K. SHIGA AND T. TSUJISHITA
KŌDAI MATH. SEM. REP.
28 (1977), 214—225

DIFFERENTIAL REPRESENTATIONS OF VECTOR FIELDS

BY KOJI SHIGA AND TORU TSUJISHITA

In the present paper we are concerned with Lie algebra representations of $\mathcal{A}(M)$, where $\mathcal{A}(M)$ denotes the Lie algebra of vector fields on a smooth manifold M. It is well-known that the Lie derivatives acting on various tensor spaces give rise to representations of $\mathcal{A}(M)$. Another type of representations of $\mathcal{A}(M)$ arises from the considerations of the adjoint representation of $\mathcal{A}(M)$ on the space of differential operators with a finite order. These examples lead us to introduce a notion of differential representations of $\mathcal{A}(M)$. Specifically, φ is called a differential representation of $\mathcal{A}(M)$ on $\Gamma(E)$, E being a certain real vector bundle, if φ is a Lie algebra representation of $\mathcal{A}(M)$ to $\mathrm{Hom}(\Gamma(E), \Gamma(E))$ with $\mathrm{supp}\,\varphi(X)\sigma\subset \mathrm{supp}\,X\cap \mathrm{supp}\,\sigma$. We do not know whether all the differential representations of $\mathcal{A}(M)$ arouse geometric interest or not. However, the differential representations mentioned above have some geometric nature, which is characterized as connection-type. Here we use the terminology "connection" in a wide sense.

The main result of the present paper can be stated as follows. If on $\Gamma(E)$ there exists a differential representation of connection type of $\mathcal{A}(M)$, then we have

$$\mathrm{Pont}(E)\subset \mathrm{PONT}(M).$$

Here $\mathrm{Pont}(E)$ denotes the subalgebra of $H^*(M;\boldsymbol{R})$ which is generated by the Pontrjagin classes $p_i(i\geqq 1)$ of E, while $\mathrm{PONT}(M)$ denotes the ideal of $H^*(M;\boldsymbol{R})$ which is generated by the Pontrjagin classes of M. This result, of course, involves that there is a topological obstruction to the existence of differential representation of connection type on $\Gamma(E)$. To find this obstruction, we extend the Chern-Weil theory on characteristic classes so as to be adaptable not only for the de Rham cohomology ring but also for more general cohomology rings. Then the theory especially applies to the Losik cohomology ring which yields a desired topological obstruction. Some analogous results will be also obtained for the Lie algebra consisting of the vector fields of type (1, 0) on a complex manifold.

The authors would like to thank K. Masuda and T. Sunada for helpful suggestions.

Received Nov. 5, 1975.

1. Generalities on differenital representations

Throughout the present paper M is assumed to be a smooth manifold with a countable basis. For any smooth real vector bundle E over M we denote by $\Gamma(E)$ the space of the smooth cross-sections of E. Let τ denote the tangent bundle of M. $\Gamma(\tau)$ is clearly identified with the space of the vector fields over M, so that $\Gamma(\tau)$ has a natural Lie algebra structure over $\boldsymbol{R}$. This Lie algebra is denoted by $\mathcal{A}(M)$.

DEFINITION. A Lie algebra representation φ of $\mathcal{A}(M)$ to Hom $(\Gamma(E), \Gamma(E))$ is called a differential representation of $\mathcal{A}(M)$ on $\Gamma(E)$ if it satisfies

$$\operatorname{supp} \varphi(X)\sigma \subset \operatorname{supp} X \cap \operatorname{supp} \sigma\,,$$

where $X \in \mathcal{A}(M)$ and $\sigma \in \Gamma(E)$.

In this section we shall briefly summarize some basic facts about the differential representations of $\mathcal{A}(M)$. Let φ be a differential representation of $\mathcal{A}(M)$ on $\Gamma(E)$. Then we can obtain a bilinear map from $\mathcal{A}(M) \times \Gamma(E)$ to $\Gamma(E)$ when we assign $\varphi(X)\sigma$ to (X, σ) $(X \in \mathcal{A}(M), \sigma \in \Gamma(E))$.

PROPOSITION 1.1. *φ gives a bilinear differential map from $\mathcal{A}(M) \times \Gamma(E)$ to $\Gamma(E)$ in the following sense:*

Let U be a closed disk with local coordinates $(x^1, \cdots, x^n)$ and let $\{e_1, \cdots, e_s\}$ be a local frame of E on U. Then φ is expressed on U in the form

$$\varphi\Big(\sum_{\iota=1}^{n} X^\iota(x)\partial_i\Big)\Big(\sum_{\mu=1}^{s} \sigma^\mu(x)e_\mu\Big) = \sum_{\alpha,\iota,A,\mu,\nu} h_{\alpha\iota A\nu}{}^{\mu}(x) D^\alpha X^\iota(x) D^A \sigma^\nu(x) e_\mu\,,$$

where $h_{\alpha\iota A\nu}{}^{\mu}(x)$ denote smooth functions on U and the indices α and A range over a finite set of multi-indices; D^α and D^A denote the partial differentiations associated to the multi-indices α and A respectively.

