_1|\leqq(\pi/3)$. Rewriting (2) with the notations (1) we have

$$F=2-5x+5x^2-\frac{5}{3}x^3+3p\eta+y^2-5px'^2-y'^2-3x'\eta'\,.$$

And it is evident that $0\leqq p\leqq1$ and $x'^2\leqq3p^2$ when $|\arg b_1|\leqq(\pi/3)$. By Grunsky's inequality with $N=2$, $x_1=0$, $x_2=1$ we have

$$\left|b_3+\frac{1}{2}b_1^2\right|\leqq\frac{1}{2}\,.$$

By taking the real part we have

$$\eta\leqq x-\frac{1}{2}x^2+\frac{1}{2}x'^2\,.$$

Hence we have

$$F\leqq 2-2x+\frac{1}{2}x^2-\frac{1}{6}x^3+y^2-\frac{7}{2}px'^2-y'^2-3x'\eta' .$$

By the area theorem

$$-2x+x^2\leqq -x'^2-2y^2-3\eta'^2 .$$

Thus we obtain

(3) $$F\leqq 2-\frac{1}{2}x^2\left(1+\frac{1}{3}x\right)-y^2-y'^2-\left(1+\frac{7}{2}p\right)x'^2-3\eta'^2-3x'\eta' .$$

Since $0\leqq p\leqq 1$,

$$-\left(1+\frac{7}{2}p\right)x'^2-3x'\eta'-3\eta'^2\leqq 0 .$$

Therefore (3) implies the desired result:

$$F\leqq 2 .$$

Equality occurs only for $x=0$.

4. Next we consider the case $n=4$. By Grunsky's inequality with $N=4$, $x_1=8b_2$, $x_2=5b_1$, $x_3=0$, $x_4=1$ we have

$$\left|b_7+6b_1b_5+6b_2b_4+\frac{49}{4}b_1^2b_3+\frac{3}{2}b_3^2+\frac{37}{2}b_1b_2^2+\frac{27}{8}b_1^4\right|$$

$$\leqq 4|b_2|^2+\frac{25}{8}|b_1|^2+\frac{1}{4} .$$

Hence we have

$$|c_7|=|b_7+6b_1b_5+6b_2b_4+15b_1^2b_3+3b_3^2+15b_1b_2^2+5b_1^4|$$

$$\leqq\left|\frac{11}{4}b_1^2b_3+\frac{3}{2}b_3^2-\frac{7}{2}b_1b_2^2+\frac{13}{8}b_1^4\right|$$

$$+4|b_2|^2+\frac{25}{8}|b_1|^2+\frac{1}{4} .$$

Further by using Grunsky's inequality with $N=2$, $x_1=0$, $x_2=1$

$$\left|b_3+\frac{1}{2}b_1^2\right|\leqq\frac{1}{2} .$$

we have

$$|c_7|\leqq\left|\frac{3}{2}b_3^2-\frac{7}{2}b_1b_2^2+\frac{1}{4}b_1^4\right|+4|b_2|^2+\frac{9}{2}|b_1|^2+\frac{1}{4} .$$

We put

(4) $$F=\Re\left(\frac{3}{2}b_3^2-\frac{7}{2}b_1b_2^2+\frac{1}{4}b_1^4\right)+4|b_2|^2+\frac{9}{2}|b_1|^2+\frac{1}{4} .$$

Now it is sufficient to prove that $F\leqq 5$ for $|\arg b_1|\leqq(\pi/4)$. Rewriting (4) with the notations (1) we have

$$
\begin{aligned}
F=&5-10x+6x^2-x^3+\frac{1}{4}x^4+\left(4-\frac{7}{2}p\right)y_2+\frac{3}{2}\eta^2 \\
&+\left(\frac{9}{2}-\frac{3}{2}p^2+\frac{1}{4}x'^2\right)x'^2+\left(4+\frac{7}{2}p\right)y'^2-\frac{3}{2}\eta'^2+7x'y'y .
\end{aligned}\tag{5}
$$

