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METRICS AND CONNECTIONS ON THE COTANGENT BUNDLE

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§1. Introduction.

Let M be an n-dimensional differentiable manifold of class C^∞ and T^*M its cotangent bundle, which is a $2n$-dimensional differentiable manifold. The problem of extending structures on M to T^*M has been the subject of a number of papers. An account of these can be found in Yano and Ishihara [12]. Starting from a torsion-free linear connection on M, Patterson and Walker [5] have shown how to construct a metric on T^*M, a process which they called the *Riemann extension.* Using the Riemann extension, Yano and Patterson [13, 14] have defined the *complete* and *horizontal lifts* of linear connections on M to T^*M. On the other hand, Tondeur [8] and Sato [7] have constructed a metric on T^*M from a metric on M, the construction being the analogue of the metric of Sasaki for the tangent bundle TM [6].

When a linear connection is given on M, we may view T^*M as an almost product manifold. Linear connections on an almost product manifold have been studied by Walker [9], Yano [10] and Davies [2], among others. In the present paper, we shall further consider the metrics and connections on T^*M mentioned above, bringing in the general theory of linear connections on an almost product manifold whenever possible. In this way, the relations between the various metrics and connections become clearer, and we obtain a new linear connection on T^*M, namely the *intermediate lift,* which, in some sense, lies somewhere between the complete and horizontal lift. Refering to the "adapted frames" on T^*M, we have computed the components of the curvature tensors of the various linear connections on T^*M, as well as that of their covariant derivatives.

Similar considerations to the metrics and connections on the tangent bundle TM can be found in the papers of Davies [2] and Yano and Davies [11]. In fact, our intermediate lift is the cotangent bundle analogue of the connection $\mathring{\nabla}$ on TM appearing in [2, §4].

As to notations and definitions, we shall generally follow that in [12]. In particular:

1) Indices $a, b, c, \cdots; h, i, j, \cdots$ have range in $\{1, \cdots, n\}$, while indices $A, B, C, \cdots; \lambda, \mu, \nu, \cdots$ have range in $\{1, \cdots, n; n+1, \cdots, 2n\}$. We put $\bar{\imath}=n+i$. Summation over repeated indices is always implied. Entries of matrices are written

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as $A_i{}^j$, A_{ji} or A^{ji}, and in all cases, j is the row index while i is the column index.

2) $\pi: T^*M \to M$ denotes the canonical projection of T^*M onto M. Coordinate systems in M are denoted by (U, x^h), where U is the coordinate neighbourhood and x^h the coordinate functions. (U, x^h) induces in a natural way a coordinate system $\{\pi^{-1}(U), (x^h, p_h)\}$ in T^*M which is called the induced coordinate system. We shall sometimes write p_h as $x^{\bar{h}}$ and (x^h, p_h) as (x^A).

3) Components in (U, x^h) of geometric objects in M will be referred to simply as components. If we want to emphasize (U, x^h), we shall say components in U. The same applies to geometric objects in T^*M.

4) ∇ denotes a linear connection on M with components Γ_{ji}^h. Its covariant differentiation will again be denoted by the same symbol ∇. The curvature tensor R of ∇ has components $R_{kji}{}^h$. We assume throughout that ∇ is torsion-free.

5) g denotes a metric on M with components g_{ji}. The covariant differentiation in (M, g) is denoted by ∇^g. As usual, $\left\{ {h \atop j\ i} \right\}$ is the Christoffel symbol for g_{ji} and $[g^{ji}]$ is the inverse of the matrix $[g_{ji}]$. The curvature tensor K of (M, g) has components $K_{kji}{}^h$.

§2. T^*M as an almost product manifold.

Let V be the field of n-planes tangent to the fibres of T^*M. It is an integrable ditribution on T^*M, which we called the vertical distribution. A torsion-free linear connection ∇ on M determines uniquely on T^*M an n-dimensional distribution complementary to V. This distribution will be called the horizontal distribution associated with ∇ and is denoted by H. The pair (H, V) defines an almost product structure on T^*M and turns it into an almost product manifold.

Let $\{\pi^{-1}(U), (x^h, p_h)\}$ be an induced coordinate system in T^*M. The horizontal distribution H restricted to $\pi^{-1}(U)$ is spanned by the n independent vector fields

$$D_j = \frac{\partial}{\partial x^j} + \Gamma_{ji}\frac{\partial}{\partial p_i}, \tag{2.1}$$

where

$$\Gamma_{ji} = p_h \Gamma_{ji}^h. \tag{2.2}$$

The vertical distribution V restricted to $\pi^{-1}(U)$ is spanned by the n independent vector fields

$$D_{\bar{j}} = \frac{\partial}{\partial p_j}. \tag{2.3}$$

It follows that $\{D_\lambda\} = \{D_j, D_{\bar{j}}\}$ constitute a frame on $\pi^{-1}(U)$. As the frame is adapted to the almost product structure (H, V), we call them *adapted frames* *on* $\pi^{-1}(U)$. In what follows, we usually refer our tensors and other geometric objects on T^*M to their components with respect to the adapted frame. Such components will be called the *frame components on* $\pi^{-1}(U)$, or more simply *frame components*, to distinguish them from the usual components.

