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ON r-FOLD (4, 2)- AND f-PRODUCTS

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In this paper we treat a generalization of the notion of an f-structure [8] and a $(4, \pm 2)$-structure [9] on a manifold. We obtain the so-called f- or (4, 2)-products (or structures) with factor C^2 and this is done in the same way as A. GRAY and R. B. BROWN have done [2] to obtain the r-fold vector cross products (or structures) starting from an almost complex structure.

The greatest part of this paper is devoted to the algebraic viewpoint and gives algebraic properties. In the second part we give some details concerning such structures on a manifold, we obtain certain theorems in relation with induced structures on submanifolds and relations with the curvature. Finally we consider a certain generalization of the Nijenhuis tensor of a structure.

1. In [8] K. YANO has considered a structure on a manifold, called an f-structure and defined as follows:

DEFINITION 1. Let M^n be a differentiable manifold of class C^∞ and $f\neq 0$ a tensor field of type (1, 1) and of class C^∞. f defines an *f-structure* if it satisfies

$$f^3+f=0 \tag{1}$$

and is of constant rank.

K. YANO, C. HOUH and B. CHEN treated in [9] another structure containing the *f-structure* as a special case.

DEFINITION 2. Let M^n be an n-dimensional differentiable manifold of class C^∞ and let there be given a tensor field $\phi\neq 0$ of type (1, 1) and of class C^∞. ϕ is a $(4, \pm 2)$-structure on M^n if $n=2m$ and if ϕ is such that

$$\phi^4\pm\phi^2=0\,, \qquad \operatorname{rank}\phi=\frac{1}{2}\operatorname{rank}\phi^2+m\,. \tag{2}$$

It is easy to see that an almost complex structure on M^n is a special case of an f-structure. These almost complex structures have been generalized by A. GRAY and R. B. BROWN [2], [5]. They considered the so-called vector cross product structures defined as follows:

DEFINITION 3. Let V denote an n-dimensional vector space over an arbitrary field of characteristic not two and let $\langle\, ,\rangle : V\times V\rightarrow F$ denote a symmetric

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nondegenerate bilinear form. An *r-fold vector cross·product* on V is a multilinear map

$$X: V^r \to V: (a_1, a_2, \cdots, a_r) \mapsto X(a_1, a_2, \cdots, a_r), \qquad 1 \leqq r \leqq n,$$

such that

(3) (i) $\langle X(a_1, a_2, \cdots, a_r), a_i \rangle = 0, \qquad \forall i \in \{1, 2, \cdots, r\}$;

(4) (ii) $\langle X(a_1, a_2, \cdots, a_r), X(a_1, a_2, \cdots, a_r) \rangle = \det(\langle a_i, a_j \rangle)$,

$$i, j \in \{1, 2, \cdots, r\}.$$

We mention that B. ECKMANN [3] and G. WHITEHEAD [7] have considered the vector cross products from the topological standpoint taking a continuous instead of a multilinear map.

As we have done in [6] (see also [1]), such a vector cross product may be generalized in the following sense.

DEFINITION 4. Let V denote an n-dimensional vector space over an arbitrary field of characteristic not two and let $\langle , \rangle : V \times V \to F$ denote a symmetric nondegenerate bilinear form. An *r-fold vector π-product with factor C^2* on V is a multilinear map

$$X: V^r \to V: (a_1, a_2, \cdots, a_r) \mapsto X(a_1, a_2, \cdots, a_r),$$

$1 \leqq r \leqq n$, such that

(5) (i) $\langle X(a_1, a_2, \cdots, a_r), a_i \rangle = 0, \qquad \forall i \in \{1, 2, \cdots, r\}$;

(6) (ii) $\langle X(a_1, a_2, \cdots, a_r), X(a_1, a_2, \cdots, a_r) \rangle = C^2 \det(\langle a_i, a_j \rangle)$,

$$i, j \in \{1, 2, \cdots, r\} \text{ and } C^2 \neq 0 \in F.$$

The case $r=1$, $C^2=-1$ gives the almost product structures.

It is interesting for further considerations that we have shown [6] that this definition is equivalent with the following.

DEFINITION 5. Let V denote an n-dimensional vector space over an arbitrary field of characteristic not two and let $\langle , \rangle : V \times V \to F$ denote a symmetric nondegenerate bilinear form. An *r-fold vector π-product with factor C^2* on V is a multilinear map

$$X: V^r \to V: (a_1, a_2, \cdots, a_r) \mapsto X(a_1, a_2, \cdots, a_r),$$

$1 \leqq r \leqq n$, such that

(7) (i) $\langle X(a_1, \cdots, a_i, \cdots, a_r), x \rangle + \langle X(a_1, \cdots, x, \cdots, a_r), a_i \rangle = 0$,

$$\forall i \in \{1, 2, \cdots, r\} \text{ and } \forall x \in V;$$

(ii) $X(X(a_1, a_2, \cdots, a_r), a_2, \cdots, a_r)=$

(8)
$$-C^2\begin{vmatrix} a_1 & a_2 & \cdots & a_r \\ \langle a_2, a_1\rangle & \langle a_2\, a_2\rangle & \cdots & \langle a_2, a_r\rangle \\ \vdots & \vdots & \cdots & \vdots \\ \langle a_r, a_1\rangle & \langle a_r, a_2\rangle & \cdots & \langle a_r, a_r\rangle \end{vmatrix}$$

with $C^2\neq 0\in F$.

2. The purpose of this paper is to generalize the $(4, \pm 2)$-structures, and in a special case the f-structures, in the same way as is done for an almost complex or product structure.

To do this we start with the following definition.

DEFINITION 6. Let V denote a vector space of dimension n over an arbitrary field F of characteristic not two and $\langle, \rangle: V\times V\rightarrow F$ a symmetric, nondegenerate bilinear from. An *r-fold* (4, 2)*-product with factor* C^2 on V is a multilinear map

$$X: V^r\rightarrow V: (a_1, a_2, \cdots, a_r)\mapsto X(a_1, a_2, \cdots, a_r),$$

$1\leqq r\leqq n$ such that

(9) (i) $\langle X(a_1, a_2, \cdots, a_r), a_i\rangle=0, \quad \forall i\in\{1, 2, \cdots, r\};$

(10) (ii)
$$\langle X(X(a_1, a_2, \cdots, a_r), a_2, \cdots, a_r), X(X(a_1, a_2, \cdots, a_r), a_2, \cdots, a_r)\rangle$$
$$-C^2 \det(\langle a_{i'}, a_{j'}\rangle)\langle X(a_1, a_2, \cdots, a_r), X(a_1, a_2, \cdots, a_r)\rangle=0,$$
$$i', j'\in\{2, 3, \cdots, r\} \text{ and } C^2\neq 0\in F.$$

Sometimes it is easier to replace (10) by another expression. This can be done by the following theorem.

THEOREM 1. *Condition* (*ii*) *in the definition of an r-fold* (4, 2)*-product with factor* C^2 *can be replaced by*

(11)
$$X^4\{a_1(a_2, \cdots, a_r)^4\}+C^2 \det(\langle a_{i'}, a_{j'}\rangle)X^2\{a_1(a_2, \cdots, a_r)^2\}=0$$

i.e.

(11′)
$$X(X(X(X(a_1, a_2, \cdots, a_r), a_2, \cdots, a_r), a_2, \cdots, a_r), a_2, \cdots, a_r)$$
$$+C^2 \det(\langle a_{i'}, a_{j'}\rangle)X(X(a_1, a_2, \cdots, a_r), a_2, \cdots, a_r)=0.$$

Proof.

