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ON THE EMBEDDING OF THE CAYLEY PLANE INTO THE EXCEPTIONAL LIE GROUP OF TYPE F_4

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§ 1. Introduction.

The embedding map φ of the Cayley plane Π into the exceptional Lie group of type $\boldsymbol{F}_4$ (this group will be denoted as $\boldsymbol{F}_4$) is explicitely constructed by I. Yokota [4]. First we study some properties of φ and show that $\varphi(A)$, $A\in\Pi$, is a reflection map in the exceptional Jordan algebra $\mathfrak{J}$. Hence, having an idea of the reflection, we consider the structure of $\boldsymbol{F}_4$. In § 4, motivated by the fact that $\boldsymbol{F}_4/\mathrm{Spin}(9)$ is homeomorphic to Π, we show that any element of $\boldsymbol{F}_4$ can be decomposed into the factors of $\varphi(A)$ and Spin(9). We also show that $\boldsymbol{F}_4=\mathrm{Aut}\,(\Pi, *)$ which means that a non-singular linear transformation α of $\mathfrak{J}$ belongs to $\boldsymbol{F}_4$ if and only if α preserves the reflection $*$ on Π.

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§ 2. Preliminaries.

Let $\mathfrak{C}$ be the Cayley algebra over the field $\boldsymbol{R}$ of real numbers and let $\mathfrak{J}$ be the set of all 3×3 Hermitian matrices

$$X=X(\xi, u)=\begin{pmatrix} \xi_1 & u_3 & \bar{u}_2 \\ \bar{u}_3 & \xi_2 & u_1 \\ u_2 & \bar{u}_1 & \xi_3 \end{pmatrix}$$

with coefficients in $\mathfrak{C}$. Define the Jordan product $\circ$ in $\mathfrak{J}$ by $X\circ Y=\frac{1}{2}(XY+YX)$. And define the trace, the inner product and the triple inner product by

$$\begin{aligned} &\mathrm{tr}\,(X)=\xi_1+\xi_2+\xi_3 && \text{for } X=X(\xi, u)\in\mathfrak{J} \\ (1)\qquad &(X, Y)=\mathrm{tr}\,(X\circ Y) && \\ &\mathrm{tr}\,(X, Y, Z)=(X\circ Y, Z) && \text{for } X, Y, Z\in\mathfrak{J}. \end{aligned}$$

A non-singular linear transformation α of $\mathfrak{J}$ is said to be an automorphism of $\mathfrak{J}$ if $\alpha(X\circ Y)=\alpha X\circ\alpha Y$ for $X, Y\in\mathfrak{J}$. The group $\boldsymbol{F}_4$ of all automorphisms of $\mathfrak{J}$

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is a 52-dimensional exceptional simple Lie group of type $\boldsymbol{F}_4$. Any element of $\boldsymbol{F}_4$ makes each of (1) invariant and $\boldsymbol{F}_4$ has two subgroups Spin(9), Spin(8) defined by the sets of all elements of $\boldsymbol{F}_4$ which make E_1, E_1 and E_2 invariant respectively, where E_i is the element in Π and is the matrix with $\xi_i=1$ and all remaining terms are zero. The homogeneous space $\boldsymbol{F}_4/\mathrm{Spin}(9)$ is homeomorphic to the Cayley plane Π which is composed of the element X of $\mathfrak{J}$ with $X^2=X$ and $\mathrm{tr}\,(X)=1$.

