Y. MATSUYAMA
KŌDAI MATH. SEM. REP.
28 (1977), 144–149

COMPLEX HYPERSURFACES OF THE PRODUCT OF TWO COMPLEX SPACE FORMS

BY YOSHIO MATSUYAMA

§ 0. Introduction

Recently, Ludden and Okumura [2] have showed that a complete hypersurface M of the product $S^n \times S^n$ of two n-spheres whose tangent space is invariant under the almost product structure on $S^n \times S^n$ (for simplicity, we say that M is invariant) is the product of S^n and a hypersurface of S^n. Using the fact, they showed that $S^{n-1}(1) \times S^n(1)$ and

$$S^m(\sqrt{m/(n-1)}) \times S^{n-m-1}(\sqrt{(n-m-1)/(n-1)}) \times S^n(1)$$

are the only compact orientable invariant minimal hypersurfaces of $S^n \times S^n$ satisfying trace $H^2 \leqq n-1$. On the other hand, the present author [3] obtained some results of the same type in the case where the ambient space is the product $P_n(C) \times P_n(C)$ of two complex projective n-spaces, i. e., a complete invariant Kaehler hypersurface of $P_n(C) \times P_n(C)$ is the product of $P_n(C)$ and a Kaehler hypersurface of $P_n(C)$, and $P_{n-1}(C) \times P_n(C)$ and $Q_{n-1}(C) \times P_n(C)$ are the only compact invariant Kaehler hypersurfaces of $P_n(C) \times P_n(C)$ with constant scalar curvature, where $Q_{n-1}(C)$ is the complex quadric.

In the present paper, we consider the following problems:

(1) Is an invariant hypersurface M of the product $M_1 \times M_2$ of two Riemannian manifolds the product of M_1 (resp. M_2) and a hypersurface of M_2 (resp. M_1)?

(2) Are the conditions that M is invariant and that the restriction of an almost product structure to (the tangent space of) the hypersurface and the second fundamental form of the hypersurface are commutative equivalent?

In § 1, we review some fundamental formulas for a complex hypersurface of the product of two complex manifolds and obtain a result: a complex hypersurface M of the product of two complex space forms is invariant under the curvature transformation ([1]) if and only if M is invariant under the almost product structure (Proposition 1). In § 2, we show that (1) is true in the case of a complex hypersurface of the product of two complex manifolds (Theorem 2). In § 3, we show that (2) is true in the case of a complex hypersurface of the product $M^n(c_1) \times M^m(c_2)(c_1, c_2 \geqq 0, c_1^2 + c_2^2 \neq 0)$ of two complex space forms (Theorem 3).

Received Sept. 29, 1975

We add that Proposition 1 (resp. Theorem 2) can be proven for real hypersurfaces of the product of two real space forms (resp. Riemannian manifolds).

Finally, the author is grateful to Professors M. Obata and K. Ogiue for their useful criticism.

§ 1. Preliminaries

Let M^n be a complex Kaehler n-manifold, and consider $M^n \times M^m$. We denote by $\bar{g}$, $\bar{J}$ and F the product Riemannian metric, the product complex structure and the almost product structure on $M^n \times M^m$. Then they satisfy the following ([3]):

$$F^2=I, \qquad \text{trace } F=2n-2m,$$

$$\bar{g}(F\bar{X}, \bar{Y})=\bar{g}(\bar{X}, F\bar{Y}), \qquad \bar{\nabla}_{\bar{X}}F=0,$$

$$F\bar{J}=\bar{J}F, \qquad \bar{J}^2=-I,$$

$$\bar{g}(\bar{J}\bar{X}, \bar{J}\bar{Y})=\bar{g}(\bar{X}, \bar{Y}), \qquad \bar{\nabla}_{\bar{X}}\bar{J}=0,$$

where $\bar{\nabla}$ denotes the operator of covariant differentiation with respect to $\bar{g}$.