I. Suppose (9) and (10) are given. Linearizing (9) we obtain for $\forall b\in V$

(12)
$$\langle X(a_1, a_2, \cdots, a_r), b\rangle+\langle X(b, a_2, \cdots, a_r), a_1\rangle=0.$$

Doing this also for (10) we get

$$(13)\qquad \langle X^2\{a_1(a_2, \cdots, a_r)^2\}, X^2\{b(a_2, \cdots, a_r)^2\}\rangle -C^2 \det(\langle a_{i'}, a_{j'}\rangle)\langle X(a_1, a_2, \cdots, a_r), X(b, a_2, \cdots, a_r)\rangle=0\,.$$

Employing (12) in (13) we obtain

$$(14)\qquad \langle X^2\{a_1(a_2, \cdots, a_r)^2\}, X^2\{b(a_2, \cdots, a_r)^2\}\rangle +C^2 \det(\langle a_{i'}, a_{j'}\rangle)\langle X^2\{a_1(a_2, \cdots, a_r)^2\}, b\rangle=0\,.$$

Substituting b by $X(b, a_2, \cdots, a_r)$ in (12), it becomes

$$(15)\qquad \langle X(a_1, a_2, \cdots, a_r), X(b, a_2, \cdots, a_r)\rangle+\langle X^2\{b(a_2, \cdots, a_r)^2\}, a_1\rangle=0$$

and substituting in this relation a_1 by $X^2\{a_1(a_2, \cdots, a_r)^2\}$ we get

$$(16)\qquad \langle X^3\{(a_1(a_2, \cdots, a_r)^3\}, X(b, a_2, \cdots, a_r)\rangle +\langle X^2\{b(a_2, \cdots, a_r)^2\}, X^2\{a_1(a_2, \cdots, a_r)^2\}\rangle=0\,.$$

(14) and (16) give together

$$(17)\qquad \langle X^3\{a_1(a_2, \cdots, a_r)^3\}, X(b, a_2, \cdots, a_r)\rangle -C^2 \det(\langle a_{i'}, a_{j'}\rangle)\langle X^2\{a_1(a_2, \cdots, a_r)^2\}, b\rangle=0$$

and with the help of (12) we obtain finally

$$\langle X^4\{a_1(a_2, \cdots, a_r)^4\}+C^2 \det(\langle a_{i'}, a_{j'}\rangle)X^2\{a_1(a_2, \cdots, a_r)^2\}, b\rangle=0\,.$$

$\langle, \rangle$ being nondegenerate we may conclude that (11) is proved.

II. Suppose now that (9) and (11) are given. Then (12) is still valid and substituting in this relation a_1 by $X(a_1, a_2, \cdots, a_r)$ and b by $X^2\{a_1(a_2, \cdots, a_r)^2\}$ we get

$$(18)\qquad \langle X^4\{a_1(a_2, \cdots, a_r)^4\}, a_1\rangle-\langle X^2\{a_1(a_2, \cdots, a_r)^2\}, X^2\{a_1(a_2, \cdots, a_r)^2\}=0\,.$$

On the other side, we have with the help of (12)

$$(19)\qquad \langle X^2\{a_1(a_2, \cdots, a_r)^2\}, a_1\rangle+\langle X(a_1, a_2, \cdots, a_r), X(a_1, a_2, \cdots, a_r)\rangle=0\,.$$

Multiplying (19) by $C^2 \det(\langle a_{i'}, a_{j'}\rangle)$ and adding by (18), we get with (11)

$$\langle X^2\{a_1(a_2, \cdots, a_r)^2, X^2\{a_1(a_2, \cdots, a_r)^2\} -C^2 \det(\langle a_{i'}, a_{j'}\rangle)\langle X(a_1, a_2, \cdots, a_r), X(a_1, a_2, \cdots, a_r)\rangle=0$$

which is the required relation (10).

3. We prove now some other interesting properties of an r-fold (4, 2)-product.

THEOREM 2. *An r-fold* (4, 2)*-product is an antisymmetric multilinear map.*

Proof. With the help of (12) we have for $\forall z$

$$\begin{aligned}\langle X(a_1, \cdots, x, \cdots, y, \cdots, a_r), z\rangle &= -\langle X(a_1, \cdots, z, \cdots, y, \cdots, a_r), x\rangle \\ &= \langle X(a_1, \cdots, z, \cdots, x, \cdots, a_r), y\rangle \\ &= -\langle X(a_1, \cdots, y, \cdots, x, \cdots, a_r), z\rangle .\end{aligned}$$

The required result follows from the fact that $\langle , \rangle$ is nondegenerate.

THEOREM 3. *A multilinear map such that*

$$\langle X(a_1, a_2, \cdots, a_r), a_i\rangle = 0, \qquad \forall i \in \{1, 2, \cdots, r\}$$

satisfies also

$$\langle X^{2p+1}\{a_1(a_2, \cdots, a_r)^{2p+1}\}, a_1\rangle = 0. \tag{20}$$

Proof. In this case (12) is valid and substituting a_1 by $X^2\{a_1(a_2, \cdots, a_r)^2\}$ we get

$$\langle X^3\{a_1(a_2, \cdots, a_r)^3\}, b\rangle + \langle X(b, a_2, \cdots, a_r), X^2\{a_1(a_2, \cdots, a_r)^2\}\rangle = 0.$$

Taking $b = a_1$ we arrive at

$$\langle X^3\{a_1(a_2, \cdots, a_r)^3\}, a_1\rangle = 0. \tag{21}$$

Linearizing (21) we get

$$\langle X^3\{a_1(a_2, \cdots, a_r)^3\}, b\rangle + \langle X^3\{b(a_2, \cdots, a_r)^3\}, a_r\rangle = 0. \tag{22}$$

In the same manner as for (12), (22) gives rise to

$$\langle X^5\{a_1(a_2, \cdots, a_r)^5\}, a_1\rangle = 0. \tag{23}$$

Continuing this we arrive finally at the desired result.

We prove now that the construction given above is indeed a generalization of the r-fold vector π-product.

THEOREM 4. *An r-fold vector π-product with factor C^2 is an r-fold* (4, 2)-*product with the same factor.*

Proof. It follows from (8) that

$$X^3\{a_1(a_2, \cdots, a_r)^3\} + C^2 \det(\langle a_{i'}, a_{j'}\rangle) X(a_1, a_2, \cdots, a_r) = 0 \tag{24}$$

and so

$$X^4\{a_1(a_2, \cdots, a_r)^4\} + C^2 \det(\langle a_{i'}, a_{j'}\rangle) X^2\{a_1(a_2, \cdots, a_r)^2\} = 0.$$

4. In this section we shall give a relationship of an r-fold (4, 2)-product with the f-structures and therefore we mention that (24) is an interesting relation.

First we consider the case $r=1$. A 1-fold (4, 2)-product with factor C^2 is thus a linear map such that

$$\forall a \in V \langle Xa, a\rangle = 0\,, \qquad X^4 + C^2X^2 = 0\,. \tag{25}$$

This is equivalent with

$$\forall a \in V \langle Xa, a\rangle = 0\,, \qquad \langle X^2a, X^2a\rangle - C^2\langle Xa, Xa\rangle = 0\,. \tag{26}$$

We have

THEOREM 5. *A 1-fold (4, 2)-product with factor C^2 is a linear map satisfying*

$$\begin{aligned} &\text{(i)} \quad \langle Xa, a\rangle = 0 \qquad \text{for } \forall a \in V\,; \\ &\text{(ii)} \quad X^3 + C^2X = 0 \end{aligned} \tag{27}$$

if $\langle, \rangle$ is positive definite. The converse is also true without restriction for $\langle, \rangle$.