The coframe $\{D^j, D^{\bar{j}}\}$ on $\pi^{-1}(U)$ dual to the adapted frame is given by

$$D^j = dx^j, \tag{2.4}$$

$$D^{\bar{j}} = -\Gamma_{ji} dx^i + dp_j. \tag{2.5}$$

The component matrix of the adapted frame and its coframe are

$$L = \begin{bmatrix} \delta_{ji} & 0 \\ \Gamma_{ji} & \delta_{ji} \end{bmatrix} \quad \text{and} \quad L^{-1} = \begin{bmatrix} \delta_{ji} & 0 \\ -\Gamma_{ji} & \delta_{ji} \end{bmatrix} \tag{2.6$_1$}$$

respectively. We write

$$L = [L_\mu^A] \quad \text{and} \quad L^{-1} = [L_A^\nu] \tag{2.6$_2$}$$

to indicate their entries.

The "non-holonomic objects" $\Omega_{\lambda\mu}{}^\nu$ of the adapted frame are defined by

$$[D_\lambda, D_\mu] = \Omega_{\lambda\mu}{}^\nu D_\nu, \tag{2.7}$$

i.e., by

$$\Omega_{\lambda\mu}{}^\nu = [D_\lambda(L_\mu^A) - D_\mu(L_\lambda^A)] L_A^\nu. \tag{2.8}$$

Using (2.8), (2.1), (2.3) and (2.6), we get the following as the possibly nonzero components of $\Omega_{\lambda\mu}{}^\nu$:

$$\begin{aligned} &\Omega_{ji}{}^{\bar{h}} = p_a R_{jih}{}^a, \\ &\Omega_{j\bar{i}}{}^{\bar{h}} = -\Omega_{\bar{i}j}{}^{\bar{h}} = \Gamma_{hi}^j. \end{aligned} \tag{2.9}$$

The projection tensors of T^*M onto H and V will again be denoted by H and V. They are tensors of type (1, 1) on T^*M whose frame component matrices are

$$H: \begin{bmatrix} \delta_{ji} & 0 \\ 0 & 0 \end{bmatrix}, \quad V: \begin{bmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{bmatrix},$$

and satisfy

$$H^2 = H,\ V^2 = V,\ HV = VH = 0,\ H + V = I, \tag{2.10}$$

where I is the identity tensor. The usual expression for the torsion tensor $S = S_{H,V}$ associated with H and V (see [3, p. 37]) reduces, by virtue of (2.10) and the integrability of V, to

$$S(\tilde{X}, \tilde{Y}) = -2V[H\tilde{X}, H\tilde{Y}]$$

where $\tilde{X}, \tilde{Y}$ are arbitrary vector fields on T^*M. It follows that the only possibly non-zero frame component of S is

(2.11) $$S_{ji}{}^{\bar h} = -2\Omega_{ji}{}^{\bar h} = -2p_a R_{jih}{}^a .$$

Let $\tilde{\nabla}$ be an arbitrary linear connection on T^*M whose components in $\pi^{-1}(U)$ are $\tilde{\Gamma}^A_{CB}$. The *frame components* of $\tilde{\nabla}$ in $\pi^{-1}(U)$ are defined by

(2.12) $$\tilde{\Gamma}^\nu_{\lambda\mu} = [D_\lambda(L^A_\mu) + \tilde{\Gamma}^A_{CB} L^C_\lambda L^B_\mu] L^\nu_A .$$

If $\tilde{X}$ is a vector field on T^*M whose frame components are $\tilde{X}^\nu$, then

(2.13) $$\tilde{\nabla}_\lambda \tilde{X}^\nu = D_\lambda(\tilde{X}^\nu) + \tilde{\Gamma}^\nu_{\lambda\mu} \tilde{X}^\mu$$

are exactly the frame components of the covariant derivative $\tilde{\nabla}\tilde{X}$ of $\tilde{X}$. There are formulas analogous to (2.13) for tensor fields of other types. The frame components of the torsion tensor $\tilde{T}$ and the curvature tensor $\tilde{R}$ of $\tilde{\nabla}$ are given by

(2.14) $$\tilde{T}^\nu_{\lambda\mu} = \tilde{\Gamma}^\nu_{\lambda\mu} - \tilde{\Gamma}^\nu_{\mu\lambda} - \Omega_{\mu} ,$$

(2.15) $$\tilde{R}_{\omega\lambda\mu}{}^\nu = D_\omega(\tilde{\Gamma}^\nu_{\lambda\mu}) - D_\lambda(\tilde{\Gamma}^\nu_{\omega\mu}) + \tilde{\Gamma}^\nu_{\omega\tau}\tilde{\Gamma}^\tau_{\lambda\mu} - \tilde{\Gamma}^\nu_{\lambda\tau}\tilde{\Gamma}^\tau_{\omega\mu} - \Omega_{\omega\lambda}{}^\tau \tilde{\Gamma}^\nu_{\tau\mu} .$$