Now, let M be a complex hypersurface of $M^n \times M^m$, and B the differential of the immersion i of M into $M^n \times M^m$. Let g and J be the induced Riemannian metric and the induced complex structure on M, respectively, and let ∇ denote the operator of covariant differentiation with respect to (the Riemannian connection of) g. Let X, Y and Z be tangent to M and N a unit normal vector. Then we have the following relations ([3]):

(1.1) $$FBX=BfX+u(X)N+\tilde{u}(X)\bar{J}N$$

(1.2) $$FN=BU+\lambda N,$$

$$g(U, X)=u(X), \qquad g(JU, X)=\tilde{u}(X),$$

$$\tilde{u}(X)=-u(JX), \qquad Jf=fJ,$$

(1.3) $$\bar{\nabla}_{BX}BY=B\nabla_X Y+h(X, Y)N+k(X, Y)JN,$$

(1.4) $$\bar{\nabla}_{BX}N=-BHX+s(X)JN,$$

$$h(X, Y)=g(HX, Y), \qquad k(X, Y)=g(JHX, Y),$$

$$HJ=-JH, \qquad \text{trace } H=\text{trace } HJ=0,$$

(1.5) $$f^2X=X-u(X)U+u(JX)JU,$$

(1.6) $$fU=-\lambda U,$$

(1.7) $$u(U)=g(U, U)=1-\lambda^2,$$

(1.8) $$(\nabla_Y f)X=h(Y, X)U+k(Y, X)JU+u(X)HY-u(JX)JHY$$

$$\nabla_X U = -fHX + \lambda HX + s(X)JU\,, \tag{1.9}$$

$$X\cdot\lambda = -2h(X, U) = -2u(HX)\,, \tag{1.10}$$

$$\text{trace } f = 2n - 2m - 2\lambda\,, \tag{1.11}$$

$$\text{trace } fH = \text{trace } f(\nabla_X H) = 0\,, \tag{1.12}$$

where f; u, $\tilde{u}$; U; λ; h, k and s define a symmetric linear transformation of the tangent bundle of M, two 1-forms, a vector field, a function on M, the second fundamental tensors of the hypersurface and a normal connection form, respectively.

If u is identically zero, then M is said to be an invariant hypersurface, that is, the tangent space $T_x(M)$ is invariant under F. We can easily see by (1.7) that this is equivalent to $\lambda^2 = 1$.

We denote by $M^n(c)$ an n-dimensional complex space form with constant holomorphic curvature c. In this section, we assume from now on that the ambient manifold is $M^n(c_1)\times M^m(c_2)(c_1{}^2+c_2{}^2\neq 0)$. Then the curvature tensor $\bar{R}$ of $M^n(c_1)\times M^m(c_2)$ and the Codazzi equation for a complex hypersurface are given by ([3], [4])

$$\begin{aligned}\bar{R}(\bar{X}, \bar{Y})\bar{Z} = \frac{1}{16}(c_1+c_2)\{&\bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y} + \bar{g}(\bar{J}\bar{Y}, \bar{Z})\bar{J}\bar{X}\\ &-\bar{g}(\bar{J}\bar{X}, \bar{Z})\bar{J}\bar{Y} + 2\bar{g}(\bar{X}, \bar{J}\bar{Y})\bar{J}\bar{Z} + \bar{g}(F\bar{Y}, \bar{Z})F\bar{X}\\ &-\bar{g}(F\bar{X}, \bar{Z})F\bar{Y} + \bar{g}(F\bar{J}\bar{Y}, \bar{Z})F\bar{J}\bar{X}\\ &-\bar{g}(F\bar{J}\bar{X}, \bar{Z})F\bar{J}\bar{Y} + 2\bar{g}(F\bar{X}, \bar{J}\bar{Y})F\bar{J}\bar{Z}\}\\ +\frac{1}{16}(c_1-c_2)\{&\bar{g}(\bar{Y}, \bar{Z})F\bar{X} - \bar{g}(\bar{X}, \bar{Z})F\bar{Y} + \bar{g}(\bar{J}\bar{Y}, \bar{Z})F\bar{J}\bar{X}\\ &-\bar{g}(\bar{J}\bar{X}, \bar{Z})F\bar{J}\bar{Y} + 2\bar{g}(\bar{X}, \bar{J}\bar{Y})F\bar{J}\bar{Z} + \bar{g}(F\bar{Y}, \bar{Z})\bar{X} - \bar{g}(F\bar{X}, \bar{Z})\bar{Y}\\ &+\bar{g}(F\bar{J}\bar{Y}, \bar{Z})\bar{J}\bar{X} - \bar{g}(F\bar{J}\bar{X}, \bar{Z})\bar{J}\bar{Y} + 2\bar{g}(F\bar{X}, \bar{J}\bar{Y})\bar{J}\bar{Z}\}\,,\end{aligned} \tag{1.13}$$