Proof. The proof of the converse is trivial.

Starting from the second relation (26) and substituting a by $(X^2+C^2)a$ we get

$$\forall a \in V \langle Aa, Aa\rangle = 0 \tag{28}$$

where

$$A = X^3 + C^2X\,. \tag{29}$$

Thus we have

$$\forall a \in V\ Aa = 0$$

and then $A=0$.

Before proving an analogous result in the general case we give

DEFINITION 7. Let V denote a vector space of dimension n over an arbitrary field F of characteristic not two and $\langle, \rangle : V \times V \to F$ a symmetric, nondegenerate bilinear form. An *r-fold f-product with factor C^2* on V is a multilinear map

$$X : V^r \to V : (a_1, a_2, \cdots, a_r) \mapsto X(a_1, a_2, \cdots, a_r)\,,$$

$1 \leqq r \leqq n$, such that

$$\text{(i)} \quad \langle X(a_1, a_2, \cdots, a_r), a_i\rangle = 0\,, \qquad \forall i \in \{1, 2, \cdots, r\}\,; \tag{30}$$

$$\text{(ii)} \quad X^3\{a_1(a_2, \cdots, a_r)^3\} + C^2 \det(\langle a_{i'}, a_{j'}\rangle) X(a_1, a_2, \cdots, a_r) = 0\,, \tag{31}$$

$C^2 \neq 0 \in F$ and $i', j' \in \{2, 3, \cdots, r\}$.

Now we can prove

THEOREM 6. *An r-fold (4, 2)-product with factor C^2 is an r-fold f-product with the same factor if $\langle, \rangle$ is positive definite. The converse is true in the general case.*

Proof. The proof of the converse is trivial.

Substitute now a_1 by

$$X^2\{a_1(a_2, \cdots, a_r)^2\} + C^2 \det(\langle a_{i'}, a_{j'}\rangle)a_1 \tag{32}$$

in (10). We obtain

$$\begin{aligned}&\langle X^3\{a_1(a_2, \cdots, a_r)^3\} + C^2 \det(\langle a_{i'}, a_{j'}\rangle)X(a_1, a_2, \cdots, a_r),\\ &\quad X^3\{a_1(a_2, \cdots, a_r)^3\} + C^2 \det(\langle a_{i'}, a_{j'}\rangle)X(a_1, a_2, \cdots, a_r)\rangle = 0\end{aligned} \tag{32'}$$

and this gives the required result for the positive definite case.

Remark. We return to the case $r=1$ and remark that we have

$$\forall a\, \langle Aa, a\rangle = 0, \qquad A^2 = 0. \tag{33}$$

In the positive definite case this has $A=0$ as a consequence but a simple counterexample shows that this is not always true in the other cases. Indeed, taking

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ resp. } \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

as matrix for A resp. $\langle , \rangle$, a simple calculation shows that (33) is satisfied.

5. We return now to the general case.

THEOREM 7. *For an r-fold* (4, 2)*-product condition* (11) *may be replaced by*

$$X^2\{a_1(a_2, \cdots, a_r)^2\} = C^2 \det(\langle a_{i'}, a_{j'}\rangle)\alpha - C^2 \begin{vmatrix} a_1 & a_2 & \cdots & a_r \\ \langle a_2, a_1\rangle & \langle a_2\, a_2\rangle & \cdots & \langle a_2, a_r\rangle \\ \vdots & \vdots & \cdots & \vdots \\ \langle a_r, a_1\rangle & \langle a_r, a_2\rangle & \cdots & \langle a_r, a_r\rangle \end{vmatrix} \tag{34}$$

where

$$X^2\{\alpha(a_2, \cdots, a_r)^2\} = 0. \tag{35}$$

Proof. Indeed, we have from (34)

$$\begin{aligned}X^4\{a_1(a_2, \cdots, a_r)^4\} &= C^2 \det(\langle a_{i'}, a_{j'}\rangle)X^2\{\alpha(a_2, \cdots, a_r)^2\}\\ &\quad - C^2 \det(\langle a_{i'}, a_{j'}\rangle)X^2\{a_1(a_2, \cdots, a_r)^2\}.\end{aligned} \tag{36}$$

Remark. Note that

$$\det(\langle a_{i'}, a_{j'}\rangle) = \langle \hat{a}_1 \wedge a_2 \wedge \cdots \wedge a_r, \hat{a}_1 \wedge a_2 \wedge \cdots \wedge a_r\rangle.$$

THEOREM 8. *For an r-fold f-product condition* (31) *may be replaced by* (34) *where*

$$X(\alpha, a_2, \cdots, a_r)=0\,. \tag{37}$$

From (34) we obtain further

$$\langle X(a_1, a_2, \cdots, a_r), \alpha\rangle=0 \tag{38}$$

and consequently

$$\langle X(\alpha, a_2, \cdots, a_r), a_1\rangle=0\,. \tag{39}$$

It follows also immediately

$$\langle X(\alpha, a_2, \cdots, a_r), a_{i'}\rangle=0\,. \tag{40}$$

The same relation gives

$$\langle \alpha, a_{i'}\rangle=0 \tag{41}$$

and

$$\begin{aligned} C^2 \det(\langle a_{i'}, a_{j'}\rangle)\langle a_1, \alpha\rangle &= C^2 \det(\langle a_\imath, a_\jmath\rangle) \\ &\quad -\langle X(a_1, a_2, \cdots, a_r), X(a_1, a_2, \cdots, a_r)\rangle\,. \end{aligned} \tag{42}$$

Using (10) we obtain further

$$\langle X(a_1, a_2, \cdots, a_r), X(\alpha, a_2, \cdots, a_r)\rangle=0 \tag{43}$$

and thus

$$\langle X^2\{a_1(a_2, \cdots, a_r)^2\}, \alpha\rangle=0\,. \tag{43'}$$

This relation gives

$$\det(\langle a_{i'}, a_{j'}\rangle)\langle \alpha, \alpha\rangle-\left\langle \alpha, \begin{vmatrix} a_1 & a_2 & \cdots & a_r \\ \langle a_2, a_1\rangle & \langle a_2, a_2\rangle & \cdots & \langle a_2, a_r\rangle \\ \vdots & \vdots & \cdots & \vdots \\ \langle a_r, a_1\rangle & \langle a_r, a_2\rangle & \cdots & \langle a_r, a_r\rangle \end{vmatrix}\right\rangle=0\,. \tag{44}$$

Finally we get

$$\langle \alpha, \alpha\rangle=\langle \alpha, a_1\rangle\,. \tag{45}$$

THEOREM 9. *If X defines an r-fold* (4, 2)*-product with factor C^2 then*

$$Z(a_1, a_2, \cdots, a_r)=\frac{1}{C^2 \det(\langle a_{i'}, a_{j'}\rangle)}X^3\{a_1(a_2, \cdots, a_r)^3\} \tag{46}$$

defines also an r-fold (4, 2)*-product with the same factor.*

Proof. Using (34) and (35) we obtain

$$Z(a_1, a_2, \cdots a_r) = -X(a_1, a_2, \cdots, a_r) + X(\alpha, a_2, \cdots, a_r) \tag{47}$$

and this shows that Z is a multilinear map since α depends linearly from a_1. An easy calculation shows further

$$Z^4\{a_1(a_2, \cdots, a_r)^4\} + C^2 \det(\langle a_{i'}, a_{j'}\rangle) Z^2\{a_1(a_2, \cdots, a_r)^2\} = 0\,. \tag{48}$$

