K. MATSUMOTO
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φ-TRANSFORMATIONS ON A K-CONTACT RIEMANNIAN MANIFOLD

BY KŌZI MATSUMOTO

§ 0. **Introduction.** It is very interesting to make reserches on the subject of manifolds admitting a tensor field invariant under a certain transformation. Now, S. Tanno has studied φ-transformations on almost contact Riemannian manifolds and given several important conclusions ([4]). The main purpose of the present paper is to prove Theorems 2.2, 3.1, 4.2 and 4.4.

§ 1. **Preriminaries.** Let M be a $(2n+1)$-dimensional differentiable manifold satisfying the second axiom of countability. In this paper, manifolds, geometric objects and mappings we consider are assumed to be differentiable and of class C^∞. If there exists a tensor field $\varphi_j{}^i$ of type (1.1), contravariant and covariant vector fields ξ^i and η_i on M which satisfy the following conditions:

$$\xi^i\eta_i=1\,, \tag{1.1}$$

$$\varphi_r{}^i\varphi_j{}^r=-\delta_j{}^i+\xi^i\eta_j\,, \tag{1.2}$$

then M is said to have an almost contact structure and called an almost contact manifold. The suffices $k, j, \cdots, i$ run over the range $\{1, 2, \cdots, 2n+1\}$ and the summation convension will be used. For an almost contact structure the following identities are established ([3]):

$$\varphi_r{}^i\xi^r=0\,, \qquad \eta_r\varphi_j{}^r=0\,. \tag{1.3}$$

Let M be an almost contact manifold. Then there exists a positive definite Riemannian metric g_{ji} such that

$$\eta_i=g_{ir}\xi^r\,, \tag{1.4}$$

$$g_{sr}\varphi_j{}^s\varphi_i{}^r=g_{ji}-\eta_j\eta_i\,. \tag{1.5}$$

Such a metric tensor g_{ji} is called an associated Riemannian metric with the given almost contact structure. If a differentiable manifold M admits tensor fields $(\varphi_j{}^i, \xi^i, \eta_i, g_{ji})$ such that g_{ji} is a Riemannian metric associated with the almost contact structure, then M is called an almost contact Riemannian manifold.

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An odd dimensional differentiable manifold M (dim $M=2n+1$) is said to have a contact structure and to be a contact manifold if there exists a 1-form η on M such that $\eta\wedge(d\eta)^n\neq 0$ everywhere on M, where $d\eta$ means the exterior derivative of η and the symbol $\wedge$ means the exterior multiplication. Then η is called a contact form of M. A contact manifold has an almost contact metric structure $(\varphi_j{}^i, \xi^i, \eta_i, g_{ji})$ such that

$$g_{ir}\varphi_j{}^r=\varphi_{ji}=\frac{1}{2}(\partial_j\eta_i-\partial_i\eta_j),$$

η_i denoting components of η, where the operator ∂ denotes the partial differentiation with respect to the local coordinates. An almost contact structure constructed from a contact form η is called a contact (metric) structure associated with η. An almost contact (Riemannian) manifold constructed from a contact form η is called a contact (Riemannian) manifold associated with η.

A contact metric structure (contact Riemannian manifold) is called a K-contact metric structure (K-contact Riemannian manifold), if its contact form η_j determines a unit Killing vector field $\xi^i=g^{ir}\eta_r$. If M is a K-contact Riemannian manifold, the following identities are established ([3]):

$$\nabla_j\eta_i=\varphi_{ji}, \qquad \nabla_j\xi^i=\varphi_j{}^i, \tag{1.6}$$

$$\nabla_r\varphi_i{}^r=2n\eta_i, \tag{1.7}$$

$$\nabla_k\varphi_{ji}+\nabla_j\varphi_{ik}+\nabla_i\varphi_{kj}=0, \tag{1.8}$$

where the operator ∇ is the covariant differentiation with respect to g_{ji}.