Let $\tilde{C}$ be a curve in T^*M whose equation in $\pi^{-1}(U)$ is $x^A = x^A(t)$. The frame components of its velocity vector are then $L^\nu_A(dx^A/dt) = (D^\nu/dt)$. It can be shown that $\tilde{C}$ is a geodesic of $\tilde{\nabla}$ iff on each $\pi^{-1}(U)$, we have

(2.16) $$\frac{d}{dt}\left(\frac{D^\nu}{dt}\right) + \tilde{\Gamma}^\nu_{\lambda\mu} \frac{D^\lambda}{dt} \frac{D^\mu}{dt} = 0 .$$

When a linear connection $\tilde{\nabla}$ is given on an almost product manifold, Walker [9] has considered the conditions for the distributions of the almost product manifold to be path-parallel, relative parallel, etc. These conditions have been reformulated by Yano [10] in terms of the connection components relative to the frames adapted to the almost product structure. In the case of T^*M, the conditions of parallelism for H and V with respect to a linear connection $\tilde{\nabla}$ are:

(2.17)
H is path-parallel iff $\tilde{\Gamma}^{\bar h}_{ji} + \tilde{\Gamma}^{\bar h}_{ij} = 0$,

H is parallel along V iff $\tilde{\Gamma}^{\bar h}_{\bar j i} = 0$,

H is parallel iff $\tilde{\Gamma}^{\bar h}_{\lambda i} = 0$,

V is path-parallel iff $\tilde{\Gamma}^{h}_{\bar j \bar i} + \tilde{\Gamma}^{h}_{\bar i \bar j} = 0$,

V is parallel along H iff $\tilde{\Gamma}^{h}_{j \bar i} = 0$,

V is parallel iff $\tilde{\Gamma}^{h}_{\lambda \bar i} = 0$.

Here, all parallelisms are with respect to $\tilde{\nabla}$ and $\tilde{\Gamma}^\nu_{\lambda\mu}$ are the frame components of $\tilde{\nabla}$ as defined in (2.12).

Walker and Yano have also explicitly constructed linear connections on an almost product manifold satisfying certain conditions of parallelism. We now adapt these constructions to T^*M. We first follow Davies [2] and consider the A-tensor and the B-tensor associated with a linear connection $\tilde{\nabla}$ on T^*M. They are tensors of type (1, 2) on T^*M whose frame components $A_{\lambda\mu}{}^{\nu}$ and $B_{\lambda\mu}{}^{\nu}$ are respectively:

$$
\begin{aligned}
&A_{ji}{}^{h}=0, && A_{ji}{}^{\bar{h}}=\tilde{\Gamma}^{\bar{h}}_{ji}.\\
&A_{j\bar{i}}{}^{h}=\tilde{\Gamma}^{h}_{j\bar{i}}, && A_{j\bar{i}}{}^{\bar{h}}=0,\\
&A_{\bar{j}i}{}^{h}=0, && A_{\bar{j}i}{}^{\bar{h}}=\tilde{\Gamma}^{\bar{h}}_{\bar{j}i},\\
&A_{\bar{j}\bar{i}}{}^{h}=\tilde{\Gamma}^{h}_{\bar{j}\bar{i}}, && A_{\bar{j}\bar{i}}{}^{\bar{h}}=0.
\end{aligned}
\tag{2.18}
$$

$$
\begin{aligned}
&B_{ji}{}^{h}=0, && B_{ji}{}^{\bar{h}}=-\frac{1}{2}(\tilde{\Gamma}^{\bar{h}}_{ji}+\tilde{\Gamma}^{\bar{h}}_{ij}),\\
&B_{ji}{}^{h}=-\tilde{\Gamma}^{h}_{j\bar{i}}, && B_{\bar{j}i}{}^{\bar{h}}=-\tilde{\Gamma}^{\bar{h}}_{\bar{i}j},\\
&B_{\bar{j}i}{}^{h}=-\tilde{\Gamma}^{h}_{i\bar{j}}, && B_{\bar{j}i}{}^{h}=-\tilde{\Gamma}^{\bar{h}}_{\bar{j}i},\\
&B_{\bar{j}\bar{i}}{}^{h}=-\frac{1}{2}(\tilde{\Gamma}^{h}_{\bar{j}\bar{i}}+\tilde{\Gamma}^{h}_{\bar{i}\bar{j}}), && B_{\bar{j}\bar{i}}{}^{\bar{h}}=0.
\end{aligned}
\tag{2.19}
$$

The construction of linear connections on T^*M mentioned above are then:

LEMMA 2.1. *Let $\tilde{\nabla}$ be a torsion-free linear connection on T^*M and B its associated B-tensor. With respect to the linear connection $\tilde{\nabla}+B$, H is parallel along V, V is parallel along H and both H, V are path-parallel.*

LEMMA 2.2. *Let $\tilde{\nabla}$ be a torsion-free linear connection on T^*M and A its associated A-tensor. With respect to the linear connection $\tilde{\nabla}-A$, both H and V are parallel.*

§3. The Riemann extension and the complete lift.