Finally

$$\langle Z(a_1, a_2, \cdots, a_r), a_i\rangle = \frac{1}{C^2 \det(\langle a_{i'}, a_{j'}\rangle)} \langle X^3\{a_1(a_2, \cdots, a_r)^3\}, a_i\rangle = 0 \tag{49}$$

because of Theorem 3 and the simple remark

$$\langle X^p\{a_1(a_2, \cdots, a_r)^p\}, a_{i'}\rangle = 0\,, \qquad \forall i' \in \{2, 3, \cdots, r\}\,. \tag{50}$$

COROLLARY.
(a)

$$\frac{1}{\det(\langle a_{i'}, a_{j'}\rangle)} X^3\{a_1(a_2, \cdots, a_r)^3\} + \frac{1}{\det(\langle a_{i''}, a_{j''}\rangle)} X^3\{a_2(a_1, a_3, \cdots, a_r)^3\} = 0\,, \tag{51}$$
$$i', j' \in \{2, 3, \cdots, r\}\,, \qquad i'', j'' \in \{1, 3, \cdots, r\}\,.$$

(b)

$$X^3\{a_1(a_2, \cdots, a_t, \cdots, a_s, \cdots, a_r)^3\} + X^3\{a_1(a_2, \cdots, a_s, \cdots, a_t, \cdots, a_r)^3\} = 0\,. \tag{52}$$

This follows immediately from (49) in the same way as for Theorem 2.

Before formulating the following theorem we generalize the notion of an almost tangent structure.

DEFINITION 8. Let V denote a vector space of dimension n over an arbitrary field F of characteristic not two and let $\langle , \rangle : V \times V \to F$ denote a symmetric, nondegenerate bilinear form. An *r-fold tangent product* on V is a multilinear map

$$X : V^r \to V : (a_1, a_2, \cdots, a_r) \mapsto X(a_1, a_2, \cdots, a_r)\,,$$

$1 \leqq r \leqq n$, such that

$$\begin{aligned} &\text{(i)} \quad \langle X(a_1, a_2, \cdots, a_r), a_i\rangle = 0\,, \qquad \forall i \in \{1, 2, \cdots, r\}\,; \\ &\text{(ii)} \quad X^2\{a_1(a_2, \cdots, a_r)^2\} = 0\,. \end{aligned} \tag{53}$$

In relation with the defined r-fold products we have now

THEOREM 10. *An r-fold* (4, 2)*-product X defines always an r-fold tangent product Y by*

$$Y(a_1, a_2, \cdots, a_r) = \frac{1}{C^2 \det(\langle a_{i'}, a_{j'}\rangle)} X^3\{a_1(a_2, \cdots, a_r)^3\} + X(a_1, a_2, \cdots, a_r)\,. \tag{54}$$

Proof. We have, with the help of (49) and (50)

$$\langle Y(a_1, a_2, \cdots, a_r), a_i\rangle = 0, \qquad \forall i \in \{1, 2, \cdots, r\}.$$

As a consequence of (46) and (47) we get

$$Y(a_1, a_2, \cdots, a_r) = X(\alpha, a_2, \cdots, a_r)$$

which shows that Y is a multilinear map. A straightforward calculation shows finally

$$Y^2\{a_1(a_2, \cdots, a_r)^2\} = 0.$$

6. For completeness we remark that Theorem 2 remains valid for an r-fold f-product and that we have also

THEOREM 11. *An r-fold vector π-product with factor C^2 is an r-fold f-product with the same factor.*

This is proved by means of (24).

7. Let us now suppose that M^n is an n-dimensional differentiable manifold (i.e. C^∞) equipped with a pseudo-Riemannian metric $\langle , \rangle$ and let $\chi(M^n)$ denote the Lie algebra of vector fields on M^n. Suppose further that M^n has a globally defined r-fold vector π-, (4, 2)- or f-product X which is differentiable (i.e. C^∞). Then X is a tensor field on M^n of type $(1, r)$ and we say that X defines an r-fold vector π-, (4, 2)- or f-structure on M^n.

From the definitions and theorems given above it is evident that every such a structure X determines global differential forms ϕ and ψ of degree $r+1$ by the formulas

$$\phi(a_1, a_2, \cdots, a_{r+1}) = \langle X(a_1, a_2, \cdots, a_r), a_{r+1}\rangle, \tag{55}$$

$$\begin{aligned}\psi(a_1, a_2, \cdots, a_{r+1}) &= \langle Z(a_1, a_2, \cdots, a_r), a_{r+1}\rangle \\ &= \frac{1}{C^2 \det(\langle a_{i'}, a_{j'}\rangle)}\langle X^3\{a_1(a_2, \cdots, a_r)^3\}, a_{r+1}\rangle\end{aligned} \tag{56}$$

for $a_1, a_2, \cdots, a_{r+1} \in \chi(M^n)$.

If is easy to prove the following theorem:

THEOREM 12.

1. *X defines an r-fold f-structure with factor C^2 if and only if*

$$\phi + \psi = 0. \tag{57}$$

2. *X defines an r-fold (4, 2)-structure with factor C^2 if ond only if*

$$\psi(X(a_1, a_2, \cdots, a_r), a_2, \cdots, a_{r+1}) + \phi(X(a_1, a_2, \cdots, a_r), a_2, \cdots, a_{r+1}) = 0 \tag{58}$$

for $a_1, a_2, \cdots, a_{r+1} \in \chi(M^n)$.

3. *X defines an r-fold vector π-structure with factor* C^2 *if and only if*

$$\phi(X(a_1, a_2, \cdots, a_r), a_2, \cdots, a_{r+1}) = -C^2 \begin{vmatrix} \langle a_1, a_{r+1}\rangle & \langle a_2, a_{r+1}\rangle & \cdots & \langle a_r, a_{r+1}\rangle \\ \langle a_2, a_1\rangle & \langle a_2, a_2\rangle & \cdots & \langle a_2, a_r\rangle \\ \vdots & \vdots & \cdots & \vdots \\ \langle a_r, a_1\rangle & \langle a_r, a_2\rangle & \cdots & \langle a_r, a_r\rangle \end{vmatrix} \tag{59}$$

for $a_1, a_2, \cdots, a_{r+1} \in \mathfrak{X}(M^n)$.

4. *X defines an r-fold f-structure if and only if*

$$\phi(\alpha, a_2, \cdots, a_{r+1}) = 0 \tag{60}$$

for $\forall a_{r+1} \in \mathfrak{X}(M^n)$.

5. *X defines an r-fold* (4, 2)*-structure if and only if*

$$\phi(X(\alpha, a_2, \cdots, a_r), a_2, \cdots, a_{r+1}) = 0 \tag{61}$$

for $\forall a_{r+1} \in \mathfrak{X}(M^n)$.