An almost contact metric structure is called a Sasakian structure if it admits a unit Killing vector field η_i satisfying the following relations:

$$\nabla_k\varphi_{ji}=\eta_j g_{ki}-\eta_i g_{kj}, \tag{1.9}$$

or equivalently

$$R_{kjih}\xi^k=g_{ji}\eta_h-g_{jh}\eta_i, \tag{1.10}$$

where $R_{kji}{}^h$ denotes the curvature tensor field of g_{ji}.

A K-contact Riemannian manifold is called an η-Einstein manifold, if the Ricci tensor field R_{ji} of g_{ji} has the following form:

$$R_{ji}=ag_{ji}+b\eta_j\eta_i, \tag{1.11}$$

where a and b are constant satisfying $a+b=2n$ for $n>1$.

A Sasakian manifold is said to be of constant φ-holomorphic sectional curvature k, if the curvature tensor field has the following form ([2]):

$$\begin{aligned} R_{kjih}=\frac{k+3}{4}(g_{kh}g_{ji}-g_{ki}g_{jh})-\frac{k-1}{4}(&\eta_k\eta_h g_{ji}+g_{kh}\eta_j\eta_i \\ &-\eta_k\eta_i g_{jh}-g_{ki}\eta_j\eta_h-\varphi_{kh}\varphi_{ji}+\varphi_{ki}\varphi_{jh}+2\varphi_{kj}\varphi_{ih}). \end{aligned} \tag{1.12}$$

§2. **A certain tensor field invariant under a φ-transformation.** Let M be a $(2n+1)$-dimensional almost contact Riemannian manifold with structure tensor fields $(\varphi_j{}^i, \xi^i, \eta_i, g_{ji})$ and f be a diffeomorphism of M. If f leaves the tensor field $\varphi_j{}^i$ invariant, then we say that f is a φ-transformation of M. For the φ-transformation of a contact Riemannian manifold, the following proposition was proved by S. Tanno ([4]):

PROPOSITION 2.1. *Let M be a contact Riemannian manifold. If f is a φ-transformation of M, then there exists a positive constant α such that*

$$\bar{\xi}^i=\alpha\xi^i, \qquad \bar{\eta}_i=\alpha\eta_i, \qquad \bar{\varphi}_{ji}=\alpha\varphi_{ji}, \tag{2.1}$$
$$\bar{g}_{ji}=\alpha g_{ji}+\alpha(\alpha-1)\eta_j\eta_i,$$

where $f_*\xi=\bar{\xi}$, $f^*\eta=\bar{\eta}$ and so on.

From (2.1), we can easily obtain

$$\bar{g}^{ji}=\frac{1}{\alpha}g^{ji}-\frac{\alpha-1}{\alpha^2}\xi^j\xi^i. \tag{2.2}$$

Because of (2.1) and (2.2), we have

$$\overline{\left\{{i \atop k\ j}\right\}}=\left\{{i \atop k\ j}\right\}+\frac{\alpha-1}{2}g^{ir}\varDelta_{kjr}-\frac{\alpha-1}{\alpha}\left\{{r \atop k\ j}\right\}\eta_r\xi^i \tag{2.3}$$
$$-\frac{(\alpha-1)^2}{2\alpha}\xi^i\xi^r\varDelta_{kjr},$$

where $\overline{\left\{{i \atop k\ j}\right\}}$ and $\left\{{i \atop k\ j}\right\}$ are respectively the Christoffel symbols formed of $\bar{g}_{ji}$ and g_{ji} and where $\varDelta_{jih}$ is defined by

$$\varDelta_{jih}=\partial_j(\eta_i\eta_h)+\partial_i(\eta_h\eta_j)-\partial_h(\eta_j\eta_i). \tag{2.4}$$

Hereafter our manifold is assumed to be a K-contact Riemannian manifold. Then (2.4) can be written as

$$\varDelta_{jih}=2\left(\eta_j\varphi_{ih}+\eta_i\varphi_{jh}+\left\{{r \atop j\ i}\right\}\eta_r\eta_h\right). \tag{2.5}$$