We take ε^{16} as the space of pairs of octanion numbers (x, y) with $|x|^2+|y|^2<1$ and ε^8 as the space of octanion numbers x with $|x|<1$. The cellular decomposition of Π is given in [3]: Π has three disjoint cells e^0, e^8, e^{16} where e^0 is the point E_3 and e^8, e^{16} respectively are the sets of all points X such that $X=X(a_i\bar{a}_j)$ with $a_1=0$, $a_2=(1-|x|^2)^{1/2}$, $a_3=x$ for $x\in\varepsilon^8$ and $X=X(a_i\bar{a}_j)$ with $a_1=(1-|x|^2-|y|^2)^{1/2}$, $a_2=x$, $a_3=y$ for $(x, y)\in\varepsilon^{16}$. The cellular map g of the closure of ε^8 into Π is defined by $g(x)=X(a_i\bar{a}_j)$ with $a_1=0$, $a_2=(1-|x|^2)^{1/2}$, $a_3=x$. And the other cellular map f of the closure of ε^{16} into Π is defined similarly. The line $\bar{e}^8$ is the union of the cells e^0, e^8 and is homeomorphic to a 8-dimensional sphere S^8, composed of the element (ξ, u) of $\boldsymbol{R}\times\mathfrak{C}$ with $\left(\xi-\frac{1}{2}\right)^2+|u|^2=\frac{1}{4}$, by the correspondence of the element $X=(a_i\bar{a}_j)$ of $\bar{e}^8$ to the element $((a_2)^2, a_3\bar{a}_2)$ of $\boldsymbol{R}\times\mathfrak{C}$. If we define the Hopf map ν of the boundary $\mathrm{Bd}\,(\varepsilon^{16})$ of ε^{16} onto the sphere S^8 by $\nu(x, y)=(|x|^2, y\bar{x})$, then map η, restricted f to $\mathrm{Bd}\,(\varepsilon^{16})$, may be also regarded as the Hopf map because we can have a following commutative diagram.

$$\begin{tikzcd}
\mathrm{Bd}\,(\varepsilon^{16}) \arrow[r, "\eta"] \arrow[dr, "\nu"'] & \bar{e}^8 \arrow[d, "\approx \text{homeo}."] \\
& S^8
\end{tikzcd}$$

§ 3. Embedding map $\varphi: \Pi \to \boldsymbol{F}_4$.

The map φ of Π into $\boldsymbol{F}_4$ is defined in [4] such that

$$\begin{aligned}\varphi(A)X&=16A\times(A\times X)+4A\circ X-3X\\&=X-4A\circ X+4A(A, X)\qquad \text{for } A\in\Pi \text{ and } X\in\mathfrak{J},\end{aligned}$$

where $2X\times Y=2X\circ Y-\mathrm{tr}\,(X)Y-\mathrm{tr}\,(Y)X+(\mathrm{tr}\,(X)\,\mathrm{tr}\,(Y)-(X, Y))E$ for $X, Y\in\mathfrak{J}$ and for the unit matrix E.

For given $A\in\Pi$, let S_A be a $\boldsymbol{R}$-vector space consists of the element X of $\mathfrak{J}$ with $A\circ X=A(A, X)$. Then it holds that $Z\in\mathfrak{J}$ belongs to the space S_A if and only if $(A\circ X, Z)=(A, X)(A, Z)$ for $X\in\mathfrak{J}$.

PROPOSITION 3.1.
(1) *$\varphi(A)$ is a reflection map across S_A.*

(2) *If* $\alpha\in \boldsymbol{F}_4$ *and* $A\in \Pi$, $\alpha\varphi(A)\alpha^{-1}=\varphi(\alpha A)$.
(3) *For* $A\in \Pi$, $\varphi^2(A)=1$ (1: *identity map in* $\mathfrak{J}$).
(4) *The group* $\boldsymbol{F}_4$ *is generated by* $\varphi(A)$, $A\in\Pi$.

REMARK. (1) This map $\varphi(A)$ is also obtained as the Freudenthal's Perspectivity $\Pi^{\kappa}_{A,B}$ with $A=B$ and $\kappa=-1$ [1, (11.3.3)]. (2) The Cayley plane Π is a symmetric space relative to φ (cf. O. Loos [2]). (3) If the domain of φ be extended to $\mathfrak{J}$, it holds that the element X of $\mathfrak{J}$ with $\varphi(X)\in \boldsymbol{F}_4$ and $X\neq 0$ exists in Π (the converse of [4]).

§ 4. Main results.

This section is devoted to show that any element of $\boldsymbol{F}_4$ can be decomposed into the factors of $\varphi(A)$ and Spin(9), and that $\boldsymbol{F}_4=\mathrm{Aut}\,(\Pi, *)$.

Let $S^{15}=\left\{(x, y)\in\mathfrak{C}\times\mathfrak{C}\,\middle|\,|x|^2+|y|^2=\frac{1}{2}\right\}(\subset\varepsilon^{16})$, $\bar{\eta}=f\,|\,S^{15}$ and $e^{16}_*=e^{16}-\{\bar{\eta}(S^{15})\cup E_1\}$, where f is the cellular map of ε^{16} defined in § 2. Then $\bar{\eta}$ is a homeomorphism of S^{15}.