6. *For* $\forall a_1, a_2, \cdots, a_{r+1} \in \mathfrak{X}(M^n)$ *we have*

$$\text{a)}\quad \psi(a_1, a_2, \cdots, a_{r+1}) = \phi(\alpha - a_1, a_2, \cdots, a_{r+1})\,; \tag{62}$$

$$\text{b)}\quad \psi(a_1, a_2, \cdots, a_{r+1}) \tag{63}$$

$$= \frac{-1}{C^2 \det(\langle a_{i'}, a_{j'}\rangle)} \phi(X(a_1, a_2, \cdots, a_r), a_2, \cdots, a_r, X(a_{r+1}, a_2, \cdots, a_r))\,.$$

8. Let ∇ denote a linear connection on M^n. Then we have

$$(\nabla_x\phi)(a_1, a_2, \cdots, a_{r+1}) = x\{\phi(a_1, a_2, \cdots, a_{r+1})\} - \sum_{i=1}^{r+1} \phi(a_1, \cdots, \nabla_x a_i, \cdots, a_{r+1})\,,$$

and an easy calculation shows that

$$(\nabla_x\phi)(a_1, a_2, \cdots, a_{r+1}) = x\{\langle X(a_1, a_2, \cdots, a_r), a_{r+1}\rangle\} \tag{64}$$
$$-\langle \nabla_x X(a_1, a_2, \cdots, a_r), a_{r+1}\rangle - \langle X(a_1, a_2, \cdots, a_r), \nabla_x a_{r+1}\rangle\,.$$

Doing the same for the $(r+1)$-form ψ we obtain, using (47),

$$(\nabla_x\psi)(a_1, a_2, \cdots, a_{r+1}) = x\{\langle X(\alpha - a_1, a_2, \cdots, a_r), a_{r+1}\rangle\} \tag{65}$$
$$-\langle \nabla_x X(\alpha - a_1, a_2, \cdots, a_r), a_{r+1}\rangle - \langle X(\alpha - a_1, a_2, \cdots, a_r), \nabla_x a_{r+1}\rangle$$
$$+\langle (\nabla_x X)(\alpha - a_1, a_2, \cdots, a_r), a_{r+1}\rangle\,.$$

We may conclude:

THEOREM 13. *If* ∇ *is a linear connection on* M^n *which is also metric, then we have*

$$(\nabla_x\phi)(a_1, a_2, \cdots, a_{r+1}) = \langle (\nabla_x X)(a_1, a_2, \cdots, a_r), a_{r+1}\rangle\,, \tag{66}$$

$$(67)\qquad (\nabla_x\phi)(a_1, a_2, \cdots, a_{r+1})=\langle(\nabla_x X)(\alpha-a_1, a_2, \cdots, a_r), a_{r+1}\rangle, \\ =(\nabla_x\phi)(\alpha-a_1, a_2, \cdots, a_{r+1}),$$

for $x, a_1, a_2, \cdots, a_{r+1}\in\mathfrak{X}(M^n)$.

DEFINITIONS. If ∇ is a linear metric connection on M^n, X an r fold vector π-, (4, 2)- or f-product and ϕ the associated $(r+1)$-form, then
(i) X is *parallel* if and only if for $\forall x\in\mathfrak{X}(M^n)$

$$(68)\qquad \nabla_x X=0;$$

(ii) X is *nearly parallel* if and only if

$$(69)\qquad (\nabla_{a_1}X)(a_1, a_2, \cdots, a_r)=0$$

for $\forall a_1, a_2, \cdots, a_r\in\mathfrak{X}(M^n)$;
(iii) X is *almost parallel* if and only if $d\phi=0$;
(iv) X is *semiparallel* if and only if $\delta\phi=0$

We remark that it follows from (69) that

$$(70)\qquad (\nabla_x X)(a_1, a_2, \cdots, a_r)+(\nabla_{a_1}X)(x, a_2, \cdots, a_r)=0.$$

Let $\mathcal{P}$, $\mathcal{NP}$, $\mathcal{AP}$, $\mathcal{SP}$ be the classes of r-fold vector π-, (4, 2)- or f-products which are parallel, nearly parallel, almost parallel or semiparallel. We have then if ∇ denote the pseudo-Riemannian connection on M^n:

THEOREM 14. *We have the following inclusions*
(i) $\mathcal{P}\subseteqq\mathcal{NP}\subseteqq\mathcal{SP}$;
(ii) $\mathcal{P}\subseteqq\mathcal{AP}$;
(iii) $\mathcal{P}=\mathcal{NP}\cap\mathcal{AP}$,
for M^n.

Proof. We have

$$(71)\qquad (d\phi)(a_1, a_2, \cdots, a_{r+2})=\sum_{i=1}^{r+2}(-1)^{i+1}(\nabla_{a_i}\phi)(a_1, \cdots, \hat{a}_i, \cdots, a_{r+2}),$$

$$(71')\qquad (\delta\phi)(a_1, \cdots, a_r)=-\sum_i \|ei\|^{-2}(\nabla_{e_i}\phi)(e_i, a_1, a_2, \cdots, a_r).$$

If X is nearly parallel then

$$(\nabla_{a_i}X)(a_1, a_2, \cdots, a_r)=0 \Rightarrow (\nabla_{a_1}\phi)(a_1, a_2, \cdots, a_{r+1})=0$$

and it follows

$$(\nabla_{a_i}\phi)(a_1, \cdots, \hat{a}_i, \cdots, a_{r+2})=(-1)^{i-1}(\nabla_{a_1}\phi)(a_2, \cdots, a_i, \cdots, \cdots, a_{r+2}).$$

Thus

$$(72)\qquad (d\phi)(a_1, a_2, \cdots, a_{r+2})=(r+2)(\nabla_{a_1}\phi)(a_2, \cdots, a_i, \cdots, a_{r+2}).$$

Now

(i) is trivial;
(ii) follows from (72);
(iii) by (72) we get

$$(\nabla_{a_1}\phi)(a_2, \cdots, a_i, \cdots, a_{r+2})=0$$

and this gives us $\forall x\ \nabla_x X=0$.

A simple calculation proves further

THEOREM 15.

$$d(\psi+\phi)(a_1, a_2, \cdots, a_{r+2})=d\phi(\alpha, a_2, \cdots, a_{r+2}), \tag{73}$$

$$\delta(\psi+\phi)(a_1, a_2, \cdots, a_r)=\delta\phi(\alpha, a_2, \cdots, a_r); \tag{73'}$$

Using (42) we obtain without difficulties

THEOREM 16.

$$2\langle(\nabla_x X)(a_1, a_2, \cdots, a_r), X(a_1, a_2, \cdots, a_r)\rangle+C^2x\{\det(\langle a_{i'}, a_{j'}\rangle)\langle a_1, \alpha\rangle\}=0. \tag{74}$$

COROLLARY (see [6]). *If X defines an r-fold vector π-product with factor C^2, then*

$$\langle(\nabla_x X)(a_1, a_2, \cdots, a_r), X(a_1, a_2, \cdots, a_r)\rangle=0 \tag{75}$$

where ∇ is a linear metric connection.

9. We give now two theorems for (4, 2)- and f-products and which are proved in [5] for r-fold vector cross products. (Remark that the proof in [5] is independent of the factor C^2). As in [5] we suppose that the r-fold product varies continuously over the whole manifold.

