S. Eto
KŌDAI MATH. SEM. REP.
28 (1977), 159—161

A NOTE ON THE INVERSE PROBLEM FOR THE COHOMOLOGY VANISHING THEOREM

BY SHUN-ICHI ETO

In the previous paper [2], we considered some properties of strongly q-convex spaces with positive vector bundles. The cohomology vanishing theorem played an important role there.[1] An inverse problem now arises; whether the cohomology vanishing theorem characterize the strongly 1-convexity. In this paper, we prove partially affirmative answers for it.

In this paper, a complex space X is assumed to be Hausdorff and reduced. We say a complex space X is *strongly* 1-*convex* if there exists a C^∞ function ψ on X and a compact subset K of X such that ψ is strongly pseudoconvex in $X-K$ and $\{x\in X;\ \psi(x)<c\}$ is relatively compact in X for any $c\in\boldsymbol{R}$ (Andreotti-Grauert [1]).

THEOREM 1. *If on a complex space X there is a compact holomorphic divisor D such that for any coherent analytic sheaf $\mathcal{F}$ on X, we have*

$$H^i(X, \mathcal{F}\otimes\mathcal{O}([-D]^k))=0 \qquad i\geqq 1$$

for sufficiently large k, then, X is a strongly 1-*convex space. Here $[-D]$ denotes the line bundle induced from the divisor $-D$.*

Proof. We assume $D=\{U_j, f_j\}$ where $\{U_j\}$ is an open covering of X and $f_j\in\Gamma(U_j, \mathcal{O}_X)$ with $f_i f_j^{-1}\in\Gamma(U_i\cap U_j, \mathcal{O}_X^*)$. Let $\{x_n\}$ be a discrete infinite sequence in X and $\mathcal{I}$ be the ideal sheaf of $\{x_n\}$. By the assumption, for sufficiently large k the following sequence is exact.

$$\Gamma(X, \mathcal{O}([-D]^k))\longrightarrow\Gamma(X, \mathcal{O}_X/\mathcal{I}\otimes\mathcal{O}([-D]^k))\longrightarrow 0\,.$$

Now, for any $s=\{s_i\}\in\Gamma(X, \mathcal{O}([-D]^k))$, $s_i=(f_j/f_i)^k s_j$ in $U_i\cap U_j$ so we define $\hat{s}=f_i^k s_i$ in every U_i, which is a well-defined holomorphic function of X. We put $\|s(x)\|:=|\hat{s}(x)|$. By the above exact sequence there exists $s_0\in\Gamma(X, \mathcal{O}([-D]^k))$ such that $\|s_0(x_n)\|\to\infty$ hence $|\hat{s}_0(x_n)|\to\infty$ as $n\to\infty$. Therefore X is holomorphically convex.

Received Oct. 21, 1975.

[1] Theorem II in [2] is improved by the same method of Hartshone [3] as follows: *X, E and $\mathcal{F}$ are the same as in Theorem II in* [2], *then there esists a positive integer k_0 such that $\Gamma(X, \mathcal{F}\otimes\mathcal{O}(E^k))$ generates $\mathcal{F}_x\otimes\mathcal{O}_x(E^k)$ for every $x\in X$ and every positive integer $k\geqq k_0$.*

Next, take any two points x, y from $X-D$, $x\neq y$. We can find $f\in\Gamma(X, \mathcal{O}_X)$ such that $f(x)\neq f(y)$ in a similar way as above. Let $\tau: X\to Y$ be the Remmert reduction of X. Then $\Gamma(X, \mathcal{O}_X)\cong\Gamma(Y, \mathcal{O}_Y)$. Since $f(x)\neq f(y)$, $\tau(x)\neq\tau(y)$. So $\tau: X-D\to Y-\tau(D)$ is biholomorphic. Now we take a strongly 1-complete function ϕ on Y then $\phi\circ\tau$ is strongly pseudoconvex on $X-D$, hence X is strongly 1-convex. Q.E.D.

If X is a subvariety of $\boldsymbol{C}^n\times\boldsymbol{P}^m$, the cohomology vanishing theorem characterize strongly 1-concexity.