Substituting (2.5) in (2.3) and calculating the curvature tensor field $\bar{R}_{kji}{}^h$ of $\bar{g}_{ji}$, we get

$$\bar{R}_{kji}{}^h=R_{kji}{}^h+(\alpha-1)(2\varphi_{kj}\varphi_i{}^h+\varphi_{ki}\varphi_j{}^h-\varphi_{ji}\varphi_k{}^h-\eta_i\nabla^h\varphi_{kj} \tag{2.6}$$
$$+\eta_j\nabla_k\varphi_i{}^h-\eta_k\nabla_j\varphi_i{}^h)+(\alpha-1)^2(\delta_k{}^h\eta_j-\delta_j{}^h\eta_k)\eta_i.$$

From (2.1) and (2.6), we have

$$\bar{W}_{kji}{}^h=W_{kji}{}^h+(\alpha-1)(-\eta_i\nabla^h\varphi_{kj}+\eta_j\nabla_k\varphi_i{}^h-\eta_k\nabla_j\varphi_i{}^h) \tag{2.7}$$
$$+(\alpha-1)^2(\delta_k{}^h\eta_j-\delta_j{}^h\eta_k)\eta_i,$$

where we put

$$W_{kji}{}^h = R_{kji}{}^h - (2\varphi_{kj}\varphi_i{}^h + \varphi_{ki}\varphi_j{}^h - \varphi_{ji}\varphi_k{}^h). \tag{2.8}$$

Taking account of (2.1), (2.2), (2.3) and (2.5), we can easily obtain the following equations:

$$\eta_i \nabla^h \varphi_{kj} = \frac{1}{\alpha} \bar{\eta}_i \bar{\nabla}^h \bar{\varphi}_{kj} + (\alpha - 1)(\delta_k{}^h \eta_j - \delta_j{}^h \eta_k)\eta_i,$$

$$\eta_j \nabla_k \varphi_i{}^h = \frac{1}{\alpha} \bar{\eta}_j \bar{\nabla}_k \bar{\varphi}_i{}^h + (\alpha - 1)(-\delta_k{}^h + \eta_k \xi^h)\eta_j \eta_i.$$

From two equations above, we have

$$\begin{aligned}(\alpha-1)(\eta_j \nabla_k \varphi_i{}^h - \eta_k \nabla_j \varphi_i{}^h - \eta_i \nabla^h \varphi_{kj}) &= (\bar{\eta}_j \bar{\nabla}_k \bar{\varphi}_i{}^h - \bar{\eta}_k \bar{\nabla}_j \bar{\varphi}_i{}^h - \bar{\eta}_i \bar{\nabla}^h \bar{\varphi}_{kj}) - (\eta_j \nabla_k \varphi_i{}^h \\ &\quad - \eta_k \nabla_j \varphi_i{}^h - \eta_i \nabla^h \varphi_{kj}) - 2\alpha(\alpha-1)(\delta_k{}^h \eta_j - \delta_j{}^h \eta_k)\eta_i.\end{aligned} \tag{2.9}$$

Thus we have, from (2.7) and (2.9),

$$\bar{T}_{kji}{}^h = T_{kji}{}^h - (\alpha^2 - 1)(\delta_k{}^h \eta_j - \delta_j{}^h \eta_k)\eta_i, \tag{2.10}$$

where we put

$$T_{kji}{}^h = W_{kji}{}^h - (\eta_j \nabla_k \varphi_i{}^h - \eta_k \nabla_j \varphi_i{}^h - \eta_i \nabla^h \varphi_{kj}). \tag{2.11}$$

Taking account of $(2.1)_{2)}$, (2.8), (2.10) and (2.11), we have

THEOREM 2.2. *If a K-contact Riemannian manifold admits a φ-transformation, then a tensor field $Z_{kji}{}^h$ is invariant under this transformation, where the tensor field $Z_{kji}{}^h$ is defined as follows:*

$$Z_{kji}{}^h = T_{kji}{}^h + (\delta_k{}^h \eta_j - \delta_j{}^h \eta_k)\eta_i \tag{2.12}$$

or equivalently as

$$\begin{aligned}Z_{kji}{}^h = R_{kji}{}^h &- (2\varphi_{kj}\varphi_i{}^h + \varphi_{ki}\varphi_j{}^h - \varphi_{ji}\varphi_k{}^h) \\ &- \eta_j \nabla_k \varphi_i{}^h + \eta_k \nabla_j \varphi_i{}^h + \eta_i \nabla^h \varphi_{kj} + (\delta_k{}^h \eta_j - \delta_j{}^h \eta_k)\eta_i.\end{aligned} \tag{2.13}$$

