S. TIMMANN
KŌDAI MATH. SEM. REP.
28 (1976), 104—109

A BOUND FOR THE NUMBER OF AUTOMORPHISMS OF A FINITE RIEMANN SURFACE

BY STEFFEN TIMMANN

1. Introduction.

An automorphism of a Riemann surface X is a $1-1$ conformal mapping of X onto itself. Let $N(g)$ be the order of the largest group of automorphisms a compact Riemann surface of genus g can admit. Similarly we denote by $N(g, k)$ the maximal order of an automorphism group of a finite (i.e. compact bordered) Riemann surface of genus g with k boundary components; in particular $N(g, 0)=N(g)$. In this paper the following bound will be proved:

$$N(g, k)\geqq\max(6, (g/2)^{1/2}) \qquad \text{for all} \quad g,\ k\geqq 0,$$

and 6 cannot be replaced by a larger constant. This improves the lower bound $N(g, k)\geqq 4$ given by R. Tsuji [8].

2. Known results.

A. Hurwitz [3] proved $N(g)\leqq 84(g-1)$ for $g\geqq 2$ and A. M. Macbeath [5] showed that this bound is attained for infinitely many values of g. R. D. M. Accola [1] and C. Maclachlan [6] proved independently that $N(g)\geqq 8(g+1)$ and this lower bound is also exact for an infinite number of g's.

Automorphisms of finite Riemann surfaces were studied by M. Heins, K. Oikawa, R. Tsuji, T. Kato and others. M. Heins (for the case $g=0$) and K. Oikawa proved that $N(g, k)$ equals the maximal order of an automorphism group of a compact Riemann surface X_g of genus g being punctured in k distinct points ([2], [7]). They showed that a finite Riemann surface $X_{g,k}$ of genus g with k boundary components can be imbedded into a compact Riemann surface X_g in such a manner that the automorphisms of $X_{g,k}$ can be continued to automorphisms of the punctured surface $X'_{g,k}=X_g-\{P_1, \cdots, P_k\}$ where the P_j are suitably chosen distinct points of X_g. On the other hand if we endow the Riemann surface X_g $(g>1)$ with the Poincaré metric then the automorphisms of X_g are isometries. Hence there are discs D_j with equal radius and midpoint P_j such that the automorphisms of $X'_{g,k}$ are also automorphisms of the finite Riemann surface $X_{g,k}=X'_{g,k}-\bigcup D_j$. The cases $g=0, 1$ can be treated similarly.

Received Sept. 2, 1975.

It follows that $N(g, k)$ is the order of the largest automorphism group G operating on a compact Riemann surface X_g of genus g such that there are k distinct points $P_1, \cdots, P_k$ in X_g being permuted by G. The inequality $N(g, k) \leqq N(g)$ is now trivial.

For some specific values of g or k the maximal order $N(g, k)$ has been determined completely. M. Heins [2] treated the case $g=0$. Of course

$$N(0, k)=\infty \qquad \text{for} \quad k=0, 1, 2 .$$

Heins proved:

$$N(0, k)=2k \qquad \text{for} \quad k \geqq 3, \ k \neq 4, 6, 8, 12, 20 ,$$

$$N(0, 4)=12 ,$$

$$N(0, 6)=N(0, 8)=24 ,$$

$$N(0, 12)=N(0, 20)=60 .$$

Using a modification of Hurwitz' method [3] K. Oikawa [7] found the upper bound

$$N(g, k) \leqq 12(g-1)+6k \qquad \text{for} \quad 2g+k \geqq 3 . \tag{1}$$

In addition he calculated $N(1, k)$ for $k \geqq 1$ explicitly:

$$N(1, k)=\begin{cases} 6k & \text{for } k=m^2+3n^2 \\ 4k & \text{for } k=m^2+n^2 \text{ but not of the form } m^2+3n^2 \\ 3k & \text{for } k=2(m^2+3n^2) \text{ but not of the form } m^2+n^2 \\ 2k & \text{for all other } k \geqq 1 . \end{cases}$$

Hence Oikawa's bound (1) is attained for $g=1$ and infinitely many k. By looking closely at Hurwitz' proof [3] of $N(g) \leqq 84(g-1)$ one sees that Oikawa's bound is also exact for $k=12(g-1)$ and all values of g for which $N(g)=84(g-1)$. If G is an automorphism group of order $84(g-1)$ operating on X_g then X_g viewed as a covering surface of X_g/G has $12(g-1)$ branch points of order 7 (and others of order 2 and 3) and the branch points of order 7 are permuted by G.