LEMMA. *Let V be a subvariety of $\boldsymbol{P}^m$ with positive dimension and $\boldsymbol{H}$ be the hyperplane bundle over V. Then, for suitable integers i $(1\leqq i\leqq m)$ and k,*

$$\dim H^i(V, \mathcal{O}(\boldsymbol{H}^k))>0\,.$$

Proof. If $\mathcal{F}$ is a coherent sheaf on V there is a following exact sequence on V. (cf. Serre [4])

$$(*)\qquad 0\longrightarrow(\mathcal{O}(\boldsymbol{H}^{k_m}))^{p_m}\xrightarrow{h_m}\cdots\longrightarrow(\mathcal{O}(\boldsymbol{H}^{k_1}))^{p_1}\xrightarrow{h_1}(\mathcal{O}(\boldsymbol{H}^{k_0}))^{p_0}\xrightarrow{h_0}\mathcal{F}\longrightarrow 0$$

Assume that $H^i(V, \mathcal{O}(H^k))=0$ for all $i\geqq 1$ and k. From (*), there is a short exact sequence.

$$0\longrightarrow(\mathcal{O}(\boldsymbol{H}^{k_m}))^{p_m}\xrightarrow{h_m}(\mathcal{O}(\boldsymbol{H}^{k_{m-1}}))^{p_{m-1}}\longrightarrow\operatorname{Im}(h_{m-1})\longrightarrow 0$$

So $H^i(V, \operatorname{Im}(h_{m-1}))=0$ for all $i\geqq 1$. By the iteration of this way we obtain $H^i(V, \operatorname{Im}(h_0))=H^i(V, \mathcal{F})=0$ for all $i\geqq 1$. Hence V is a Stein space, which is contrary to our assumption. Q.E.D.

THEOREM 2. *Let X be a subvariety of $\boldsymbol{C}^n\times\boldsymbol{P}^m$, and B be a line bundle over X which is the pull-back bundle of the hyperplane bundle $\boldsymbol{H}$ of $\boldsymbol{P}^m$. If, for any coherent analytic sheaf $\mathcal{F}$ on X, there exists an integer k_0 such that*

$$H^i(X, \mathcal{F}\otimes\mathcal{O}(B^k))=0$$

for every $k\geqq k_0$, then X is a strongly 1-convex space.

Proof. Since X is holomorphically convex, it has the Remmert reduction $\pi: X\to Y$. For any $x\in Y$ the fibre $\pi^{-1}(x)$ is a subvariety of $\boldsymbol{P}^m$. Let $\{x_n\}$ be a discrete infinite sequence in Y. First, we shall show $\dim_C\pi^{-1}(x_n)=0$ for all sufficiently large n.

Otherwise, we can take a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $\dim_C\pi^{-1}(x_{n_k})>0$ for all x_{n_k}. By Lemma there is a subsequence $\{x_\alpha\}$ such that for some i, $1\leqq i\leqq m$ and k_α $H^i(\pi^{-1}(x_\alpha), \mathcal{O}(B^{k_\alpha}))\neq 0$ for all α. Now, we put

$$\mathcal{S}:=\bigoplus_\alpha\hat{\mathcal{O}}(B^{k_\alpha-\alpha}|_{\pi^{-1}(x_\alpha)})$$

which is a coherent analytic sheaf over X and $\operatorname{supp}(\mathcal{S})=\bigcup_\alpha\pi^{-1}(x_\alpha)$. Here $\hat{\mathcal{O}}$ denotes the trivial extension of $\mathcal{O}$ to X. Then for any integer s,

$$\dim_C H^i(X, \mathcal{S}\otimes\mathcal{O}(B^s))=\sum_{\alpha} \dim_C H^i(\pi^{-1}(x_\alpha), \mathcal{O}(B^{k_\alpha-\alpha+s}))\neq 0\,,$$

which is a contradiction.

Thus, there exists a compact set K in Y such that for every $x\in Y-X$ $\dim_C \pi^{-1}(x)=0$. Now if we take a strongly 1-complete function ϕ on Y, $\phi\circ\pi$ is strongly pseudoconvex in $X-\pi^{-1}(K)$ where $\pi^{-1}(K)$ is compact since π is proper. It follows that X is strongly 1-convex. Q.E.D.

The author doesn't know whether Theorem 2 is valid without the assumption that X is a subvariety of $\boldsymbol{C}^n\times\boldsymbol{P}^m$.

References

[1] ANDREOTTI, A. et H. GRAUERT, Théorèmes de finitude pour la cohomologie des espaces complexes, Bull. Soc. Math. France **90** (1962), 193-259.

[2] ETO, S., H. KAZAMA AND K. WATANABE, On strongly q-pseudoconvex spaces with positive vector bundles, Mem. Sci Kyushu Univ. **28** (1974), 135-146.

[3] HARTSHONE, R., Ample vector bundles, I. H. E. S. Publ. Math. **29** (1966), 63-94.

[4] SERRE, J.-P., Faisceaux algébriques cohérents, Ann. Math. **61** (1955), 197-278.

DEPARTMENT OF APPLIED SCIENCE, FACULTY OF ENGINEERING,
KYUSHU UNIVERSITY.