THEOREM 17. *Let $\bar{X}$ be an r-fold* (4, 2)*-product (resp. f-product) with factor C^2 with respect to a metric tensor $\langle,\rangle$ on a manifold M^n. Let M^{n-k} be a submanifold of M^n such that the restriction of $\langle,\rangle$ to the normal bundle (supposed to be orientable) of M^{n-k} is nondegenerate and positive definite. Then $\bar{X}$ induces an $(r-k)$-fold* (4, 2)*-product (resp. f-product) X with the same factor C^2 on M^{n-k} in a natural way.*

Proof. Let $n_1, n_2, \cdots, n_k$ be k normal vector fields defined on an open subset of M^{n-k} such that $\langle n_i, n_j\rangle=\delta_{ij}$, $i, j=2, 3, \cdots, k$, and $n_1\wedge n_2\wedge\cdots\wedge n_k$ is consistent with the orientation of the normal bundle. Define then

$$X(a_1, a_2, \cdots, a_{r-k})=\bar{X}(a_1, a_2, \cdots, a_{r-k}, n_1, n_2, \cdots, n_k) \tag{76}$$

for $a_1, a_2, \cdots, a_{r-k}\in\chi(M^{n-k})$. It is not difficult to verify conditions (9) and (11) (resp. (30) and (31)) defining the product. Note that (76) is independent of the choice of $n_1, n_2, \cdots, n_k$.

THEOREM 18. *Let S^n denote the unit sphere in R^{n+1} and let $\langle,\rangle$ denote the* *metric tensor of S^n induced from the positive definite one on R^{n+1}. If S^n has a globally defined r-fold* (4, 2)- *or f-product with factor C^2, then in the vector space sense there is an $(r+1)$-fold continuous* (4, 2)- *or f-product with the same factor on R^{n+1}.*

Rroof. We use the construction of [5].

Let X_m denote the r-fold product on S^n at m. Define now

$$X=(R^{n+1})^{r+1}\to R^{n+1}$$

as follows. Let $a_1, a_2, \cdots, a_{r+1}\in R^{n+1}$ and write $a_{r+1}=b+e$ where b is the component of a_{r+1} normal to $a_1, a_2, \cdots, a_r$. If $b=0$ we set $X(a_1, a_2, \cdots, a_{r+1})=0$. If $b\neq 0$, let $d=\|b\|^{-1}b$ and set

$$X(a_1, a_2, \cdots, a_{r+1})=\|b\|X_d(a_1, a_2, \cdots, a_r)\,.$$

Using the definition of the products one can verify at once that X satisfies the required relations.

Note that X is linear in $a_1, a_2, \cdots, a_r$ but in general is only continuous in a_{r+1}.

10. Let M^{n-k} be a submanifold of a pseudo-Riemannian manifold M^n such that the restriction of the metric tensor $\langle , \rangle$ of M^n to M^{n-k} is nondegenerate. Let $\bar{\chi}(M^{n-k})=\{X|M^{n-k}, X\in\chi(M^n)\}$. Then we may write $\bar{\chi}(M^{n-k})=\chi(M^{n-k})\oplus\chi(M^{n-k})^{\perp}$ where $\chi(^{n-k})^{\perp}$ is the collection of vector fields normal to M^{n-k}. The configuration tensor $T:\chi(M^{n-k})\times\bar{\chi}(M^{n-k})\to\bar{\chi}(M^{n-k})$ is defined by $T_xy=\bar{\nabla}_xy-\nabla_xy$ for $x, y\in\chi(M^{n-k})$ and $T_xz=\pi\bar{\nabla}_xz$ for $x\in\chi(M^{n-k})$, $z\in\chi(M^{n-k})^{\perp}$. Hence ∇ and $\bar{\nabla}$ are the Riemannian connections of M^{n-k} and M^n resp. and π is the projection of $\bar{\chi}(M^{n-k})$ onto $\chi(M^{n-k})$. Then [4] for each $x\in\chi(M^{n-k})$, T_x is a skew-symmetric linear operator with respect to $\langle , \rangle$ and $T_xy=T_yx$ for $x, y\in\chi(M^{n-k})$.

THEOREM 19. *Let M^{n-k} and M^n be pseudo-Riemannian manifolds which satisfy the hypotheses of Theorem* 17. *Then for $x, a_1, a_2, \cdots, a_{r-k}\in\chi(M^{n-k})$ we have*

$$(\nabla_xX)(a_1, a_2, \cdots, a_{r-k})=\pi(\bar{\nabla}_x\bar{X})(a_1, \cdots, a_{r-k}, n_1, \cdots, n_k)$$
$$+\sum_{i=1}^{k}\pi\bar{X}(a_1, \cdots, a_{r-k}, n_1, \cdots T_xn_i, \cdots, n_k) \tag{77}$$

where $n_1, n_2, \cdots, n_k$ are the same as in the proof of Theorem 17.

Note that T_xn_i may be replaced by $\bar{\nabla}_xn_i$ since this is orthogonal to n_i. (77) follows immediately from the definition.

COROLLARY. *If $\bar{X}$ is parallel and M^{n-k} totally geodesic in M^n, then X is parallel.*

THEOREM 20. *Let M^{n-k} and M^n be pseudo-Riemannian manifolds which* *satisfy the hypotheses of Theorem* 19. *If* $\bar{X}$ *is nearly parallel and* M^{n-k} *totally umbilical in* M^n, *then* X *is nearly parallel.*

Proof. M^{n-k} is totally umbilical if and only if there exists for every unit normal n a function $\kappa(n)\in\mathcal{D}(M^{n-k})$ (depending on n) such that

$$T_x n=\kappa(n)x$$

for all $x\in\mathfrak{X}(M^{n-k})$. Thus, using (77),

$$(78)\qquad (\nabla_x X)(a_1, a_2, \cdots, a_{r-k})=\pi(\bar{\nabla}_x\bar{X})(a_1, a_2, \cdots, a_{r-k}, n_1, n_2, \cdots, n_k) + \sum_{i=1}^{k}\kappa_i\bar{X}(a_1, a_2, \cdots, a_{r-k}, n_1, \cdots, x, \cdots, n_k).$$

The required result follows now at once from the definition.

11. As before let M^n be a pseudo-Riemannian manifold with metric tensor $\langle , \rangle$ and Riemannian connection ∇. X is a (4, 2)-, f- or vector π-product with factor C^2 and ϕ the associated $(r+1)$-form defined by (55). The purpose of this section is to see if the relations given by A. GRAY [5] are valid for the defined structures. They introduce the curvature and relate them with the structure.

If θ is a p-form on M^n and $a, b, a_1, \cdots, a_p\in\mathfrak{X}(M^n)$ we shall need the following formulas:

$$(79)\qquad (\nabla\theta)(a\,; a_1, \cdots, a_p)=(\nabla_a\theta)(a_1, \cdots, a_p) =a\{\theta(a_1, \cdots, a_p)\}-\sum_{i=1}^{p}\theta(a_1, \cdots, \nabla_a a_i, \cdots, a_p),$$

$$(80)\qquad (\nabla^2\theta)(a\,; b\,; a_1, \cdots, a_p)=(\nabla_a(\nabla\theta))(b\,; a_1, \cdots, a_p),$$

$$(81)\qquad (R_{ab}\theta)(a_1, \cdots, a_p)=-\sum_{i=1}^{p}\theta(a_1, \cdots, R_{ab}a_i, \cdots, \cdots, a_p),$$

$$(82)\qquad (\Delta\theta)(a_1, \cdots, a_p)=\sum_{i=1}^{p}\sum_{k=1}^{p}(-1)^{i+1}\|e_k\|^{-2}(R_{a_ie_k}\theta)(e_k\,; a_1, \cdots, \hat{a}_i, \cdots, a_p) -\sum_{k=1}^{n}\|e_k\|^{-2}(\nabla^2\theta)(e_k\,; e_k\,; a_1, \cdots, a_p),$$

where $\{e_1, \cdots, e_k\}$ is an orthogonal frame field on an open subset of M^n. $\nabla\theta$ and $\nabla^2\theta$ are the first and second covariant derivatives of θ, $R_{ab}=\nabla_{[a,b]}-[\nabla_a, \nabla_b]$ and $\Delta=d\delta+\delta d$ is the Laplacian.