Next, let X^i and Y^i be vector fields on M such that they are orthogonal to ξ^i. Then transvecting (2.13) with $Y^k X^j \eta_h$, we have

$$Z_{kji}{}^h Y^k X^j \eta_h = R_{kji}{}^h Y^k X^j \eta_h. \tag{2.14}$$

On the other hand, the following proposition is well-known ([1]):

PROPOSITION 2.3. *A K-contact Riemannian manifold is a Sasakian manifold if and only if*

$$R_{kji}{}^h Y^k X^j \eta_h = 0$$

for any vector fields Y^i and X^i orthogonal to ξ^i.

Thus we have, from (2.14) and the Proposition 2.3,

THEOREM 2.4. *A necessary and sufficient condition for a K-contact Riemannian manifold to be a Sasakian manifold is that*

$$Z_{kji}{}^{h}Y^{k}X^{j}\eta_{h}=0 \tag{2.15}$$

for any vector fields Y^i and X^i orthogonal to ξ^i, which is equivalent to

$$\begin{aligned} Z_{kjih}=R_{kjih}-\eta_{j}\eta_{i}g_{kh}+g_{ih}\eta_{k}\eta_{i}+\eta_{k}\eta_{h}g_{ji} \\ -\eta_{j}\eta_{h}g_{ki}-(2\varphi_{kj}\varphi_{ih}+\varphi_{ki}\varphi_{jh}-\varphi_{ji}\varphi_{kh}) . \end{aligned} \tag{2.16}$$

§ 3. **The manifold satisfying $Z_{kji}{}^{h}=0$.** In this section, our K-contact Riemannian manifold is assumed to satisfy the condition $Z_{kji}{}^{h}=0$. Then we have from (2.16)

$$\begin{aligned} R_{kjih}=\eta_{j}\eta_{i}g_{kh}-g_{jh}\eta_{k}\eta_{i}-\eta_{k}\eta_{h}g_{ji}+\eta_{j}\eta_{h}g_{ki}+2\varphi_{kj}\varphi_{ih} \\ +\varphi_{ki}\varphi_{jh}-\varphi_{ji}\varphi_{kh} . \end{aligned}$$

Thus, we have, from (1.12) and the above equation,

THEOREM 3.1. *If a K-contact Riemannian manifold satisfies the condition $Z_{kji}{}^{h}=0$, then the manifold is a Sasakian manifold of constant φ-holomorphic sectional curvature -3.*

§ 4. **Manifolds satisfying certain conditions with respect to $\bar{g}_{ji}$.** In this section, we shall consider manifolds such that the curvature tensor field $\bar{R}_{kji}{}^{h}$ of $\bar{g}_{ji}$ satisfies some special kinds of conditions. At first, we assume that our K-contact Riemannian manifold admitting a φ-transformation is of constant curvature k with respect to $\bar{g}_{ji}$. Then by assumption we have ([5])

$$\bar{R}_{kji}{}^{h}=k(\delta_{k}^{h}\bar{g}_{ji}-\delta_{j}^{h}\bar{g}_{ki}) . \tag{4.1}$$

By using $(2.1)_{4)}$ and (4.1), we have, from (2.6),

$$\begin{aligned} R_{kji}{}^{h}=k\{\alpha(\delta_{k}^{h}g_{ji}-\delta_{j}^{h}g_{ki})+\alpha(\alpha-1)(\delta_{k}^{h}\eta_{j}-\delta_{j}^{h}\eta_{k})\eta_{i}\} \\ -(\alpha-1)(2\varphi_{kj}\varphi_{i}{}^{h}+\varphi_{ki}\varphi_{j}{}^{h}-\varphi_{ji}\varphi_{k}{}^{h}-\eta_{i}\nabla^{h}\varphi_{kj} \\ +\eta_{j}\nabla_{k}\varphi_{i}{}^{h}-\eta_{k}\nabla_{j}\varphi_{i}{}^{h})-(\alpha-1)^{2}(\delta_{k}^{h}\eta_{j}-\delta_{j}^{h}\eta_{k})\eta_{i} . \end{aligned} \tag{4.2}$$