$$\begin{aligned}(\nabla_X H)Y - (\nabla_Y H)X - s(X)JHY + s(Y)JHX\\ = \frac{1}{16}(c_1+c_2)\{u(X)fY - u(Y)fX + u(JX)fJY\\ -u(JY)fJX - 2g(JX, JY)JU\}\\ +\frac{1}{16}(c_1-c_2)\{u(X)Y - u(Y)X + u(JX)JY\\ -u(JY)JX - 2g(X, JY)JU\}\,,\end{aligned} \tag{1.14}$$

respectively. We have

RROPOSITION 1. *A complex hypersurface M of $M^n(c_1)\times M^m(c_2)$, $c_1{}^2+c_2{}^2\neq 0$, is invariant under the curvature transformation: $\bar{R}(BX, BY)T_x(M)\subset T_x(M)$ if and only if M is invariant: $FT_x(M)\subset T_x(M)$.*

Proof. Noting that $\bar{g}(\bar{R}(BX, BY)BZ, N)=g((\nabla_X^* H)Y-(\nabla_Y^* H)X, Z)$ and using (1.14), we see that the necessity is trivial, where $\nabla_X^* H=\nabla_X H-s(X)JH$([3]). Suppose that $\bar{g}(\bar{R}(BX, BY)BZ, N)=0$. If $\|U\|_x\neq 0$ at $x\in M$, then we can choose an orthonormal frame $\left\{E_i, JE_i, \frac{U}{\|U\|}, \frac{JU}{\|U\|}\right\}1\leqq i\leqq n+m-2$ in a neighborhood of x such that, at x, $fE_i=E_i$ for $1\leqq i\leqq n-1$ and $fE_i=-E_i$ for $n\leqq i\leqq n+m-2$. Replacing X and Y in (1.14) with vanishing left hand side by U and E_i respectively, we have

$$c_1(fE_i+E_i)+c_2(fE_i-E_i)=0\,.$$

Applying f to the both sides, we get

$$c_1(E_i+fE_i)+c_2(E_i-fE_i)=0\,.$$

Hence $2c_1(fE_i+E_i)=0$. Since we may choose i such that $n\leqq i\leqq n+m-2$, $c_1=0$. Similarly $c_2=0$. This is a contradiction.

COROLLARY 1. *A complex hypersurface of $M^n(c_1)\times M^m(c_2)$, $c_1^2+c_2^2\neq 0$, with parallel second fundamental form is invariant.*

COROLLARY 2. *A totally geodesic hypersurface of $M^n(c_1)\times M^m(c_2)$, $c_1^2+c_2^2\neq 0$, is invariant.*

§ 2. Invariant complex hypersurfaces of $M^n\times M^m$

In this section we assume that the complex hypersurface M is invariant. Then (1.1), (1.5), (1.7) and (1.8) can be written as

$$FBX=BfX\,, \tag{2.1}$$

$$f^2X=X\,, \tag{2.2}$$

$$1-\lambda^2=0\,, \tag{2.3}$$

$$\nabla_X f=0\,. \tag{2.4}$$

From (2.2) and (2.4) M is product manifold, say, $M=M_1\times M_2$, where $FBX_1=BX_1$ for $X_1\in T_x(M_1)$ and $FBX_2=-BX_2$ for $X_2\in T_x(M_2)$. Noting that trace $f=2n-2m-2\lambda=2n-2m-2$ (resp. $2n-2m+2$), we see that dim $T_x(M_1)=n-1$ (resp. n) and dim $T_x(M_2)=m$ (resp. $m-1$). Thus we have

THEOREM 2. *An invariant complex hypersurface of $M^n\times M^m$ is a product manifold $M'\times M^m$ (resp. $M^n\times M'$), where M' is a complex hypersurface of M^n (resp. M^m).*

§ 3. Complex hypersurfaces of $M^n(c_1)\times M^m(c_2)$ satisfying $Hf=fH$

Now we assume that a complex hypersurface M satisfies the condition Hf

$=fH$. Then differentiating $HfX=fHX$ covariantly and making use of (1.8), we get

$$(\nabla_Y H)fX+g(HY, X)HU+g(JHY, X)HJU+u(X)H^2Y-u(JX)HJHY$$
$$=g(HY, HX)U+g(JHY, HX)JU+u(HX)HY-u(JHX)JHY+f(\nabla_Y H)X$$

and hence

$$g((\nabla_Y H)fX, Z)+g(HY, X)g(HU, Z)+g(JHY, X)g(HJU, Z)+u(X)g(H^2Y, Z)-u(JX)g(HJHY, Z)$$
$$=g(HY, HX)g(U, Z)+g(JHY, HX)g(JU, Z)+u(HX)g(HY, Z)-u(JHX)g(JHY, Z)+g(f(\nabla_Y H)X, Z).$$