This is really obtained as a consequence of a more general statement as follows.

PROPOSITION 1.2. *Let E_i $(\iota=1, 2, 3)$ be vector bundles over M. Let Φ be a bilinear map of $\Gamma(E_1) \times \Gamma(E_2)$ to $\Gamma(E_3)$ with*

$$\operatorname{supp} \Phi(\xi_1, \xi_2) \subset \operatorname{supp} \xi_1 \cap \operatorname{supp} \xi_2 \qquad (\xi_i \in \Gamma(E_\iota))\,.$$

Then Φ is a bilinear differential map from $\Gamma(E_1) \times \Gamma(E_2)$ to $\Gamma(E_3)$.

Proposition 1.2 can be regarded as a generalization of the well-known Peetre's Theorem and the proof is also obtained by a slight modification of the proof of that theorem found in [4; Chap. 3].

Some examples of differential representations of $\mathcal{A}(M)$ have been described in the introduction. Actually, there exists a considerably general procedure of constructing differential representations of $\mathcal{A}(M)$ which yields those examples as special cases (cf. [5; I]). It is a remarkable fact that all such represention satisfy an important condition, which is stated in the following definition.

DEFINITION. A differential representation φ of $\mathcal{A}(M)$ on $\Gamma(E)$ is said to have connection-type if

$$\varphi(X)(f\sigma)=X(f)\sigma+f\varphi(X)\sigma \tag{1.3}$$

holds, where $X\in\mathcal{A}(M)$, $f\in C^\infty(M)$ and $\sigma\in\Gamma(E)$.

We note that there exist non-trivial differential representations of $\mathcal{A}(M)$ which have not connection-type. The examples below give such representations.

EXAMPLE 1. Let T^n be the n-dimensional torus. Take the "parameters" $(x_1, \cdots, x_n)$ on T^n induced from the coordinates of R^n. Let $E=T^n\times R^s$ with $s\geqq 2$. Take any $H_i\in gl(s;\boldsymbol{R})$ $(i=1, 2, \cdots, n)$ such that

$$H_iH_j=H_jH_i \qquad (i, j=1, 2, \cdots, n)$$

and each H_i is nilpotent. To each multi-index $\alpha=(\alpha_1, \cdots\alpha_n)$ we assign the matrix

$$H^\alpha=\frac{1}{\alpha\,!}H_1{}^{\alpha_1}H_1{}^{\alpha_2}\cdots H_n{}^{\alpha_n}$$

and write $H^\alpha=(h_{\alpha\nu}{}^\mu)$ $(\mu, \nu=1, 2, \cdots, s)$. Note that $H^\alpha=0$ if $|\alpha|$ is sufficiently large. We can then define a differential representation φ of $\mathcal{A}(T^n)$ on $\Gamma(E)$ by setting

$$\varphi(\sum X^i\partial_i)(\sum\sigma^\mu e_\mu)=\sum_{\alpha,i,\mu,\nu} h_{\alpha\nu}{}^\mu D^\alpha X^i\cdot\partial_i\sigma^\nu e_\mu .$$

EXAMPLE 2. For each multi-index α we take a matrix $K_\alpha\in gl(s;\boldsymbol{R})$ with the form

$$K_0=\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix},\quad K=\begin{pmatrix} 0 & \tilde{K}_\alpha\\ 0 & 0\end{pmatrix} \qquad \text{if } |\alpha|>0 .$$

Here I denotes the identity matrix with rank s' $(0<s'<s)$ and $\tilde{K}_\alpha$ denotes any $(s', s-s')$-matrix. Then we can define a differential representation ψ of $\mathcal{A}(T^n)$ on $\Gamma(E)$, E being the same as in Example 1, by setting

$$\psi(\sum X^i\partial_i)(\sum\sigma^\mu e_\mu)=\sum_{\alpha,\mu,\nu} k_{\alpha\nu}{}^\mu X^i\cdot D^{\alpha+(i)}\sigma^\nu\cdot e_\mu ,$$

where $K_\alpha=(k_{\alpha\nu}{}^\mu)$, (i) denotes the multi-index $(0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0)$ and the index α occurring in the summation ranges over any finite set of multi-indices.