PROPOSITION 4.1. *For* $B\in\Pi$,

(1) *If* $B\notin\bar{e}^8\cup\{E_1\}$, *there exist exactly two* A_1, A_2 *in* e^{16}_* *such that* $\varphi(A_1)E_1=B$, $\varphi(A_2)E_1=B$.

(2) *If* $B\in\bar{e}^8$, *the set* $\{A\in\Pi\,|\,\varphi(A)E_1=B\}$ *is homeomorphic to a 7-dimensional sphere and* $\bar{\eta}(S^{15})=\bigcup_B\{A\in\Pi\,|\,\varphi(A)E_1=B\}$.

(3) *If* $B=E_1$, *then* $\bar{e}^8\cup\{E_1\}=\{A\in\Pi\,|\,\varphi(A)E_1=E_1\}$.

Proof. Let $A=A(y_i\bar{y}_j)\in\Pi$, $y_1=(1-|y_2|^2-|y_3|^2)^{1/2}$, and put $\lambda=y_1{}^2$. Then

$$\varphi(A)E_1=\begin{pmatrix}(1-2\lambda)^2 & (-2+4\lambda)y_1\bar{y}_2 & (-2+4\lambda)y_1\bar{y}_3\\ (-2+4\lambda)y_2\bar{y}_1 & 4\lambda|y_2|^2 & 4\lambda y_2\bar{y}_3\\ (-2+4\lambda)y_3\bar{y}_1 & 4\lambda y_3\bar{y}_2 & 4\lambda|y_3|^2\end{pmatrix}.$$

Hence, for $B=B(x_i\bar{x}_j)\in\Pi$, $x_1=(1-|x_2|^2-|x_3|^2)^{1/2}$, the element A of Π with $\varphi(A)E_1=B$ can be given as:

(1) If $B\notin\bar{e}^8\cup\{E_1\}$, we have

$$A_1=A(y_i\bar{y}_j),\qquad y_1=\frac{\sqrt{1-x_1}}{\sqrt{2}}\qquad y_2=\frac{-x_2}{\sqrt{2}\sqrt{1-x_1}}\qquad y_3=\frac{-x_3}{\sqrt{2}\sqrt{1-x_1}},$$

$$A_2=A(y_i\bar{y}_j),\qquad y_1=\frac{\sqrt{1+x_1}}{\sqrt{2}}\qquad y_1=\frac{x_2}{\sqrt{2}\sqrt{1+x_1}}\qquad y_3=\frac{x_3}{\sqrt{2}\sqrt{1+x_1}}.$$

That A_1, A_2 exist in e^{16}_* can be seen from the fact that $A=A(y_i\bar{y}_j)\in\Pi$ exists in e^{16}_* if and only if $y_1\neq 1, \frac{1}{\sqrt{2}}$.

(2) If $B\in\bar{e}^8$, the solution is affected by the attaching of $\mathrm{Bd}\,(\varepsilon^{16})$ to $\bar{e}^8$ by

the Hopf map η. Since x_1 in B is 0, $\lambda=\frac{1}{2}$. So A is necessary to be one of the form $A=A(y_i\bar{y}_j)$, $y_1=\frac{1}{\sqrt{2}}$. Hence we have $\{A\in\Pi\mid\varphi(A)E_1=B\}\subset\bar{\eta}(S^{15})$ because $(y_2, y_3)\in S^{15}$ and $A=\bar{\eta}(y_2, y_3)$. Define a homeomorphism $h: S^{15}\to\mathrm{Bd}(\varepsilon^{16})$ by $h(x, y)=(\sqrt{2}x, \sqrt{2}y)$ and define a map H of $\bar{\eta}(S^{15})$ onto $\bar{e}^8$ by $H(A)=\varphi(A)E_1$ for $A\in\bar{\eta}(S^{15})$. Then we obtain the following commutative diagram.

$$\begin{tikzcd} S^{15} \arrow[r, "\bar{\eta}"] \arrow[d, "h"'] & \bar{\eta}(S^{15}) \arrow[d, "H"] \\ \mathrm{Bd}(\varepsilon^{16}) \arrow[r, "\eta"] & \bar{e}^8 \end{tikzcd}$$

It follows that H is the Hopf map. Therefore the inverse-image of H is a 7-dimensional sphere. The latter assertion is evident from the diagram.

(3) If $B=E_1$, it is easy to solve $\varphi(A)E_1=E_1$.