THEOREM 21. *Let* $a, b, a_1, \cdots, a_r\in\mathfrak{X}(M^n)$; *then*

(83) $$(\nabla^2\phi)(a\,;\,b\,;\,a_1, \cdots, a_r, X(a_1, \cdots, a_r))$$
$$=-\langle(\nabla_a X)(a_1, \cdots, a_r), (\nabla_b X)(a_1, \cdots, a_r)\rangle$$
$$+\frac{C^2}{2}(\nabla_a b-ab)\{\det(\langle a_{i'}, a_{j'}\rangle)\langle a_1, \alpha\rangle\}+A(a, b)$$

where $A(a, b)$ *is given by*

(83′) $$A(a, b)=\frac{C^2}{2}b\sum_{i=1}^{r}\{\lambda(a_1, \cdots, a_i+\nabla_a a_i, \cdots, a_r)$$
$$-\lambda(a_1, \cdots, a_i, \cdots, a_r)$$
$$-\lambda(a_1, \cdots, \nabla_a a_i, \cdots, a_r)\}$$

and

(83″) $$\lambda(a_1, \cdots, a_r)=\det(\langle a_{i'}, a_{j'}\rangle)\langle a_1, \alpha\rangle\,.$$

Proof. This formula is a straightforward calculation from the definition (80) of $\nabla^2\phi$ where we use formula (74) of Theorem 16. Remark that with (42)

$$\frac{C^2}{2}(\nabla_a b-ab)\{\det(\langle a_{i'}, a_{j'}\rangle)\langle a_1, \alpha\rangle\}$$
$$=\frac{C^2}{2}(\nabla_a b-ab)\{\det(\langle a_i, a_j\rangle)-\langle X(a_1, \cdots, a_r), X(a_1, \cdots, a_r)\rangle\}\,.$$

COROLLARY 1. *Let* $a, b, a_1, \cdots, a_r\in\mathfrak{X}(M^n)$; *then*

(84) $$(\nabla^2\phi)(a\,;\,b\,;\,a_1, \cdots, a_r, X(a_1, \cdots, a_r))=(\nabla^2\phi)(b\,;\,a\,;\,a_1, \cdots, a_r, X(a_1, \cdots, a_r))$$

if A *is symmetric in* a *and* b.

COROLLARY 2. *Suppose now that we have a vector* π*-structure with factor* C^2 *and let* $\langle,\rangle$ *be positive definite. If* $\nabla^2\phi=0$, *it follows from* (83) *with* $a=b$ *that*

$$(\nabla_a X)(a_1, \cdots, a_r)=0$$

and hence with (66)

$$\nabla\phi=0\,.$$

We have thus.

If X *is an* r*-fold vector* π*-structure with factor* C^2 *with respect to a positive definite* $\langle,\rangle$ *and if* ϕ *is the associated* $(r+1)$*-form, then*

$$\nabla^2\phi=0\Rightarrow\nabla\phi=0$$

and X *is parallel.*

If R_{ab} denotes the curvature operator of $\langle,\rangle$, then the *Ricci-curvature* k is defined by

(85) $$k(a, b)=\sum_{i=1}^{n}\|e_i\|^{-2}\langle R_{ae_i}b, e_i\rangle$$

for $a, b\in\mathfrak{X}(M^n)$, where $\{e_1, \cdots, e_n\}$ is any orthogonal frame field on an open subset of M^n.

As in [5] we define now the *Chern form* γ of the (4, 2)-, f- or vector π-product X.

DEFINITION. The *Chern form* γ of the (4, 2)-, f- or vector π-product with factor C^2 is the $(r+1)$-fold differential form γ defined by

$$(86)\qquad (r+1)\gamma(a_1, a_2, \cdots, a_{r+1})$$
$$=\sum_{k=1}^{n}\sum_{i<j}(-1)^{r+i+j}\|e_k\|^{-2}\langle R_{a_ia_j}e_k, X(e_k, a_1, \cdots, \hat{a}_i, \cdots, \hat{a}_j, \cdots, a_{r+1})\rangle$$

for $a_1, \cdots, a_{r+1}\in\mathfrak{X}(M^n)$ and any orthogonal frame field $\{e_1, \cdots, e_n\}$.

We have then

THEOREM 22. *Let $a_1, \cdots, a_{r+1}\in\mathfrak{X}(M^n)$ and let $\{e_1, \cdots, e_n\}$ be an orthogonal frame field on an open subset of M^n. Then*

$$(87)\qquad (\Delta\phi)(a_1, \cdots, a_{r+1})=-\sum_{k=1}^{n}\|e_k\|^{-2}(\nabla^2\phi)(e_k; e_k; a_1, \cdots, a_{r+1})$$
$$-\sum_{i=1}^{r+1}(-1)^{i+r}k(a_i, X(a_i, \cdots, \hat{a}_i, \cdots, a_{r+1}))$$
$$-(r+1)\gamma(a_1, \cdots, a_{r+1})\,.$$

Proof. This follows from a calculation of the Laplacian (82) using (81) and the definition of the Chern form γ.

THEOREM 23. *Let $a_1, \cdots, a_r\in\mathfrak{X}(M^n)$ and let $\{e_1, \cdots, e_n\}$ be an orthogonal frame field on an open subset of M^n. Then*

$$(88)\qquad (\Delta\phi)(a_1, \cdots, a_r, X(a_1, \cdots, a_r))=\sum_{k=1}^{n}\|e_k\|^{-2}\|(\nabla_{e_k}X)(a_1, \cdots, a_r)\|^2$$
$$-\frac{C^2}{2}\sum_{k=1}^{n}(\nabla_{e_k}e_k-e_ke_k)\{\langle a_2\wedge\cdots\wedge a_r, a_2\wedge\cdots\wedge a_r\rangle\langle a_1, \alpha_1\rangle\}$$
$$-(r+1)\gamma(a_1, \cdots, a_r, X(a_1, \cdots, a_r))$$
$$+k(X(a_1, \cdots, a_r), X(a_1 \cdots, a_r))-\sum_{k=1}^{n}\|e_k\|^{-2}A(e_k, e_k)$$
$$-C^2\sum_{i=1}^{r}\langle a_1\wedge\cdots\wedge\hat{a}_i\wedge\cdots\wedge a_r, a_1\wedge\cdots\wedge\hat{a}_i\wedge\cdots\wedge a_r\rangle k(a_i, \alpha_i)$$
$$+C^2\sum_{i=1}^{r}\sum_{j=1}^{r}(-1)^{i+j}\langle a_1\wedge\cdots\wedge\hat{a}_i\wedge\cdots\wedge a_r, a_1\wedge\cdots\wedge\hat{a}_j\wedge\cdots\wedge a_r\rangle k(a_i, a_j))\rangle$$

where the α_i are defined by (*following* (34))

$$(89)\quad X^2(a_i(a_1, a_2, \cdots, \hat{a}_i, \cdots, a_r)^2)=C^2\langle a_1\wedge\cdots\wedge\hat{a}_i\wedge\cdots\wedge a_r, a_1\wedge\cdots\wedge\hat{a}_i\wedge\cdots\wedge a_r\rangle\alpha_i$$

$$-C^2\sum_{j=1}^{r}(-1)^{i+j}\langle a_1\wedge\cdots\wedge\hat{a}_i\wedge\cdots\wedge a_r,\ a_1\wedge\cdots\wedge\hat{a}_j\wedge\cdots\wedge a_r\rangle a_j\,.$$