As to the line bundles, however, a straightforward calculation shows

PROPOSITION 1.3. *Let E be a line bundle over a connected manifold M. Then any non-trivial differential representation of $\mathcal{A}(M)$ on $\Gamma(E)$ is of connection-type.*

2. Characteristic classes with values in certain cohomology rings

Let V be a real vector bundle or an inductive vector bundle over M. (As to the inductive vector bundle, see [5;I].) Assume that we have a cochain complex

$$\mathcal{V}: 0 \longrightarrow \Gamma(\Lambda^0 V) \xrightarrow{d_V} \Gamma(\Lambda^1 V) \xrightarrow{d_V} \cdots \xrightarrow{d_V} \Gamma(\Lambda^p V) \xrightarrow{d_V} \cdots$$

with the multiplicative property

$$d_V(\omega \wedge \eta) = d_V\omega \wedge \eta + (-1)^p \omega \wedge d_V \eta\,, \tag{2.1}$$

where $\omega \in \Gamma(\Lambda^p V)$. We note that $\Gamma(\Lambda^0 V) = C^\infty(M)$. The cohomology group of $\mathcal{V}$ is denoted by

$$H^*(\mathcal{V}) = \sum H^p(\mathcal{V})\,,$$

which, in view of (2.1), is endowed with a ring structure.

In this section we shall define a characteristic ring of any real vector bundle as a subalgebra of $H^*(\mathcal{V})$. First we make a few comments on the cochain complex $\mathcal{V}$.

$$\operatorname{supp} d_V(\omega) \subset \operatorname{supp} \omega\,. \tag{2.2}$$

In fact, if we apply (2.1) to $f, g \in \Gamma(\Lambda^0 V)$, we have immediately $\operatorname{supp} d_V(f) \subset \operatorname{supp} f$. This, combined with (2.1), yields $f(d_V\omega) = 0$ for $f \in \Gamma(\Lambda^0 V)$ and $\omega \in \Gamma(\Lambda^p V)$ whenever $\operatorname{supp} f \cap \operatorname{supp} \omega = \phi$ holds. Hence (2.2) follows.

Let

$$\mathcal{R}: 0 \longrightarrow \Gamma(\Lambda^0 \tau^*) \xrightarrow{d} \Gamma(\Lambda^1 \tau^*) \xrightarrow{d} \cdots \xrightarrow{d} \Gamma(\Lambda^p \tau^*) \xrightarrow{d} \cdots$$

be the de Rham complex of M. Then

PROPOSITION 2.3. *There is a homomorphism $\lambda: \tau^* \to V$, which induces the homomorphism of $\mathcal{R}$ to $\mathcal{V}$. That is, we have a commutative diagram*

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{d} & \Gamma(\Lambda^{p-1}\tau^*) & \xrightarrow{d} & \Gamma(\Lambda^p \tau^*) & \xrightarrow{d} & \cdots \\
 & & \downarrow \lambda_*^{p-1} & & \downarrow \lambda_*^{p} & & \\
\cdots & \xrightarrow{d_V} & \Gamma(\Lambda^{p-1} V) & \xrightarrow{d_V} & \Gamma(\Lambda^p V) & \xrightarrow{d_V} & \cdots\,,
\end{array}$$

where λ_*^p denotes the map induced from the bundle map $\Lambda^p \lambda: \Lambda^p \tau^* \to \Lambda^p V$.

Proof. In view of (2.1) and (2.2), it is not difficult to see that $d_V | \Gamma(\Lambda^0 V)$ is a first order differential operator without constant terms. Let

$$\operatorname{Symb}(d_V | \Gamma(\Lambda^0 V))$$

denote the symbol of this differential operator. Then for any $\eta_x \in \tau_x^*$

$$\operatorname{Symb}(d_V | \Gamma(\Lambda^0 V))(\eta_x)$$

induces a linear map from $\boldsymbol{R}$ to V_x. Define the homomorphism λ from τ^* to V by setting

$$\lambda(\eta_x)=\mathrm{Symb}\,(d_V\,|\,\Gamma(\Lambda^0 V))(\eta_x)\cdot 1\,.$$

Then we have

$$\lambda_*{}^1(df)=d_V f\,, \qquad \text{for} \quad f\in C^\infty(M)\,. \tag{2.4}$$

Take any $\omega\in\Gamma(\Lambda^1\tau^*)$ and express locally ω as $\sum\omega_i(x)dx^i$. Then by (2.1) and (2.4) the following calculation is valid locally:

$$\begin{aligned} d_V(\lambda_*{}^1(\omega)) &= d_V(\lambda_*{}^1(\textstyle\sum\omega_i dx^i))=d_V(\sum\omega_i\lambda_*{}^1(dx^i)) \\ &= \textstyle\sum d_V\omega_i\wedge\lambda_*{}^1(dx^i)+\sum\omega_i d_V\lambda_*{}^1(dx^i) \\ &= \textstyle\sum\lambda_*{}^1(d\omega_i)\wedge\lambda_*{}^1(dx^i) \\ &= \lambda_*{}^2(d\omega)\,. \end{aligned}$$

But this together with (2.2) yields $d_V\lambda_*{}^1=\lambda_*{}^2 d$ throughout M. From (2.1) it then follows immediately that $d_V\lambda_*{}^{p-1}=\lambda_*{}^p d$ holds for $p=0, 1, 2, \cdots$. This completes the proof.