Starting from a torsion-free linear connection ∇ on M, Patterson and Walker [5] have constructed a metric on T^*M, namely the *Riemann extension of ∇*. If the components in U of ∇ are Γ^h_{ji}, the component matrix in $\pi^{-1}(U)$ of the Riemann extension is

$$
\begin{bmatrix} -2\Gamma_{ji} & \delta_{ji} \\ \delta_{ji} & 0 \end{bmatrix}.
\tag{3.1}
$$

It follows that the corresponding frame component matrix is

$$
\begin{bmatrix} 0 & \delta_{ji} \\ \delta_{ji} & 0 \end{bmatrix}.
\tag{3.2}
$$

Let ∇^C be the Riemannian connection on T^*M associated with the Riemann extension. Yano and Patterson [13] called ∇^C the *complete lift of* ∇ *to* T^*M.

For an arbitrary metric on T^*M whose frame component matrix on $\pi^{-1}(U)$ is $[G_{\lambda\mu}]$, the frame components $\tilde{\Gamma}^\nu_{\lambda\mu}$ of its associated Riemannian connection can be shown to be given by

$$\tilde{\Gamma}^\nu_{\lambda\mu}=\frac{1}{2}G^{\nu\sigma}(D_\lambda G_{\sigma\mu}+D_\mu G_{\sigma\lambda}-D_\sigma G_{\lambda\mu})+\frac{1}{2}(\Omega_{\lambda\mu}{}^\nu+\Omega^\nu{}_{\lambda\mu}+\Omega^\nu{}_{\mu\lambda}), \tag{3.3}$$

where $[G^{\lambda\mu}]$ is the inverse of the matrix $[G_{\lambda\mu}]$ and

$$\Omega^\nu{}_{\lambda\mu}=G^{\nu\sigma}G_{\mu\tau}\Omega_{\sigma\lambda}{}^\tau . \tag{3.4}$$

On letting $[G_{\lambda\mu}]$ to be the matrix in (3.2), we then get from (3.3) the following as the possibly non-zero frame components of the complete lift ∇^C:

$$\tilde{\Gamma}^h_{ji}=\Gamma^h_{ji},\ \tilde{\Gamma}^{\bar h}_{ji}=p_a R_{hij}{}^a,\ \tilde{\Gamma}^{\bar h}_{j\bar i}=-\Gamma^i_{jh} . \tag{3.5}$$

By using (3.5) and (2.15), we obtain the following as the possibly non-zero frame components of the curvature tensor $\tilde{R}$ of ∇^C:

$$\begin{aligned} &\tilde{R}_{kji}{}^h=R_{kji}{}^h,\ \tilde{R}_{kji}{}^{\bar h}=p_a(\nabla_k R_{hij}{}^a-\nabla_j R_{hik}{}^a), \\ &\tilde{R}_{kj\bar i}{}^{\bar h}=-R_{kjh}{}^i,\ \tilde{R}_{kji}{}^{\bar h}=-\tilde{R}_{\bar j ki}{}^{\bar h}=-R_{hik}{}^j . \end{aligned} \tag{3.6}$$

It follows from (3.6) that (T^*M, ∇^C) is locally flat iff (M, ∇) is locally flat. Furthermore, we can use (3.5) and (3.6) to compute the frame components of $\nabla^C\tilde{R}$. Its possibly non-zero frame components are found to be:

$$\begin{aligned} &\tilde{\nabla}_l\tilde{R}_{kji}{}^h=\nabla_l R_{kji}{}^h, \\ &\tilde{\nabla}_l\tilde{R}_{kji}{}^{\bar h}=p_a(\nabla_l\nabla_k R_{hij}{}^a-\nabla_l\nabla_j R_{hik}{}^a) \\ &\qquad\qquad +p_a(R_{htl}{}^a R_{kji}{}^t+R_{ktl}{}^a R_{hij}{}^t+R_{tjl}{}^a R_{hik}{}^t+R_{til}{}^a R_{kjh}{}^t), \\ &\tilde{\nabla}_l\tilde{R}_{kj\bar i}{}^{\bar h}=-\nabla_l R_{kjh}{}^i, \\ &\tilde{\nabla}_l\tilde{R}_{kji}{}^{\bar h}=-\tilde{\nabla}_l\tilde{R}_{\bar j ki}{}^{\bar h}=-\nabla_l R_{hik}{}^j, \\ &\tilde{\nabla}_{\bar l}\, R_{kji}{}^{\bar h}=\nabla_k R_{hij}{}^l-\nabla_j R_{hik}{}^l . \end{aligned} \tag{3.7}$$

From (3.7), we can get the result of Afifi [1] that (T^*M, ∇^C) is locally symmetric iff (M, ∇) is locally symmetric (cf. [12, p. 271-2]).