And it is evident that $0\leqq p\leqq 1$ and $x'^2\leqq p^2$ when $|\arg b_1|\leqq(\pi/4)$. By the area theorem

$$-9x+\frac{9}{2}x^2\leqq-\frac{9}{2}x'^2-9y^2-9y'^2-\frac{27}{2}\eta^2-\frac{27}{2}\eta'^2 .\tag{6}$$

Putting (6) into (5), we obtain

$$
\begin{aligned}
F\leqq 5-xP(x)-\left(5+\frac{7}{2}p\right)y^2-12\eta^2-\left(\frac{3}{2}p^2-\frac{1}{4}x'^2\right)x'^2& \\
-\left(5-\frac{7}{2}p\right)y'^2-15\eta'^2+7x'y'y ,&
\end{aligned}\tag{7}
$$

$$P(x)=1-\frac{3}{2}x+x^2-\frac{1}{4}x^3 .$$

Since $x'^2\leqq p^2\leqq 1$, we have

$$-\left(5+\frac{7}{2}p\right)y^2+7x'y'y-\left(5-\frac{7}{2}p\right)y'^2\leqq 0 .$$

It is easy to prove that $P(x)>0$ for $0\leqq p\leqq 1$. Therefore (7) implies that $F\leqq 5$ for $|\arg b_1|\leqq(\pi/4)$, with equality holding only for $x=0$.

5. Finally we are concerned with the case $n=5$. By Grunsky's inequality with $N=5$, $x_1=0$, $x_2=10b_2$, $x_3=(35/6)b_1$, $x_4=0$, $x_5=1$ we have

$$
\begin{aligned}
\Big|b_9+8b_1b_7+10b_2b_6+3b_3b_5+2b_4^2+\frac{81}{4}b_1^2b_5+54b_1b_2b_4+10b_1b_3^2+28b_2^2b_3& \\
+\frac{81}{4}b_1^3b_3+\frac{233}{4}b_1^2b_2^2+\frac{257}{60}b_1^5\Big|& \\
\leqq 8|b_2|^2+\frac{49}{12}|b_1|^2+\frac{1}{5} .&
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
|c_9|\leqq|b_9+8b_1b_7+8b_2b_6+8b_3b_5+4b_4^2+28b_1^2b_5+56b_1b_2b_4+28b_1b_3^2& \\
+28b_2^2b_3+56b_1^3b_3+84b_1^2b_2^2+14b_1^5|&
\end{aligned}
$$

$$\leqq\left|-2b_2b_6+5b_3b_5+2b_4^2+\frac{31}{4}b_1^2b_5+2b_1b_2b_4\right.$$
$$\left.+18b_1b_3^2+\frac{143}{4}b_1^3b_3+\frac{103}{4}b_1^2b_2^2+\frac{583}{60}b_1^5\right|$$
$$+8|b_2|_2+\frac{49}{12}|b_1|^2+\frac{1}{5}.$$

Further by using Grunsky's inequality with $N=3$, $x_1=x_2=0$, $x_3=1$

$$\left|b_5+b_1b_3+b_2^2+\frac{1}{3}b_1^3\right|\leqq\frac{1}{3},$$

we have

$$|c_9|\leqq\left|-2b_2b_6+5b_3b_5+2b_4^2+2b_1b_2b_4+18b_1b_3^2+28b_1^3b_3+18b_1^2b_2^2+\frac{107}{15}b_1^5\right|$$
$$+8|b_2|^2+\frac{20}{3}|b_1|^2+\frac{1}{5}.$$

We put

$$F=\Re\left(-2b_2b_6+5b_3b_5+2b_4^2+2b_1b_2b_4+18b_1b_3^2+28b_1^3b_3+18b_1^2b_2^2+\frac{107}{15}b_1^5\right)$$
$$(8)\qquad +8|b_2|^2+\frac{20}{3}|b_1|^2+\frac{1}{5}.$$