The resulting homomorphism of $\mathcal{R}$ to $\mathcal{V}$ is also denoted by λ.

We shall mention a method of obtaining the multiplicative cochain complexes [5; I]. Let F be a real vector bundle over M. $\Gamma(F)$ is called a Lie algebra over M if $\Gamma(F)$ admits a Lie algebra structure whose bracket rule satisfies $\mathrm{supp}\,[\xi, \eta]\subset\mathrm{supp}\,\xi\cap\mathrm{supp}\,\eta$ $(\xi, \eta\in\Gamma(F))$. Suppose that $\Gamma(F)$ has a Lie algebra representation ψ on $C^\infty(M)$ such that

$$\psi(\xi)(fg)=\psi(\xi)f\cdot g+f\cdot\psi(\xi)g\,,$$

where $\xi\in\Gamma(F)$ and $f, g\in C^\infty(M)$. Let $\{\hat{C}^*(\Gamma(F)), d\}$ be the cochain complex which gives the Lie algebra cohomology of $\Gamma(F)$ associated with the representation ψ. Consider the subcomplex of $\{\hat{C}^*(\Gamma(F)), d\}$ whose p-cochain space consists of those alternating p-linear maps L from $\Gamma(F)$ to $C^\infty(M)$ which satisfy

$$\mathrm{supp}\,L(\xi_1, \cdots, \xi_p)\subset\mathrm{supp}\,\xi_1\cap\cdots\cap\,\mathrm{supp}\,\xi_p$$

$(\xi_i\in\Gamma(F))$. Then it turns out that this subcomplex is canonically isomorphic to a multiplicative cochain complex

$$\mathcal{E}: 0\longrightarrow\Gamma(\Lambda^0(JF)^*)\xrightarrow{\ d\ }\Gamma(\Lambda^1(JF)^*)\xrightarrow{\ d\ }\cdots\xrightarrow{\ d\ }\Gamma(\Lambda^p(JF)^*)\xrightarrow{\ d\ }\cdots,$$

where we put $(JF)^*=\varinjlim\,(J^kF)^*$; $(J^kF)^*$ is the dual bundle of the k-jet bundle of F. Besides, a subcomplex of $\mathcal{E}$ often provides us with a useful example of the multiplcative cochain complex. As an example of such $\Gamma(F)$, we can take $\mathcal{A}(M)$ or a subalgebra of $\mathcal{A}(M)$ which gives a foliation of M, or a Lie algebra $\Gamma(F)$ over M which has a Lie algebra homomorphism onto $\mathcal{A}(M)$.

Assume that a multiplicative cochain complex $\mathcal{V}=\{\Gamma(\Lambda^*V), d\}$ be given.

DEFINITION. Let E be a real vector bundle. An $\boldsymbol{R}$-linear map

$$D:\Gamma(E)\longrightarrow\Gamma(V\otimes E)$$

is called a $\mathcal{V}$-connection on E if it satisfies

$$D(f\sigma)=d_V f\otimes\sigma+fD\sigma$$

where $f\in C^\infty(M)$ and $\sigma\in\Gamma(E)$.

We note that the definition of the usual connection is included as a special case when we take $\mathcal{R}$ as $\mathcal{V}$. It is easy to see that there are many $\mathcal{V}$-connections on E. Suppose that a $\mathcal{V}$-connections D on E be given. Then, as usual, we can define the connection matrix $\theta=(\theta_\nu{}^\mu)$ for a local frame $e_1, \cdots, e_s$ on an open set U: $De_\nu=\sum_\mu \theta_\nu{}^\mu\otimes e_\mu$ $(\mu, \nu=1, \cdots, s)$ where $\theta_\nu{}^\mu\in\Gamma(\Lambda^1 V|U)$. Put

$$\Omega_\nu{}^\mu=d_V\theta_\nu{}^\mu-\sum_\kappa \theta_\nu{}^\kappa\wedge\theta_\kappa{}^\mu .$$

Then the matrix $(\Omega_\nu{}^\mu)$ gives rise to a global section of $\Gamma(\Lambda^2 V\otimes\mathrm{Hom}(E, E))$, which we denote by Ω. Ω may be called the $\mathcal{V}$-curvature of D. Let I^p denote the space of adjoint invariant homogeneous polynomial functions of degree p on $\mathfrak{gl}(s;\boldsymbol{R})$. If $P\in I^p$, then $P(\Omega)$ gives an element of $\Gamma(\Lambda^{2p}V)$. It is an essential point for our arguments that the Chern-Weil theory on charactiristic classes remains true in our situation. Actually we have the following proposition.

PROPOSITION 2.5. i) *$P(\Omega)$ is a cocycle.*

ii) *The cohomology class $[P(\Omega)]\in H^{2p}(\mathcal{V})$ defined by $P(\Omega)$ is independent of the choice of $\mathcal{V}$-connections.*

To prove this, we have only to verify that the method used in developing the Chern-Weil theory in [2] also applies to this case. But the verification is immediate, since the formal calculus of differential forms used there remains valid owing to (2.1). Hence the assertion follows.