A curve $\tilde{C}$ in T^*M can be regarded in a natural way as its projection $C=\pi\tilde{C}$ onto M together with a field of covectors along C. We wish to consider the condition for $\tilde{C}$ to be a geodesic in (T^*M, ∇^C). As usual, we confine ourselves to $\pi^{-1}(U)$ and let the equations of $\tilde{C}$ be $x^h=x^h(t)$, $p_h=p_h(t)$. Then the equation of C is $x^h=x^h(t)$ and $p_h(t)$ gives the components of the covector field along C. We recall that (D^ν/dt) are the frame components of the velocity vector of $\tilde{C}$ and that $D^{\bar j}$ is just the usual covariant differential

$$\delta p_j=dp_j-p_a\Gamma^a_{ji}dx^i .$$

From (2.16) and (3.5), we get immediately the following as the condition for $\hat{C}$ to be a geodesic in (T^*M, ∇^C):

$$\frac{d^2x^h}{dt^2}+\Gamma^h_{ji}\frac{dx^j}{dt}\frac{dx^i}{dt}=0\,,$$
(3.8)
$$\frac{\delta^2 p_h}{dt^2}+p_a R_{hji}{}^a\frac{dx^j}{dt}\frac{dx^i}{dt}=0\,.$$

Here, $(\delta^2 p_h/dt^2)=(\delta/dt)(\delta p_h/dt)$ is the usual second intrinsic derivative of the covector field along C. The first condition in (3.8) means that C is a geodesic in (M, ∇).

For the special case of $\nabla=\nabla^g$, i.e., ∇ is the Riemannian connection of a metric g on M, we can interpret the second condition in (3.8) as follows. Let us consider the vector field along C whose components are $y^h(t)=g^{ha}p_a(t)$. Then, equation $(3.8)_2$ is easily seen to be equivalent to

$$\frac{\delta^2 y^k}{dt^2}+K_{aji}{}^k y^a\frac{dx^j}{dt}\frac{dx^i}{dt}=0\,, \tag{3.9}$$

where $(\delta^2 y^k/dt^2)=(\delta/dt)(\delta y^k/dt)$ is the usual second intrinsic derivative for $y^k(t)$. In this case, (3.9) is exactly the condition for the vector field along C to be a Jacobi field in (M, ∇^g).

§4. Horizontal lift of linear connection.

Let R be the curvature tensor of ∇. The local tensor fields

$$p_a R_{hij}{}^a\frac{\partial}{\partial p_h}\otimes dx^j\otimes dx^i$$

on each $\pi^{-1}(U)$ piece together to form a tensor field of type (1, 2) on T^*M, which we denote by γR. In [14], Yano and Patterson define the *horizontal lift* ∇^H *of* ∇ *to* T^*M to be the linear connection

$$\nabla^H=\nabla^C-\gamma R\,.$$

Since the only possibly non-zero frame component of γR is $(\gamma R)_{ji}{}^{\bar{h}}=p_a R_{hij}{}^a$, it follows from (3.5) that the non-zero frame components of ∇^H are

$$\tilde{\Gamma}^h_{ji}=\Gamma^h_{ji},\ \tilde{\Gamma}^{\bar{h}}_{j\bar{i}}=-\Gamma^i_{jh}\,. \tag{4.1}$$

From (4.1) and (2.14), it can be shown that the only posssibly non-zero frame components of the torsion tensor $\tilde{T}$ of ∇^H are

$$\tilde{T}_{ji}{}^{\bar{h}}=-\Omega_{ji}{}^{\bar{h}}=-p_a R_{jih}{}^a\,. \tag{4.2}$$

Comparing (4.2) with (2.11), we found that the torsion tensor $\tilde{T}$ of ∇^H is exactly half that of the torsion tensor $S=S_{H,V}$ associated with the projection tensors H and V. On the other hand, a routine calculation using (4.1) and (3.2) shows that the metric tensor of the Riemann extension of ∇ is parallel with respect to ∇^H. Hence,

PROPOSITION 4.1. *∇^H is the metric connection of the Riemann extension of ∇ with torsion equal to* $(1/2)S_{H,V}$.

Using (4.1) and (2.15) we found that the possibly non-zero frame components of the curvature tensor $\tilde{R}$ of ∇^H are

$$\tilde{R}_{kji}{}^h=\tilde{R}_{kj\bar{i}}{}^{\bar{h}},\ \tilde{R}_{kj\bar{i}}{}^{\bar{h}}=-R_{kjh}{}^i\,. \tag{4.3}$$

From (4.1) and (4.3), we can compute the frame components of $\nabla^H\tilde{R}$, the following of which are possibly non-zero:

$$\tilde{\nabla}_l\tilde{R}_{kji}{}^h=\nabla_l R_{kji}{}^h,\ \tilde{\nabla}_l\tilde{R}_{kj\bar{i}}{}^{\bar{h}}=-\nabla_l R_{kjh}{}^i\,. \tag{4.4}$$

Thus, (T^*M, ∇^H) is locally flat (resp. locally symmetric) iff (M, ∇) is locally flat (resp. locally symmetric).