Transvecting (4.2) with η_h, we have

$$R_{kji}{}^{h}\eta_{h}=(k\alpha-\alpha+1)(\eta_{k}g_{ji}-\eta_{j}g_{ki}) . \tag{4.3}$$

Since η_i is a Killing vector field, (4.3) can be written as

$$\nabla_{i}\varphi_{jk}=-(k\alpha-\alpha+1)(\eta_{k}g_{ji}-\eta_{j}g_{ki}) . \tag{4.4}$$

Substituting (4.4) in (4.2), we have

(4.5) $$R_{kji}{}^{h}=k\{\alpha(\delta_k{}^h g_{ji}-\delta_j{}^h g_{ki})+\alpha(\alpha-1)(\delta_k{}^h\eta_j-\delta_i{}^h\eta_k)\eta_i\}$$
$$-(\alpha-1)[2\varphi_{kj}\varphi_i{}^h+\varphi_{ki}\varphi_j{}^h-\varphi_{ji}\varphi_k{}^h$$
$$+(k\alpha-\alpha+1)\{(\delta_k{}^h\eta_j-\delta_j{}^h\eta_k)\eta_i-(\xi^h g_{ki}-\eta_i\delta_k{}^h)\eta_j$$
$$+(\xi^h g_{ji}-\eta_i\delta_j{}^h)\eta_k\}]-(\alpha-1)^2(\delta_k{}^h\eta_j-\delta_j{}^h\eta_k)\eta_i\,.$$

Transvecting (4.5) with η_h, we have

(4.6) $$R_{kji}{}^h\eta_h=\{k\alpha-(\alpha-1)(k\alpha-\alpha+1)\}(\eta_k g_{ji}-\eta_j g_{ki})\,.$$

Comparing (4.3) with (4.6), we have

(4.7) $$k=1\,,$$

where we assume that the φ-transformation is non-isometric. Tnus we have

THEOREM 4.1. *If a K-contact Riemannian manifold admitting a non-isometric φ-transformation is of constant curvature k with respect to $\bar{g}_{ji}$, then $k=1$.*

Substituting (4.7) in (4.3) and (4.4), we have

(4.8) $$R_{kji}{}^h\eta_h=\eta_k g_{ji}-\eta_j g_{ki}\,,$$

(4.8)′ $$\nabla_i\varphi_{kj}=\eta_k g_{ji}-\eta_j g_{ki}\,,$$

respectively. Hence our K-contact Riemannian manifold is Sasakian. Substituting (4.8)′ in (4.2), we have

(4.9) $$R_{kjih}=\alpha(g_{kh}g_{ji}-g_{jh}g_{ki})+(1-\alpha)(2\varphi_{kj}\varphi_{ih}+\varphi_{ki}\varphi_{jh}$$
$$-\varphi_{ji}\varphi_{kh}-\eta_i\eta_k g_{jh}+\eta_j\eta_i g_{kh}-\eta_j\eta_h g_{ki}+\eta_k\eta_h g_{ji})\,.$$

Thus our manifold is a Sasakian manifold with constant φ-holomorphic sectional curvature $4\alpha-3$.

Conversely, if our manifold admitting a φ-transformation is assumed to be a Sasakian manifold with constant φ-holomorphic sectional curvature $4\alpha-3$, then (2.6) can be written as

$$\bar{R}_{kji}{}^h=\alpha(\delta_k{}^h g_{ji}-\delta_j{}^h g_{ki})+\alpha(\alpha-1)(\delta_k{}^h\eta_j-\delta_j{}^h\eta_k)\eta_i\,.$$