We say that $[P(\Omega)]$ is the $\mathcal{V}$-characteristic class of E associated to P. We remark that the $\mathcal{R}$-characteristic class of E coincides with the (real) Pontrjagin class of E. Set $I=\sum_{p\geq 0} I^p$. Then I forms an algebra over $\boldsymbol{R}$. The image of I through the map $P\to[P(\Omega)]$ forms a subalgebra of $H^*(\mathcal{V})$, which is called the $\mathcal{V}$-characteristic ring of E and denoted by $\mathrm{Pont}(E;\mathcal{V})$. We put $\mathrm{Pont}(E)=\mathrm{Pont}(E;\mathcal{R})$.

It should be noted that these characteristic classes possess a functorial property with respect to the "coefficient region" $\mathcal{V}$. To state this more precisely, let $\mathcal{V}=\{\Gamma(\Lambda^* V), d_V\}$ and $\mathcal{W}=\{\Gamma(\Lambda^* W), d_W\}$ be two multiplicative cochain complexes. Suppose that there exists a homomorphism

$$\kappa: V\longrightarrow W$$

such that $\Lambda^p\kappa: \Lambda^p V\to\Lambda^p W$ $(p=0, 1, 2, \cdots)$ induce a homomorphism of $\mathcal{V}$ to $\mathcal{W}$. Then we have the homomorphism

$$\kappa_*: H^*(\mathcal{V})\longrightarrow H^*(\mathcal{W}) .$$

On the other hand, for any $\mathcal{V}$-connection D on E, the composition of maps

$$\Gamma(E) \xrightarrow{D} \Gamma(V\otimes E) \xrightarrow{\kappa\otimes 1} \Gamma(W\otimes E)$$

induces a $\mathcal{W}$-connection on E, which is denoted by $\kappa^*(D)$. Let Ω denote the $\mathcal{V}$-curvature of D and $\kappa_*(\Omega)$ the $\mathcal{W}$-curvature of $\kappa_*(D)$. Then we have

$$\kappa_*[P(\Omega)]=[P(\kappa_*(\Omega))]\,. \tag{2.6}$$

Apply (2.6) to the map λ stated in Proposition 2.3. Then we have established the following proposition.

PROPOSITION 2.7. $\mathrm{Pont}\,(E\,;\mathcal{V})=\lambda_*\,\mathrm{Pont}\,(E)$.

Even in the case where the basic field is the complex number field, the discussions proceed in parallel with the above. It is, however, relevant to the customary usage that in this case we adopt the notation $\mathrm{Chern}\,(E\,;\mathcal{V})$ instead of $\mathrm{Pont}\,(E\,;\mathcal{V})$. For example, Proposition is replaced by

$$\mathrm{Chern}\,(E\,;\mathcal{V})=\lambda_*\,\mathrm{Chern}\,(E) \tag{2.8}$$

in the complex case.

3. Topological obstructions

We first recall the basic properties of the Losik complex [3], [5]. The Losik complex $\mathcal{L}$ is defined to be a multiplicative cochain complex which is obtained from the Lie algebra $\mathcal{A}(M)$ and its canonical representation of $\mathcal{A}(M)$ on $C^\infty(M)$, according to the procedure explained in Section 2. Thus we have

$$\mathcal{L}: 0 \longrightarrow \Gamma(\Lambda^0(J\tau)^*) \xrightarrow{d} \Gamma(\Lambda^1(J\tau)^*) \xrightarrow{d} \cdots \xrightarrow{d} \Gamma(\Lambda^p(J\tau)^*) \xrightarrow{d} \cdots .$$

The complex $\mathcal{L}$ canonically includes the de Rham complex $\mathcal{R}$ as a subcomplex. The inclusion map $\mathcal{R}\to\mathcal{L}$ coincides with the map λ which we have introduced in Proposition 2.3. The structure of the cohomology ring $H^*(\mathcal{L})$ is completely determined by M.V. Losik [3]: There is a ring isomorphism

$$H^*(\mathcal{L})\cong H^*(B(\tau^{\boldsymbol{C}})\,;\boldsymbol{R})\,,$$

where $B(\tau^{\boldsymbol{C}})$ denotes the $U(n)$-bundle over M associated to $\tau\otimes\boldsymbol{C}$ ($n=\dim M$). Moreover we have

(3.1) $\mathrm{Ker}\,\lambda_*=\mathrm{PONT}\,(M)$, where $\mathrm{PONT}\,(M)$ denotes the ideal of $H^*(M\,;\boldsymbol{R})$ generated by the Pontrjagin classes p_i of M ($i=1, 2, \cdots, [n/4]$).