Let $\tilde{C}$ be a curve in T^*M whose equations in $\pi^{-1}(U)$ are $x^h=x^h(t)$, $p_h=p_h(t)$. The condition for $\tilde{C}$ to be a geodesic, namely (2.16), in the case of ∇^H, reduces to

$$\begin{aligned}&\frac{d^2x^h}{dt^2}+\Gamma^h_{ji}\frac{dx^j}{dt}\frac{dx^i}{dt}=0\,, \quad \text{and}\\ &\frac{\delta^2 p_h}{dt^2}=\frac{d}{dt}\Big(\frac{\delta p_h}{dt}\Big)-\Gamma^i_{jh}\frac{dx^j}{dt}\frac{\delta p_i}{dt}=0\,.\end{aligned} \tag{4.5}$$

Thus, $\tilde{C}$ is a geodesic in (T^*M, ∇^H) if its projection onto M is a geodesic and its associated covector field has vanishing second intrinsic derivative in (M, ∇).

§5. The intermediate lift.

Let us consider the parallelisms of H and V. If we compare the frame components of ∇^C as listed in (3.5) with conditions (2.17) for parallelism, we found that V is path-parallel and is parallel along H in (T^*M, ∇^C). Whereas H is parallel along V, it is not in general path-parallel. However, we can construct from ∇^C a linear connection on T^*M having this property as well, bearing in mind that ∇^C is torsion-free so that we can use Lemma 2.1.

To do this, we first obtain from (2.19) and (3.5) the frame components of the B-tensor B associated with ∇^C. We find that the only possibly non-zero frame component is

$$B_{ji}{}^{\bar{h}}=-\frac{1}{2}p_a(R_{hij}{}^a+R_{hji}{}^a)\,. \tag{5.1}$$

It follows that the possibly non-zero frame components of the linear connection ∇^C+B are

$$\tilde{\Gamma}^h_{ji}=\Gamma^h_{ji},\ \tilde{\Gamma}^{\bar{h}}_{ji}=\frac{1}{2}p_aR_{jih}{}^a,\ \tilde{\Gamma}^{\bar{h}}_{j\bar{i}}=-\Gamma^i_{jh}\,. \tag{5.2}$$

Thus,

PROPOSITION 5.1. *Let ∇^I be the linear connection on T^*M which is obtained from ∇^C according to Lemma 2.1, i.e., $\nabla^I=\nabla^C+B$. The possibly non-zero frame components of ∇^I are given by (5.2). With respect to ∇^I, H is parallel along V, V is parallel along H and both H, V are path-parallel.*

From (5.1), we see that the skew-symmetric part of the B-tensor of ∇^C is zero. It then follows from $\nabla^I=\nabla^C+B$ that ∇^I is torsion-free. Hence, the only case when ∇^I is metrical with respect to the Riemann extension of ∇ is when $\nabla^I=\nabla^C$.

Let us compare (5.2) and the conditions (2.17) for parallelism. We notice that V is parallel in (T^*M, ∇^I) but H is not (unless ∇ is locally flat). Since ∇^I is torsion-free, we can obtain from it a linear connection with respect to which both H and V are parallel by Lemma 2.2. From (2.18) and (5.1), we first obtain the following as the only possibly non-zero frame components of the A-tensor A associated with ∇^I:

$$A_{ji}{}^{\bar h}=\frac{1}{2}p_a R_{jih}{}^a . \tag{5.3}$$

The linear connection ∇^I-A constructed according to Lemma 2.2 has

$$\tilde\Gamma^h_{ji}=\Gamma^h_{ji},\ \tilde\Gamma^{\bar h}_{j\bar i}=-\Gamma^i_{jh}$$

as its non-zero frame components. But this is exactly that appearing in (4.1). Hence,

PROPOSITION 5.2. *The horizontal lift ∇^H is obtainable from ∇^I by Lemma 2.2, i.e., $\nabla^H=\nabla^I-A$.*

Since ∇^I is obtained from ∇^C and ∇^H from ∇^I, we shall call ∇^I the *intermediate lift of ∇ to T^*M.*

We can compute the curvature tensor $\tilde R$ of ∇^I in the usual way. Its possibly non-zero frame components are found to be

$$\begin{aligned}&\tilde R_{kji}{}^h=R_{kji}{}^h,\ \tilde R_{kji}{}^{\bar h}=\frac{1}{2}p_a(\nabla_k R_{jih}{}^a-\nabla_j R_{kih}{}^a)\,,\\ &\tilde R_{kj\bar i}{}^{\bar h}=-R_{kjh}{}^i,\ \tilde R_{k\bar j i}{}^{\bar h}=-\tilde R_{j\bar k i}{}^{\bar h}=-\frac{1}{2}R_{kih}{}^j\,.\end{aligned} \tag{5.4}$$