We shall use Lemma 4.2 and Proposition 4.3 for the decomposition of any element of Spin (9).

LEMMA 4.2. *For $\alpha\in\boldsymbol{F}_4$, the following two conditions are equivalent.* (1) $\alpha\in$ Spin(9), (2) αE_2, $\alpha E_3\in\bar{e}^8$.

Let $S^7=\left\{x\in\mathfrak{C}\,\middle|\,|x|=\frac{1}{\sqrt{2}}\right\}$, $\omega=g|S^7$ and $e^8_*=e^8-\{\omega(S^7)\cup E_2\}$, where g is the cellular map of ε^8 defined in § 2. Then ω is a homeomorphism of S^7.

PROPOSITION 4.3. *For $B\in\bar{e}^8$,*

(1) *If $B\neq E_2$, E_3, there exist exactly two A_1, A_2 in e^8_* such that $\varphi(A_1)E_2=B$, $\varphi(A_2)E_2=B$.*

(2) *If $B=E_3$, $\omega(S^7)=\{A\in\bar{e}^8\mid\varphi(A)E_2=E_3\}$.*

(3) *If $B=E_2$, $\{E_2, E_3\}=\{A\in\bar{e}^8\mid\varphi(A)E_2=E_2\}$.*

Proof. The proof is similar to that of Proposition 4.1. If $B\neq E_2$, E_3, for $B=B(x_i\bar{x}_j)\in\bar{e}^8$ with $x_2=(1-|x_3|^2)^{1/2}$, the element A of $\bar{e}^8$ with $\varphi(A)E_2=B$ can be given as

$$A_1=A(y_i\bar{y}_j),\qquad y_1=0\qquad y_2=\frac{\sqrt{1-x_2}}{\sqrt{2}}\qquad y_3=\frac{-x_3}{\sqrt{2}\sqrt{1-x_2}},$$

$$A_2=A(y_i\bar{y}_j),\qquad y_1=0\qquad y_2=\frac{\sqrt{1+x_2}}{\sqrt{2}}\qquad y_3=\frac{x_3}{\sqrt{2}\sqrt{1+x_2}}.$$

THEOREM 4.4. *For $\alpha\in\boldsymbol{F}_4$,*

(1) *If $\alpha\notin$ Spin (9) and $\alpha E_1\notin\bar{e}^8$, there exist exactly two pairs (A_1, β), (A_2, γ) in $e^{16}_*\times$Spin (9) such that $\alpha=\varphi(A_1)\beta$, $\alpha=\varphi(A_2)\gamma$.*

(2) *If $\alpha\notin$Spin (9) and $\alpha E_1\in\bar{e}^8$, the set $\{A\in\Pi\mid\alpha=\varphi(A)\beta$ for some $\beta\in$ Spin(9)$\}$ is homeomorphic to a 7-dimensional sphere.*

(3) *If $\alpha\in$ Spin (9), $\alpha\notin$ Spin (8) and $\alpha E_2\neq E_3$, there exist exactly two pairs*

(A_1, β), (A_2, γ) *in* $e^8_* \times \mathrm{Spin}(8)$ *such that* $\alpha=\varphi(A_1)\beta$, $\alpha=\varphi(A_2)\gamma$.

(4) *If* $\alpha\in\mathrm{Spin}(9)$, $\alpha\notin\mathrm{Spin}(8)$ *and* $\alpha E_2=E_3$, $\omega(S^7)=\{A\in\bar{e}^8 \mid \alpha=\varphi(A)\beta$ *for some* $\beta\in\mathrm{Spin}(8)\}$.

(5) *If* $\alpha\in\mathrm{Spin}(8)$, *for* E_i, *there exists* β_i *in* $\mathrm{Spin}(8)$ *such that* $\alpha=\varphi(E_i)\beta_i$ $(i=1, 2, 3)$.

Proof. This Theorem can be proved by Proposition 4.1, 4.3. For instance, in case of (1), if we put $\alpha E_1=B$, we can get A_i in Π such that $\varphi(A_i)E_1=B$ $(i=1, 2)$ by Proposition 4.1 (1). Hence, put $\beta=\varphi(A_1)\alpha$ and $\gamma=\varphi(A_2)\alpha$, then $\beta, \gamma \in\mathrm{Spin}(9)$. Therefore we have exactly two pairs (A_1, β), (A_2, γ) in $e^{16}_*\times\mathrm{Spin}(9)$ such that $\alpha=\varphi(A_1)\beta$, $\alpha=\varphi(A_2)\gamma$.