Let E be a real vector bundle over M. Assume that on $\Gamma(E)$ we have a differential representation φ of connection type of $\mathcal{A}(M)$. Then φ defines an $\mathcal{L}$-connection D_φ as follows:

$$D_\varphi(\sigma)(j^\infty X)=\varphi(X)\sigma\,,$$

where j^∞ means the jet extension map $\Gamma(\tau)\to\Gamma(J\tau)$. Let Ω_φ denote the $\mathcal{L}$-curvature of D_φ. Then a direct calculation yields

$$(\varphi(X)\varphi(Y)-\varphi(Y)\varphi(X)-\varphi([X, Y]))\sigma=\Omega_\varphi(X, Y)\tau,$$

whence we have $\Omega_\varphi=0$. In view of Proposition 2.5, this implies

$$\mathrm{Pont}(E;\mathcal{L})=0.$$

By Proposition 2.7, this can be rewritten as

$$\lambda_*\,\mathrm{Pont}(E)=0.$$

Hence, referring to (3.1), we have obtained the following theorem.

THEOREM 3.2. *If on $\Gamma(E)$ there is a differential representation of connection-type of $\mathcal{A}(M)$, then we have*

$$\mathrm{Pont}(E)\subset\mathrm{PONT}(M).$$

Let $\tilde{M}$ be the underlying topological manifold of M. According to the topological invariance of rational Pontrjagin classes, PONT (M) is independent of the differential structures introduced on $\tilde{M}$. Hence if a vector bundle E satisfies the condition

$$\mathrm{Pont}(E)\not\subset\mathrm{PONT}(M)$$

then there is no differential representation of connection type of vector fields on $\Gamma(E)$, even when we change a differential structure of $\tilde{M}$. For example, a Bott generator E on S^{4n} which gives a generator of $\tilde{K}(S^{4n})$ satisfies this condition; here, of course, E is regarded as a real bundle.

If we consider the category of the complex vector bundles, in view of (2.8) the corresponding theorem holds in the following form:

THEOREM 3.3. *Let E be a complex vector bundle over M. If on $\Gamma(E)$ there is a differential **C**-linear representation of connection type of $\mathcal{A}^C(M)$, then we have*

$$\mathrm{Chern}(E)\subset\mathrm{PONT}_C(M),$$

where we put $\mathcal{A}^C(M)=\mathcal{A}(M)\otimes C$, being identified with $\Gamma(\tau\otimes C)$, and $\mathrm{PONT}_C(M)$ *denotes the ideal of $H^*(M;C)$ generated by the complex Pontrjagin classes of M.*

As to the complex line bundles, we can obtain a more exact result.

THEOREM 3.4. *Let E be a complex line bundle over a connected manifold M. Then in order that we have a non-trivial differential **C**-linear representation of $\mathcal{A}^C(M)$ on $\Gamma(E)$ it is necessary and sufficient that the first complex Chern class of E vanishes:* $c_1(E)=0$.

Proof. Necessity: We first note that Proposition 1.2 holds true for the **C**-linear representations of $\mathcal{A}^C(M)$. Hence if we have a non-trivial differential **C**-linear representation φ of $\mathcal{A}^C(M)$ on $\Gamma(E)$, then φ is necessarily of connection-type. From Theorem 3.3 it follows that $c_1(E)=0$.

Sufficiency: It is known that if $c_1(E)=0$, then we can choose as the transition functions of E locally constant functions on M. (If we identify E with an element of $H^1(M, D^*)$, the condition $c_1(E)=0$ implies that E is contained in the image of $H^1(M, C^*)$ through the inclusion map $\iota: C^*\to D^*$. Here D^* and C^* denote the sheaf of germs of non-zero smooth functions on M and of non-zero complex numbers, respectively.) Hence the natural representation of $\mathcal{A}^C(M)$ on $C^\infty(M)\otimes \boldsymbol{C}$ extends to a representation of $\mathcal{A}^C(M)$ on $\Gamma(E)$, which is clearly non-trivial. This completes the proof.

Now we wish to obtain analogous theorems for a complex manifold. Let M be a complex manifold with $\dim_{\boldsymbol{C}} M=n$. Let $\tau\otimes \boldsymbol{C}=T\oplus\bar{T}$ be the canonical splitting of $\tau\otimes\boldsymbol{C}$ to the holomorphic and the anti-holomorphic tangent bundles. Putting $\mathcal{A}_\partial(M)=\Gamma(T)$, we consider it as a Lie subalgebra of $\mathcal{A}^C(M)$. Let E be a holomorphic vector bundle over M. A differential representation φ of $\mathcal{A}_\partial(M)$ on $\Gamma(E)$ is called holomorphic if it satisfies the following two conditions:

(3.5) If $X\in\mathcal{A}_\partial(M)$ and $\sigma\in\Gamma(E)$ are holomorphic on an open set U of M, then $\varphi(X)\sigma$ is holomorphic on U.