It follows that (T^*M, ∇^I) is locally flat if (M, ∇) is locally flat. Furthermore, the possibly non-zero frame components of $\nabla^I\tilde R$ are

$$\begin{aligned}&\tilde\nabla_l\tilde R_{kji}{}^h=\nabla_l R_{kji}{}^h\,,\\ &\tilde\nabla_l\tilde R_{kji}{}^{\bar h}=\frac{1}{2}p_a(\nabla_l\nabla_k R_{jih}{}^a-\nabla_l\nabla_j R_{kih}{}^a)\\ &\qquad+\frac{1}{2}p_a\Big(R_{lth}{}^a R_{kji}{}^t+\frac{1}{2}R_{klt}{}^a R_{jih}{}^t\end{aligned} \tag{5.5}$$

$$+\frac{1}{2}R_{ljt}{}^{a}R_{kih}{}^{t}+R_{lit}{}^{a}R_{kjh}{}^{t}\Big),$$

$$\tilde{\nabla}_{l}\tilde{R}_{kji}{}^{\bar h}=-\nabla_{l}R_{kjh}{}^{i},$$

$$\tilde{\nabla}_{l}\tilde{R}_{kj\bar i}{}^{\bar h}=-\tilde{\nabla}_{l}\tilde{R}_{jk\bar i}{}^{\bar h}=-\frac{1}{2}\nabla_{l}R_{kih}{}^{j},$$

$$\tilde{\nabla}_{\bar l}\tilde{R}_{kji}{}^{\bar h}=\frac{1}{2}(\nabla_{k}R_{jih}{}^{l}-\nabla_{j}R_{kih}{}^{l}).$$

Thus, there is no nice condition for (T^*M, ∇^I) to be locally symmetric.

From (5.3), we see that the symmetric part of the A-tensor of ∇^I is zero. It then follows from $\nabla^H=\nabla^I-A$ that ∇^H and ∇^I have the same geodesics.

§6. The metric $T^*(g, \nabla)$.

Let g be a metric and ∇ a torsion-free linear connection on M. We shall construct from g and ∇ a metric on T^*M, which we denote by $T^*(g, \nabla)$. The line element of $T^*(g, \nabla)$ on $\pi^{-1}(U)$ is taken to be

$$g_{ji}dx^j dx^i+g^{ji}\delta p_j \delta p_i \tag{6.1}$$

where $\delta p_j=dp_j-p_a\Gamma^a_{ji}dx^i$ is the usual covariant differential appearing in (2.5). It is easily seen that (6.1) indeed defines a global metric on T^*M and that the frame component matrix of $T^*(g, \nabla)$ is

$$\begin{bmatrix} g_{ji} & 0 \\ 0 & g^{ji} \end{bmatrix}. \tag{6.2}$$

We would like to establish conditions for ∇^C, ∇^I and ∇^H to be metrical with respect to $T^*(g, \nabla)$. Let us denote by G the metric tensor of $T^*(g, \nabla)$ and by $[G_{\lambda\mu}]$ the matrix in (6.2). By a simple calculation, the possibly non-zero frame components of $\nabla^C G$, $\nabla^I G$ and $\nabla^H G$ are found respectively to be

$$\nabla_k{}^C G_{ji}=\nabla_k g_{ji},\quad \nabla_k{}^C G_{j\bar i}=\nabla_k{}^C G_{\bar i j}=-p_b R_{ajk}{}^{b}g^{ai},\quad \nabla_k{}^C G_{\bar j\bar i}=\nabla_k g^{ji}. \tag{6.3}$$

$$\nabla_k{}^I G_{ji}=\nabla_k g_{ji},\quad \nabla_k{}^I G_{j\bar i}=\nabla_k{}^I G_{\bar i j}=-\frac{1}{2}p_b R_{kja}{}^{b}g^{ai},\quad \nabla_k{}^I G_{\bar j\bar i}=\nabla_k g^{ji}. \tag{6.4}$$

$$\nabla_k{}^H G_{ji}=\nabla_k g_{ji},\ \nabla_k{}^H G_{\bar j\bar i}=\nabla_k g^{ji}. \tag{6.5}$$

From (6.3) and (6.4), we get

PROPOSITION 6.1. *Let g be a metric and ∇ a torsion-free linear connection on M. Then, the following conditions are equivalent:*

(a) *∇^C is metrical with respect to $T^*(g, \nabla)$,*

(b) *∇^I is metrical with respect to $T^*(g, \nabla)$,*

(c) *∇ is the Riemannian connection of g and is locally flat.*

In this case, $\nabla^C=\nabla^I=\nabla^H$ and is the Riemannian connection of $T^*(g, \nabla)$. It is also the Riemannian connection of the Riemann extension of ∇.