Now we study the structure of $\boldsymbol{F}_4$ with an idea of reflection. Define Aut $(\Pi, *)$ be the set of non-singular linear transformations α of $\mathfrak{J}$ such that $\alpha(A*B)=\alpha A*\alpha B$ for $A, B\in\Pi$, where this $*$ is defined in $\mathfrak{J}$ by $X*Y=\varphi(X)Y=Y-4X\circ Y+4X(X, Y)$ for $X, Y\in\mathfrak{J}$. Then our aim is to prove $\boldsymbol{F}_4=\mathrm{Aut}(\Pi, *)$.

LEMMA 4.5. *Let* α *be a non-singular linear transformation of* $\mathfrak{J}$, *then the following two conditions are equivalent.* (1) $\alpha\in\boldsymbol{F}_4$, (2) $\alpha(A\circ B)=\alpha A\circ\alpha B$ *for* $A, B\in\Pi$.

LEMMA 4.6. *For* $A\in\Pi$, *there exists a sequence* $\{B_n\}$ *such that* $B_n\in\Pi$, $B_n\neq A$ *for every* $n\in\boldsymbol{N}$ *and* $\lim B_n=A$.

THEOREM 4.7. $\boldsymbol{F}_4=\mathrm{Aut}(\Pi, *)$.

Proof. Let $\alpha\in\boldsymbol{F}_4$, then we have $\alpha(A*B)=\alpha(\varphi(A)B)=\alpha\varphi(A)\alpha^{-1}\alpha B=\varphi(\alpha A)\alpha B=\alpha A*\alpha B$ for $A, B\in\Pi$. Conversely, assume $\alpha\in\mathrm{Aut}(\Pi, *)$, then for $A, B\in\Pi$, we get $\alpha(A*B)=\alpha A*\alpha B$ and $\alpha(B*A)=\alpha B*\alpha A$. Namely, we have $-\alpha(A\circ B)+\alpha A(A, B)=-\alpha A\circ\alpha B+\alpha A(\alpha A, \alpha B)$ and $-\alpha(B\circ A)+\alpha B(B, A)=-\alpha B\circ\alpha A+\alpha B(\alpha B, \alpha A)$. From these equations, we see that $\alpha(A-B)(A, B)=\alpha(A-B)(\alpha A, \alpha B)$. Hence, if $A\neq B$, we obtain $(A, B)=(\alpha A, \alpha B)$ because α is a non-singular linear transformation of $\mathfrak{J}$. Next, let A be fixed arbitrary, then we prove that $(\alpha A, \alpha A)=1$. Define two maps $f, g: \Pi\to\boldsymbol{R}$ by $f(B)=(A, B)$, $g(B)=(\alpha A, \alpha B)$ for $B\in\Pi$. Then, by the continuity of α and of the inner product, both f, g are continuous. For this A, let $\{B_n\}$ satisfy the condition of Lemma 4.6, then $\lim B_n=A$ and $f(B_n)=g(B_n)$. Hence, we have $(\alpha A, \alpha A)=g(A)=g(\lim B_n)=\lim g(B_n)=\lim f(B_n)=f(\lim B_n)=f(A)=1$. So, we get that $(A, B)=(\alpha A, \alpha B)$ for $A, B\in\Pi$. From this fact and $-\alpha(A\circ B)+\alpha A(A, B)=-\alpha A\circ\alpha B+\alpha A(\alpha A, \alpha B)$, we obtain $\alpha(A\circ B)=\alpha A\circ\alpha B$. Since A, B are arbitrary, by Lemma 4.5, we conclude that $\alpha\in\boldsymbol{F}_4$.

REMARK. Define a new multiplication $*$ in the Lie group $\boldsymbol{F}_4$ by $\alpha*\beta=\alpha\beta^{-1}\alpha$ for $\alpha, \beta\in\boldsymbol{F}_4$, then we can know that $\boldsymbol{F}_4=\varphi(\Pi)*\mathrm{Spin}(9)$ by Theorem 4.4, [5, Theorem 7.2] and also know that φ is a representation of (projective space $\Pi, *$) into (Lie group $\boldsymbol{F}_4, *$).

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