(3.6) If $g\in C^\infty(M)$ is anti-holomorphic on an open set U of M, then $\varphi(gX)=g\varphi(X)$ on U.

A differential representation φ of $\mathcal{A}_\partial(M)$ on $\Gamma(E)$ is said to have connection-type if the condition stated in (1.3) holds for $X\in\mathcal{A}_\partial(M)$. Let CHERN (M) denote the ideal of $H^*(M; \boldsymbol{C})$ generated by the Chern classes c_i of M ($i=1, 2, \cdots, n$).

THEOREM 3.7. *If on $\Gamma(E)$ there is a holomorphic representation of connection-type of $\mathcal{A}_\partial(M)$, then we have*

$$\text{Chern}\,(E)\subset\text{CHERN}\,(M)$$

Before entering into the proof, we need information on the cohomology ring of a certain cochain complex. Let $J_\partial(T)$ denote the projective vector bundle $J_\partial(T)=\varprojlim J_\partial{}^k(T)$, where $J_\partial{}^k(T)$ is the k-jet bundle of $\Gamma(T)$ along the partial derivatives with respect to $\partial/\partial z^1, \cdots, \partial/\partial z^n$. Consider the Lie algebra cohomology of $\mathcal{A}_\partial(M)$ associated with the canonical representation through the ∂-differentiation on $\Gamma(\Lambda^q\bar{T})$, where q is a fixed integer. The corresponding cochain complex contains a subcomplex

$$\cdots \longrightarrow \Gamma(\Lambda^p(J_\partial T)^*\otimes\Lambda^q\bar{T}) \xrightarrow{\partial} \Gamma(\Lambda^{p+1}(J_\partial T)^*\otimes\Lambda^q\bar{T}) \longrightarrow \cdots$$

(cf. [6]). On the other hand, each $\Lambda^p(J_\partial T)^*$ is the inductive limit of holomorphic vector bundles, whence we can obtain the complex

$$\cdots \longrightarrow \Gamma(\Lambda^p(J_\partial T)^*\otimes\Lambda^q\bar{T}) \xrightarrow{\bar\partial} \Gamma(\Lambda^p(J_\partial T)^*\otimes\Lambda^{q+1}\bar{T}) \longrightarrow \cdots$$

by the use of the $\bar\partial$-operators. Thus we have the double complex

$$\mathcal{L}_\partial=\{\bigoplus_{p+q=r}\Gamma(\Lambda^p(J_\partial T)^*\otimes\Lambda^q\bar{T}),\ d\}\qquad(r=0,1,2,\cdots).$$

Note that d is given by $d=\partial+(-1)^p\bar\partial$ on $\Gamma(\Lambda^p(J_\partial T)^*\otimes\Lambda^q\bar{T})$. In fact, we have a ring isomorphism

$$H^*(\mathcal{L}_\partial)\cong H^*(B(T),\boldsymbol{C}),$$

where $B(T)$ denotes the principal $U(n)$-bundle over M associated with T. Besides, letting $\lambda:\mathcal{R}^C\to\mathcal{L}_\partial$ be the canonical inclusion map, we have

(3.8) $$\operatorname{Ker}\lambda_*=\mathrm{CHERN}\,(M),$$

where $\mathcal{R}^C$ denotes the "complexification" of the de Rham complex.

Proof of Theorem 3.7. Suppose that on $\Gamma(E)$ we have a holomorphic representation φ of connection-type of $\mathcal{A}_\partial(M)$. Then we define an $\mathcal{L}_\partial$-connection D_φ on E by

$$D_\varphi(\sigma)(j_\partial{}^\infty X_1)=\varphi(X_1)\sigma\,,\qquad\text{for } X_1\in\Gamma(T)\,,$$
$$D_\varphi(\sigma)(X_2)=X_2\sigma\,,\qquad\text{for } X_2\in\Gamma(\bar{T})\,,$$

where $\sigma\in\Gamma(E)$ and $j_\partial{}^\infty$ denotes the jet extension map of $\Gamma(T)$ to $\Gamma(J_\partial T)$. We note that this is well-defined. In fact, since φ is holomorphic, if $j_\partial{}^\infty X_1=j_\partial{}^\infty X_2$ at a point p_0, we have $\varphi(X_1)\sigma=\varphi(X_2)\sigma$ at p_0. Since φ is holomorphic and has connection-type, we have $\bar\partial\theta_{\alpha\beta}=0$ for the connection matrix $\theta=(\theta_{\alpha\beta})$ of D_φ with respect to a holomorphic local frame of E. It is then easy to verify that the $\mathcal{L}_\partial$-curvature of D_φ vanishes. Hence the assertion follows from (2.8) and (3.8).

It is well-known that on a complex torus there are many holomorphic vector bundles whose Chern classes do not vanish. Theorem 3.3 means that on these vector bundles there exists no holomorphic representation of connection-type.