Now it is sufficient to prove that $F\leqq 14$ for $|\arg b_1|\leqq(\pi/5)$. Rewriting (8) with the notations (1) we have

$$F=14-49x+78x^2-\frac{214}{3}x^3+\frac{107}{3}x^4-\frac{107}{15}x^5+28p^3\eta$$
$$+(8+18p^2-18x'^2)y^2+18p\eta^2+2\xi^2+2py\xi-2y\phi+5\eta\varphi$$
$$(9)\qquad +\left(\frac{20}{3}-\frac{214}{3}p^3+\frac{107}{3}px'^2\right)x'^2+(8-18p^2+18x'^2)y'^2-18p\eta'^2$$
$$-2\xi'^2+(-84p^2+28x'^2)x'\eta'-2py'\xi'+2y'\phi'-5\eta'\varphi'$$
$$+y(-72px'y'-2x'\xi')+\eta(-84px'^2-36x'\eta')-2x'y'\xi.$$

And it is evident that $0\leqq p\leqq 1$ and $x'^2<0.53p^2$ when $|\arg b_1|\leqq(\pi/5)$. Here we make use of Golusin's inequality. We put $N=2$, $x_1=0$, $x_2=1$ in Golusin's inequality. Then we have

$$|2b_2|^2+2|2b_3+b_1^2|^2\leqq 2.$$

Rewriting this we have

$$(10)\qquad 4p^2\eta\leqq 4x-6x^2+4x^3-x^4-2y^2-4\eta^2-(2p^2+x'^2)x'^2-2y'^2-4\eta'^2$$
$$-8px'\eta'+4x'^2\eta.$$

Putting (10) into (9), we have

$$
\begin{aligned}
F \leqq\, & 14-21x+8x^2-\frac{4}{3}x^3+\frac{2}{3}x^4-\frac{2}{15}x^5 \\
& +(8-14p+18p^2-18x'^2)y^2-10p\eta^2+2\xi^2+2py\xi-2y\phi+5\eta\varphi \\
& +\Big(\frac{20}{3}-\frac{256}{3}p^3+\frac{86}{3}px'^2\Big)x'^2+(8-14p-18p^2+18x'^2)y'^2 \\
& -46p\eta'^2-2\xi'^2+(-140p^2+28x'^2)x'\eta'-2py'\xi'+2y'\phi'-5\eta'\varphi' \\
& +y(-72px'y'-2x'\xi')+\eta(-56px'^2-36x'\eta')-2x'y'\xi\,.
\end{aligned}
$$

By the area theorem

$$
\begin{aligned}
-21x+\frac{21}{2}x^2 \leqq & -\frac{21}{2}x'^2-21y^2-21y'^2-\frac{63}{2}\eta^2-\frac{63}{2}\eta'^2 \\
& -42\xi^2-42\xi'^2-\frac{105}{2}\varphi^2-\frac{105}{2}\varphi'^2-63\phi^2-63\phi'^2\,.
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
& F\leqq 14-x^2P(x)-Q\,, \\
& P(x)=\frac{5}{2}+\frac{4}{3}x-\frac{2}{3}x^2+\frac{2}{15}x^3\,, \\
& Q=(13+14p-18p^2+18x'^2)y^2+(31.5+10p)\eta^2+40\xi^2+52.5\varphi^2 \\
\text{(11)}\qquad & \quad +63\phi^2-2py\xi+2y\phi-2\cdot 2.5\eta\varphi \\
& \quad +(3.833+85.333p^3-28.667px'^2)x'^2+(13+14p+18p^2-18x'^2)y'^2 \\
& \quad +(31.5+46p)\eta'^2+44\xi'^2+52.5\varphi'^2+63\phi'^2+2(70p^2-14x'^2)x'\eta' \\
& \quad +2py'\xi'-2y'\phi'+2\cdot 2.5\eta'\varphi' \\
& \quad +2\cdot 36px'y'y+2x'\xi'y+2\cdot 28px'^2\eta+2\cdot 18x'\eta'\eta+2x'y'\xi\,.
\end{aligned}
$$

It is evident that $P(x)>0$ for $0\leqq p\leqq 1$. In order to prove $Q\geqq 0$, we first observe

$$
\begin{aligned}
& 0.12\eta^2-2\cdot 2.5\eta\varphi+52.5\varphi^2\geqq 0\,, \\
& 0.12\eta'^2+2\cdot 2.5\eta'\varphi'+52.5\varphi'^2\geqq 0\,, \\
& 12\eta^2+2\cdot 18x'\eta'\eta+27x'^2\eta'^2\geqq 0\,, \\
& (19.38+10p)\eta^2+2\cdot 28px'^2\eta+26.685px'^4\geqq 0\,.
\end{aligned}
$$