Proof. We start from formula (87) and use (83) and (89). The formula (89) follows immediately from (34). This can be written as follows

$$X^2\{a_1(a_2,\cdots,a_r)^2\}=C^2\langle\hat{a}_1\wedge a_2\wedge\cdots\wedge a_r,\ \hat{a}_1\wedge a_2\wedge\cdots\wedge a_r\rangle\alpha_1 \tag{90}$$
$$+C^2\sum_{j=1}^{r}(-1)^j\langle\hat{a}_1\wedge a_2\wedge\cdots\wedge a_r,\ a_1\wedge\cdots\wedge\hat{a}_j\wedge\cdots\wedge a_r\rangle a_j\,.$$

12. THEOREM 24. *$a\in\mathfrak{X}(M^n)$ is an infinitesimal automorphism of the* (4, 2)-, *f- or vector π-structure if and only if for $\forall b_1,\cdots,b_r\in\mathfrak{X}(M^n)$*

$$[a,\ X(b_1,b_2,\cdots,b_r)]=\sum_{i=1}^{r}X(b_1,\cdots,b_{i-1},[a,b_i],b_{i+1},\cdots,b_r)\,. \tag{91}$$

Proof. a is an infinitesimal automorphism iff for $\forall b_1,\cdots,b_r$

$$(\mathcal{L}_aX)(b_1,b_2,\cdots,b_r)=0\,. \tag{92}$$

This is equivalent to

$$\mathcal{L}_aX(b_1,b_2,\cdots,b_r)=\sum_{i=1}^{r}X(b_1,\cdots,b_{i-1},\mathcal{L}_ab_i,b_{i+1},\cdots,b_r)\,. \tag{93}$$

Formula (91) follows now easily.

THEOREM 25. *Let X and Y be two $(1, r)$-tensor fields on M^n, then the following formula defines a $(1, 2r)$-tensor field S: for all $a_1,a_2,\cdots,a_r,b_1,b_2,\cdots,b_r\in\mathfrak{X}(M^n)$ we have*

$$\begin{aligned}
&S(a_1,a_2,\cdots,a_r;b_1,b_2,\cdots,b_r)\\
&=[X(a_1,a_2,\cdots,a_r),\ Y(b_1,b_2,\cdots,b_r)]+[Y(a_1,a_2,\cdots,a_r),\ X(b_1,b_2,\cdots,b_r)]\\
&\quad-\sum_{i=1}^{r}X(a_1,\cdots,a_{i-1},[a_i,Y(b_1,b_2,\cdots,b_r)],a_{i+1},\cdots a_r)\\
&\quad-\sum_{i=1}^{r}Y(a_1,\cdots,a_{i-1},[a_i,X(b_1,b_2,\cdots,b_r)],a_{i+1},\cdots,a_r)\\
&\quad-\sum_{i=1}^{r}X(b_1,\cdots,b_{i-1},[Y(a_1,a_2,\cdots,a_r),b_i],b_{i+1},\cdots,b_r)\\
&\quad-\sum_{i=1}^{r}Y(b_1,\cdots,b_{i-1},[X(a_1,a_2,\cdots,a_r),b_i],b_{i+1},\cdots,b_r)\\
&\quad+\sum_{i,j=1}^{r}X(a_1,\cdots,a_{j-1},Y(b_1,\cdots,b_{i-1},[a_j,b_i],b_{i+1},\cdots,b_r),a_{j+1},\cdots,a_r)\\
&\quad+\sum_{i,j=1}^{r}Y(a_1,\cdots,a_{j+1},X(b_1,\cdots,b_{i-1},[a_j,b_i],b_{i+1},\cdots,b_r),a_{j+1},\cdots,a_r)\,.
\end{aligned} \tag{94}$$

The proof of this theorem is immediate.

It is easy to see that in the case $r=1$ the tensor S defined by (94) is the Nijenhuis tensor. Therefore we define

DEFINITION. The tensor S defined by (94) is the *Nijenhuis tensor* of the two $(1, r)$-tensors X and Y.

The tensor S defined by (94), where $X=Y$ is a (4, 2)-, f- or vector π-structure, is called the *Nijenhuis tensor* of the structure.

We obtain then as a generalization of a known theorem (see KOBAYASHI and NOMIZU II, p. 128):

THEOREM 26. *Let $a_1, a_2, \cdots, a_r \in \mathfrak{X}(M^n)$ be infinitesimal automorphisms of the structure defined by X. Then $X(a_1, a_2, \cdots, a_r)$ is an infinitesimal automorphism of X if and only if*

$$\forall b_1, b_2, \cdots, b_r \in \mathfrak{X}(M^n) : S(a_1, a_2, \cdots, a_r ; b_1, b_2, \cdots, b_r)=0\,, \tag{95}$$

where S is the Nijenhuis tensor of this structure.

Proof. a_j is an infinitesimal automorphism if and only if for $\forall b_1, \cdots, b_r \in \mathfrak{X}(M^n)$

$$[a_j, X(b_1, b_2, \cdots, b_r)]=\sum_{i=1}^{r} X(b_1, \cdots, b_{i-1}, [a_j, b_i], b_{i+1}, \cdots, b_r)\,. \tag{96}$$

$X(a_1, a_2, \cdots, a_r)$ is also an infinitesimal automorphism if and only if for $\forall b_1, \cdots, b_r \in \mathfrak{X}(M^n)$

$$\begin{aligned}&[X(a_1, \cdots, a_r), X(b_1, \cdots, b_r)]\\&\quad=\sum_{i=1}^{r} X(b_1, \cdots, b_{i-1}, [X(a_1, \cdots, a_r), b_i], b_{i+1}, \cdots, b_r)\,.\end{aligned} \tag{97}$$

The Nijenhuis tensor of the structure is defined by (94) and we find so

$$\begin{aligned}\frac{1}{2}S(a_1, a_2, \cdots, a_r ; b_1, b_2, \cdots, b_r)&=[X(a_1, a_2, \cdots, a_r), X(b_1, b_2, \cdots, b_r)]\\&-\sum_{i=1}^{r} X(a_1, \cdots, a_{i-1}, [a_i, X(b_1, b_2, \cdots, b_r)], a_{i+1}, \cdots, a_r)\\&-\sum_{i=1}^{r} X(b_1, \cdots, b_{i-1}, [X(a_1, a_2, \cdots, a_r), b_i], b_{i+1}, \cdots, b_r)\\&+\sum_{i,j=1}^{r} X(a_1, \cdots, a_{j-1}, X(b_1, \cdots, b_{i-1}, [a_j, b_i], b_{i+1}, \cdots, b_r), a_{j+1}, \cdots, a_r)\,.\end{aligned} \tag{98}$$

It follows now from (96)

$$\sum_{i=1}^{r} X(a_1, \cdots a_{i-1}, [a_i, X(b_1, b_2, \cdots, b_r)], a_{i+1}, \cdots, a_r) \tag{99}$$

$$=\sum_{i,j=1}^{r} X(a_1, \cdots, a_{i-1}, X(b_1, \cdots, b_{j-1}, [a_i, b_j], b_{j+1}, \cdots, b_r), a_{i+1}, \cdots, a_r).$$

With the help of (97), (98) and (99) the proof is now immediate.

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