By virtue of $(2.1)_{4)}$ and the above equation, we get

$$\bar{R}_{kji}{}^h=\delta_k{}^h g_{ji}-\delta_j{}^h\bar{g}_{ki}\,.$$

Thus we have

THEOREM 4.2. *If a K-contact Riemannian manifold M admitting a non-isometric φ-transformation is of constant curvature with respect to $\bar{g}_{ji}$, then its curvature is equal to 1 and M is a Sasakian manifold of constant φ-holomorphic sectional curvature $4\alpha-3$. Conversely, if a Sasakian manifold M admitting a φ-transformation is of constant φ-holomorphic sectional curvature $4\alpha-3$, then M is of constant curvature 1 with respect to $\bar{g}_{ji}$.*

From (2.6), we have

$$\bar{R}_{ji}=R_{ji}-2(\alpha-1)g_{ji}+2(\alpha-1)(n\alpha+n+1)\eta_j\eta_i\,, \tag{4.10}$$

where the tensor fields $\bar{R}_{ji}$ and R_{ji} are the Ricci tensor fields of $\bar{g}_{ji}$ and g_{ji} respectively.

We assume that our manifold is an Einstein manifold with respect to $\bar{g}_{ji}$, that is, that

$$\bar{R}_{ji}=a\bar{g}_{ji}\,, \tag{4.11}$$

where a is constant ([5]). Then by virtue of $(2.1)_{4)}$ and (4.11), we have, from (4.10),

$$R_{ji}=2(n\alpha+\alpha-1)g_{ji}-2(\alpha-1)(n+1)\eta_j\eta_i\,, \tag{4.12}$$

and hence we see that our K-contact Riemannian manifold is an η-Einstein manifold.

Conversely, if we assume that the K-contact Riemannian manifold is an η-Einstein manifold and $\dim M>3$, then we have

$$R_{ji}=bg_{ji}+(2n-b)\eta_j\eta_i\,. \tag{4.13}$$

Substituting (4.13) in (4.10), we have

$$\bar{R}_{ji}=\frac{b-2(\alpha-1)}{\alpha}\bar{g}_{ji}+\frac{2n\alpha-b+2\alpha-2}{\alpha}\bar{\eta}_j\bar{\eta}_i\,. \tag{4.14}$$

From (4.14), if the constant b satisfies the following relation:

$$b=2(n\alpha+\alpha-1)\,, \tag{4.15}$$

then the manifold is an Einstein manifold with respect to $\bar{g}_{ji}$ and the Ricci tensor field of g_{ji} is given by (4.12). Thus we have

THEOREM 4.3. *If a K-contact Riemannian manifold M admitting a φ-transformation is an Einstein manifold with respect to $\bar{g}_{ji}$, then M is an η-Einstein manifold. Conversely, if a K-contact Riemannian manifold M admitting a φ-transformation is an η-Einstein manifold and satisfies the relation* (4.15), *then M is an Einstein manifold with respect to $\bar{g}_{ji}$, where we assume that* $\dim M>3$.

At last, we assume that our K-contact Riemannian manifold is conformally flat with respect to $\bar{g}_{ji}$ ([5]), that is, that

$$\bar{R}_{kji}{}^{h}=\frac{1}{2n-1}(\delta_k^h\bar{R}_{ji}-\delta_j^h\bar{R}_{ki}+\bar{R}_k{}^h\bar{g}_{ji}-\bar{R}_j{}^h\bar{g}_{ki}) -\frac{\bar{R}}{2n(2n-1)}(\delta_k^h\bar{g}_{ji}-\delta_j^h\bar{g}_{ki})\,, \tag{4.16}$$

where $\bar{R}$ denotes the scalar curvature of $\bar{g}_{ji}$. Then we have

$$(4.17)\qquad \bar{R}_{kji}{}^h=\frac{1}{2n-1}(\delta_k{}^hR_{ji}-\delta_j{}^hR_{ki}+R_k{}^hg_{ji}-R_j{}^hg_{ki})$$
$$-\frac{R+6n(\alpha-1)}{2n(2n-1)}(\delta_k{}^hg_{ji}-\delta_j{}^hg_{ki})$$
$$+\frac{2(n+1)(\alpha-1)}{2n-1}(\eta_kg_{ji}-\eta_jg_{ki})\xi^h$$
$$+\frac{\alpha-1}{2n-1}(R_k{}^h\eta_j-R_j{}^h\eta_k)\eta_i$$
$$+\frac{(\alpha-1)\{2n(2n\alpha+2n-\alpha+3)-R\}}{2n(2n-1)}(\delta_k{}^h\eta_j-\delta_j{}^h\eta_k)\eta_i,$$