Further we have

$$
\begin{aligned}
(3.833+85.333p^3-55.352px'^2)x'^2+2(70p^2-14x'^2)x'\eta' & \\
+(31.38+46p-27x'^2)\eta'^2\geqq 0\,. &
\end{aligned}
$$

Indeed we consider the discriminant $\varDelta$ of this quadratic from. Then

$$\varDelta > 120.279+176.318p+2677.749p^3-974.682p^4$$
$$-(103.491+1736.946p+586.192p^2+2303.991p^3)x'^2$$
$$+(1494.504p-196)x'^4 .$$

If $x'^2 \leqq 0.45p^2$ and $0 \leqq p \leqq 1$, then

$$\varDelta > 120.279+176.318p+2677.749p^3-947.682p^4$$
$$-0.45p^2(103.491+1736.946p+586.192p^2+2303.991p^3)$$
$$+(0.45)^2p^4(1494.504p-196)$$
$$\geqq 120.279+176.318p-46.571p^2$$
$$+1896.123p^3-1278.159p^4-734.159p^5 > 0 .$$

If $0.45p^2 \leqq x'^2 < 0.53p^2$, then $0 \leqq p < 0.84$, whence

$$\varDelta > 120.279+176.318p-54.851p^2+1757.167p^3-1340.421p^4-801.31p^5 > 0 .$$

Thus we have the desired inequality. By using these inequalities we have

$$Q \geqq (13+14p-18p^2+18x'^2)y^2+40\xi^2+63\phi^2-2py\xi+2y\phi$$
$$+(13+14p+18p^2-18x^2)y'^2+44\xi'^2+63\phi'^2+2py'\xi'-2y'\phi'$$
$$+2\cdot 36px'y'y+2x'\xi'y+2x'y'\xi .$$

Further by using the inequalities

$$\xi^2+2x'y'\xi+x'^2y'^2 \geqq 0 ,$$
$$4\xi'^2+2x'\xi'y+0.25x'^2y^2 \geqq 0 ,$$
$$36x'^2y^2+2\cdot 36px'y'y+36p^2y'^2 \geqq 0 ,$$

we have

$$Q \geqq (13+14p-18p^2-18.25x'^2)y^2+39\xi^2+63\phi^2-2py\xi+2y\phi$$
$$+(13+14p-18p^2-19x'^2)y'^2+40\xi'^2+63\phi'^2+2py'\xi'-2y'\phi' .$$

We consider the symmetric matrix associated with the quadratic form

$$(13+14p-18p^2-18.25x'^2)y^2+39\xi^2+63\phi^2-2py\xi+2y\phi :$$

$$\begin{pmatrix} 63 & 0 & 1 \\ 0 & 39 & -p \\ 1 & -p & 13+14p-18p^2-18.25x'^2 \end{pmatrix} .$$

Its principal diagonal minor determinants are

$$63,$$
$$2457,$$
$$31902+34398p-44289p^2-44840.25x'^2 \equiv \Delta .$$

If $x'^2 \leqq 0.45p^2$ and $0 \leqq p \leqq 1$, then

$$\Delta \geqq 31902+34398p-64468p^2 > 0 .$$

If $0.45p^2 \leqq x'^2 < 0.53p^2$ and $0 \leqq p < 0.84$, then

$$\Delta \geqq 31902+34398p-68055p^2 > 0 .$$

Hence it follows that $(13+14p-18p^2-18.25x'^2)y^2+39\xi^2+63\phi^2-2py\xi+2y\phi \geqq 0$ for $|\arg b_1| \leqq (\pi/5)$. Similarly it follows that $(13+14p-18p^2-19x'^2)y'^2+40\xi'^2+63\phi'^2+2py'\xi'-2y'\phi' \geqq 0$ for $|\arg b_1| \leqq (\pi/5)$. Consequently we have $Q \geqq 0$. Thus (11) implies that $F \leqq 14$ for $|\arg b_1| \leqq (\pi/5)$, with equality holding only for $x=0$.

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