Replacing Y and Z by E_i belonging to an orthonormal frame and making use of symmetric property of $\nabla_Y H$, we find

$$\sum_i \{g(fX, (\nabla_{E_i}H)E_i)+g(HX, E_i)g(HU, E_i)+g(JHX, E_i)g(HJU, E_i)+u(X)g(H^2E_i, E_i)-u(JX)g(JHE_i, HE_i)\}$$
$$=\sum_i \{g(H^2X, E_i)g(U, E_i)+g(JH^2X, E_i)g(JU, E_i)+u(HX)g(HE_i, E_i)-u(JHX)g(JHE_i, E_i)+g(f(\nabla_{E_i}H)X, E_i)\},$$

from which

$$\sum s(E_i)g(fX, JHE_i)+\frac{1}{4}c_1\lambda(n-\lambda)g(U, X)-\frac{1}{4}c_2\lambda(m+\lambda)g(U, X)+\text{trace } H^2g(U, X)$$
(3.1)
$$=2g(H^2X, U)+\sum s(E_i)g(fJHX, E_i)-\left\{\frac{1}{4}c_1(n-1+\lambda^2-\lambda)+\frac{1}{4}c_2(m-1+\lambda+\lambda^2)\right\}g(U, X),$$

because of (1.12), (1.14) and

$$0=Y\cdot\text{trace } H=\text{trace }(\nabla_Y H)=\sum g((\nabla_Y H)E_i, E_i)$$
$$=g(\sum (\nabla_{E_i}H)E_i-\sum s(E_i)JHE_i+\frac{1}{4}c_1(n-\lambda)U-\frac{1}{4}c_2(m+\lambda)U, Y),$$

i. e.,

$$\sum (\nabla_{E_i}H)E_i=\sum s(E_i)JHE_i-\frac{1}{4}c_1(n-\lambda)U+\frac{1}{4}c_2(m+\lambda)U.$$

Hence we get from (3.1)

$$2H^2U=\left\{\frac{1}{4}c_1(n-1)(1+\lambda)+\frac{1}{4}c_2(m-1)(1-\lambda)+\text{trace } H^2\right\}U. \tag{3.2}$$

TREOREM 3. *A complex hypersurface M of $M^n(c_1)\times M^m(c_2)$ ($c_1\geqq 0$, $c_2\geqq 0$, $c_1{}^2+c_2{}^2\neq 0$) is invariant if and only if $Hf=fH$.*

Proof. If $Hf=fH$ and $\|U\|_x\neq 0$ for some $x\in M$, then we can choose an orthonormal basis $\left\{E_i, JE_i, \frac{U}{\|U\|}, \frac{JU}{\|U\|}\right\}1\leqq i\leqq n+m-2$ in a neighborhood of x such that $H\frac{U}{\|U\|}=\mu\frac{U}{\|U\|}$. Then from (3.2)

$$2\mu^2\frac{U}{\|U\|}=\left\{\frac{1}{4}c_1(n-1)(1+\lambda)+\frac{1}{4}c_2(m-1)(1-\lambda)\right.$$
$$\left.+2\sum_i g(H^2E_i, E_i)+2\mu^2\right\}\frac{U}{\|U\|}.$$

Hence $c_1=c_2=0$. This is a contradiction. The converse is trivial because of (1.9).

REFERENCES

[1] B. Y. CHEN AND K. OGIUE, On totally real submanifolds, Trans. Amer. Math. Soc. **193** (1974), 257-266.
[2] G. D. LUDDEN AND M. OKUMURA, Some integral formulas and their applications to hypersurfaces of $S^n\times S^n$, J. Differential Geometry **9** (1974), 617-631.
[3] Y. MATSUYAMA, Complex hypersurfaces of $P_n(C)\times P_n(C)$, Tôhoku Math. J. **28** (1976), 541-552.
[4] S. TACHIBANA, Some theorems on locally product Riemannian manifold, Tôhoku Math. J. **12** (1960), 281-292.

DEPARTMENT OF MATHEMATICS
TOKYO METROPOLITAN UNIVERSITY