where we used the following equations:

$$\bar{R}_j{}^i=\frac{1}{\alpha}R_j{}^i-\frac{2(\alpha-1)}{\alpha}\delta_j{}^i+\frac{2(\alpha-1)(n+1)}{\alpha}\eta_j\xi^i$$

and

$$\bar{R}=\frac{1}{\alpha}R-\frac{2n(\alpha-1)}{\alpha}.$$

Substituting (4.17) in (2.6), we have

$$(4.18)\qquad R_{kji}{}^h=\frac{1}{2n-1}\{\delta_k{}^hR_{ji}-\delta_j{}^hR_{ki}+R_k{}^hg_{ji}-R_j{}^hg_{ki}$$
$$+2(\alpha-1)(n+1)(\eta_kg_{ji}-\eta_jg_{ki})\xi^h+(\alpha-1)(R_k{}^h\eta_j$$
$$-R_j{}^h\eta_k)\eta_i\}-\frac{R+6n(\alpha-1)}{2n(2n-1)}(\delta_k{}^hg_{ji}-\delta_j{}^hg_{ki})$$
$$+\frac{(\alpha-1)(8n^2+4n-R)}{2n(2n-1)}(\delta_k{}^h\eta_j-\delta_j{}^h\eta_k)\eta_i$$
$$-(\alpha-1)(2\varphi_{kj}\varphi_i{}^h+\varphi_{ki}\varphi_j{}^h-\varphi_{ji}\varphi_k{}^h-\eta_i\nabla^h\varphi_{kj}$$
$$+\eta_j\nabla_k\varphi_i{}^h-\eta_k\nabla_j\varphi_i{}^h).$$

Transvecting (4.18) with $\xi^k\eta_h$, we have

$$(4.19)\qquad R_{ji}=\frac{R-2n}{2n}g_{ji}+\Big(2n-\frac{R-2n}{2n}\Big)\eta_j\eta_i.$$

Substituting (4.19) in (4.18) and transvecting η_h with the equation thus obtained, we can see that the manifold is Sasakian. Then the curvature tensor field $R_{kji}{}^h$ has the form

$$(4.20)\qquad R_{kji}{}^h=\frac{R-2n(3\alpha-1)}{2n(2n-1)}(\delta_k{}^hg_{ji}-\delta_j{}^hg_{ki})$$
$$+\frac{-R+2n(3\alpha+2n-2)}{2n(2n-1)}\{(\delta_k{}^h\eta_j-\delta_j{}^h\eta_k)\eta_i$$
$$+(\eta_kg_{ji}-\eta_jg_{ki})\xi^h\}-(\alpha-1)(2\varphi_{kj}\varphi_i{}^h+\varphi_{ki}\varphi_j{}^h-\varphi_{ji}\varphi_k{}^h).$$

Conversely, we assume that for a Sasakian manifold admitting a φ-transformation its curvature tensor field $R_{kji}{}^h$ constructed from g_{ji} is given by (4.20). Then by a straightfoward calculation we can see that our manifold is of conformally flat with respect to $\bar{g}_{ji}$. Thus we have

THEOREM 4.4. *A necessary and sufficient condition for a K-contact Riemannian manifold admitting a φ-transformation to be conformally flat with respect to $\bar{g}_{ji}$ is that our manifold is a Sasakian manifold and the curvature tensor field of g_{ji} is given by* (4.20).

From Theorem 4.2 and (4.20), we have

COROLLARY 4.5. *If the scalar curvature of g_{ji} satisfies the following equation:*

$$R=2n(2n\alpha+2\alpha-1),$$

and if the Riemannian mauifold M with $\bar{g}_{ji}$ is conformally flat, then the manifold is of constant curvature 1 *with respect to $\bar{g}_{ji}$, where we must assume that our φ-transformation is non-isometric.*

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YAMAGATA UNIVERSITY,
YAMAGATA, JAPAN