Proposition 6.1 is especially interesting when g is positive definite. Then, $T^*(g, \nabla)$ is again positive definite. On the other hand, the Riemann extension of ∇ has n positive and n negative signs. Yet their associated Riemannian connections are still the same.

From (6.5), we similarly get

PROPOSITION 6.2. *Let g be a metric and ∇ a torsion-free linear connection on M. Then, ∇^H is metrical with respect to $T^*(g, \nabla)$ iff ∇ is the Riemannian connection of g.*

§7. The metric T^*g.

The results of Propositions 6.1 and 6.2 indicate that the interesting cases of $T^*(g, \nabla)$ occur when $\nabla=\nabla^g$, i.e., ∇ is the Riemannian connection of g. From now on, we shall assume this. The metric $T^*(g, \nabla)$ now depends solely on g and we denote it by T^*g. Its line element is still given by (6.1), but with

$$\delta p_j=dp_j-p_a\left\{ {a \atop j\ \ i} \right\}dx^i .$$

The metric T^*g was first studied by Tondeur [8], who showed that T^*g together with the canonical 2-form $dx^j\wedge dp_j$ on T^*M defined an almost Kaehlerian structure on T^*M. Let TM be the tangent bundle over M, $p: TM\rightarrow M$ the canonical projection and $\{p^{-1}(U), (x^h, y^h)\}$ the usual induced coordinate system in TM. Let Tg be the Sasaki metric on TM ([6]). It was shown by Sato [7] that the maps $p^{-1}(U)\rightarrow\pi^{-1}(U)$ defined by

$$x^h=x^h,\ p_h=g_{hj}y^j ,$$

pieced together to form an isometry $f: (TM, Tg)\rightarrow(T^*M, T^*g)$. Although f does not preserve the complete lifts of vector fields to TM and to T^*M, the metric properties of T^*g can be obtained from the corresponding properties of Tg. In particular, Kowalski [4] has shown that (TM, Tg) is never locally symmetric, and the same is thus true for (T^*M, T^*g). Note that we require a locally symmetric space to be first of all, non-flat.

From Proposition 6.2 and Proposition 4.1, we get

PROPOSITION 7.1. *Let g be a metric on M. Then, the horizontal lift $(\nabla^g)^H$ of ∇^g to T^*M is the metric connection of T^*g with torsion $(1/2)S_{H,V}$ where H is the projection tensor onto the horizontal distribution of T^*M determined by ∇^g. Furthermore, $(\nabla^g)^H$ is also metrical with respect to the Riemann extension of ∇^g.*

To see what other connections are deducible from T^*g, let us work out the frame components of its Riemannian connection. They can be obtained from

(3.3) by letting $[G_{\lambda\mu}]$ to be the matrix in (6.2). The resulting frame components are:

$$
\begin{aligned}
&\tilde{\Gamma}^{h}_{ji}=\left\{{h \atop j\ i}\right\}, && \tilde{\Gamma}^{\bar h}_{ji}=\frac{1}{2}p_a K_{jih}{}^{a},\\
&\tilde{\Gamma}^{h}_{j\bar i}=\frac{1}{2}g^{ha}g^{ib}p_c K_{ajb}{}^{c}, && \tilde{\Gamma}^{\bar h}_{j\bar i}=-\left\{{i \atop j\ h}\right\},\\
&\tilde{\Gamma}^{h}_{\bar j i}=\frac{1}{2}g^{ha}g^{jb}p_c K_{aib}{}^{c}, && \tilde{\Gamma}^{\bar h}_{\bar j i}=0,\\
&\tilde{\Gamma}^{h}_{\bar j\bar i}=0, && \tilde{\Gamma}^{\bar h}_{\bar j\bar i}=0.
\end{aligned}
\tag{7.1}
$$

We point out that the frame components in (7.1) are different from that given in [7, p. 466] because the adapted frame used there is different from that of ours.

From (7.1), we can work out the associated B-tensor of the Riemannian connection, using (2.19). The possibly non-zero frame components are

$$B_{j\bar i}{}^{h}=-\frac{1}{2}g^{ha}g^{ib}p_c K_{ajb}{}^{c}$$

and

$$B_{\bar j i}{}^{h}=-\frac{1}{2}g^{ha}g^{jb}p_c K_{aib}{}^{c}.$$

The linear connection on T^*M obtained from the Riemannian connection of T^*g according to Lemma 2.1 has therefore the following possibly non-zero frame components:

$$\tilde{\Gamma}^{h}_{ji}=\left\{{h \atop j\ i}\right\},\qquad \tilde{\Gamma}^{\bar h}_{ji}=\frac{1}{2}p_a K_{jih}{}^{a},\qquad \tilde{\Gamma}^{\bar h}_{j\bar i}=-\left\{{i \atop j\ h}\right\}.$$

Thus, finally, we have

PROPOSITION 7.2. *The linear connection obtained from the Riemannian connection of T^*g according to Lemma 2.2 is just the intermediate lift of ∇^g.*

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