Finally, we shall formulate another theorem on a complex manifold which is closely related to the Dolbeault cohomology. The natural representation of $\mathcal{A}_\partial(M)$ on $C^\infty(M)$ induces a multiplicative cochain complex

$$\mathcal{D}:\cdots\longrightarrow\Gamma(\Lambda^p(J_\partial T)^*)\xrightarrow{\ \partial\ }\Gamma(\Lambda^{p+1}(J_\partial T)^*)\longrightarrow\cdots$$

in a canonical way (cf. [6]). The structure of the cohomology ring $H^*(\mathcal{D})$ is completely determined in [6], which states that we have a ring isomorphism

$$H^*(\mathcal{D})\cong H^*(U(n);\boldsymbol{C})\otimes H^*(M;\bar{\mathcal{O}}),$$

where $\bar{\mathcal{O}}$ denotes the sheaf of germs of anti-holomorphic functions on M. The composition of maps

$$\tau^*\otimes\boldsymbol{C}\xrightarrow{\ \pi\ }T^*\xrightarrow{\ j\ }(J_\partial T)^*$$

corresponds to the map λ stated in Proposition 2.3, where π is the projection and j the canonical injection. The inclusion map $\iota:\boldsymbol{C}\to\bar{\mathcal{O}}$ induces the homomorphism

$$\iota_* : H^*(M; \boldsymbol{C}) \longrightarrow H^*(M; \bar{\mathcal{O}}) .$$

Then we have

$$\operatorname{Ker} \lambda_* = \operatorname{Ker} \iota_*$$

([6]). Let E be a smooth complex vector bundle over M. A differential representation φ of $\mathcal{A}_{\partial}(M)$ on $\Gamma(E)$ is called a partially holomorphic representation if it satisfies the condition (3.6). Then from the above results we can deduce the following theorem.

THEOREM 3.9. *If on $\Gamma(E)$ there is a partially holomorphic representation of connection type of $\mathcal{A}_{\partial}(M)$, then we have*

$$\iota_*(\text{Chern}(E)) = 0 .$$

Note that if E is a holomorphic vector bundle, then it is well-known that there is a connection of type (1, 0) on E whose curvature consists of (1, 1)-forms. From this it follows that there is a partially holomorphic representation of connection type of $\mathcal{A}_{\partial}(M)$ on any holomorphic vector bundle.

Finally, we add a remark on foliations. Let M be a smooth manifold. Assume that we have a splitting of the tangent bundle $\tau = F \oplus F_1$ such that both $\Gamma(F)$ and $\Gamma(F_1)$ give foliations of M. This is in some aspect analogous to the situation that we meet in the complex analytic case where we have the splitting $\tau \otimes \boldsymbol{C} = T \oplus \bar{T}$. In fact, it turns out that a theorem corresponding to Theorem 3.7 holds for such foliations. Of course, "holomorphic" will be here replaced by the notion of the vanishing of the partial derivatives in the normal direction to F. Once this convention is granted, the arguments proceed in parallel, and a similar theorem will be established. However, we do not further enter into details.

Added in proof: The statement (3.1) which is asserted in [3] should be replaced by the following one, which is proved essentially in [7]:

$$\operatorname{Ker} \lambda_* = \text{PONT}(M) , \tag{3.1$'$}$$

where PONT (M) denotes the ideal of $H^*(M, \boldsymbol{R})$ of the classes which are subordinate to the classes $(p_1, \cdots, p_{[n/4]})$ in the sense of [8]. Here p_i is the i-th Pontrjagin class of M.

Also the ideal CHERN (M) must be defined as the one consisting of the classes subordinate to $(c_1, \cdots, c_n)$, where c_i is the i-th Chern class of M.

Accordingly all the symbols PONT (M), CHERN (M) in the paper must be understood in the above sense.

[8] R. Thom, Operations en cohomologie réelle, Séminaire H. Cartan, E.N.S., 1954/55.

REFERENCES

[1] R. BOTT, Lectures on characteristic classes and foliations, Lecture Notes in Mathematics **279**, Springer, 1972.

[2] S. S. CHERN, Complex manifolds without potential theory, Van Nostrand Mathematical Studies **15**, Princeton, 1967.

[3] M. V. LOSIK, On the cohomologies of infinite-dimensional Lie algebras of vector fields, Functional Analysis and their Applications **4** (1970), 127-135.

[4] R. NARASHIMHAN, Analysis on real and complex manifolds, Masson & Cie, Paris, 1968.

[5] K. SHIGA, Cohomology of Lie algebras over a manifold; I, II, Jour. Math. Soc. Japan **26** (1974), 324-361, 587-607.

[6] T. TSUJISHITA, Cohomology of the Lie algebra of vector fields on a complex manifold, Inventiones Math. **34** (1976), 1-18.

[7] F. W. KAMBER AND P. TONDEUR, Characteristic invariants of foliated bundles, Manuscripta Math **11** (1974), 51-89.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MATHEMATICS,
OSAKA UNIVERSITY.