R. Tsuji [8] studied hyperelliptic Riemann surfaces and determined $N(2, k)$ completely by showing that $N(2, k)$ is a periodic function of k with period 120 and giving a table of the first 120 values of $N(2, k)$. From his results it follows in particular that

$$N(2, 59)=6 \leqq N(2, k) \leqq 48=N(2, 6) .$$

R. Tsuji also proved the best lower bound known so far valid for all g and k:

$$N(g, k) \geqq 4 . \tag{2}$$

Actually he proved this for hyperelliptic Riemmann surfaces; so for every $g \geqq 2$, $k \geqq 0$ there is a hyperelliptic surface X_g admitting an automorphism group at least of order 4 which permutes k distinct points of X_g.

T. Kato [4] found the exact values of $N(g, k)$ for $k=1, 2, 3$ and $g\geqq 1$. He proved that

$$N(g, 1)=4g+2 \quad \text{for } g\geqq 1,$$

$$N(g, 2)=8g \quad \text{for } g\geqq 1,$$

$$N(g, 3)=\begin{cases} 12g+6 & \text{for } g=0, 1 \\ 6g+3 & \text{for } g\neq 0, 1 \text{ and } j^2+j+1\equiv 0 \pmod{2g+1} \text{ has a solution} \\ 4g+14 & \text{for } g\equiv 1 \pmod 9 \text{ and } j^2+j+1\equiv 0 \pmod{2g+1} \text{ does not have a solution} \\ 4g+6 & \text{for } g\equiv 0 \pmod 3 \text{ and } j^2+j+1\equiv 0 \pmod{2g+1} \text{ does not have a solution} \\ (24g+12)/5 & \text{for } g=2, 7 \\ 4g+2 & \text{otherwise.} \end{cases}$$

All the above results have been obtained by different methods suited for the special values of g and k. In the following paragraph two lower bounds for $N(g, k)$ valid for all g and k will be given that improve Tsuji's bound (2).

3. Two new lower bounds.

PROPOSITION 1. *$N(g, k)\geqq 6$ for all g, $k\geqq 0$ and 6 cannot be replaced by a larger constant.*

Proof. The second statement follows immediately from ([2], [7], [8])

$$N(0, 3)=N(1, 1)=N(2, 59)=6.$$

To prove the first statement it is sufficient to construct for given $g, k\geqq 0$ a compact Riemann surface X_g of genus g with an automorphism φ of order 6 permuting a set of k mutually distinct points of X_g. Because of $N(g, k)\geqq 6$ for $k=0, 1$ or $g=0, 1$ ([1], [4]) we may suppose that $k>1$, $g>1$. The following three cases will be treated parallel:

$$\text{a) } 2g=3h \qquad \text{b) } 2g+1=3h \qquad \text{c) } 2g+2=3h.$$

Of course every g can be represented in one of these forms with $h\in \boldsymbol{N}$. Let X_g be the hyperelliptic Riemann surface defined by the algebraic equation

$$\text{a) } w^2=z(z^{3h}-1) \qquad \text{b) } w^2=z(z^{3h}-1) \qquad \text{c) } w^2=z^{3h}-1$$

and define the automorphism φ by

$$\text{a) } (z, w)\longmapsto (e^{2\pi i/3}z, e^{\pi i/3}w) \qquad \text{b) } (z, w)\longmapsto (e^{2\pi i/3}z, e^{\pi i/3}w)$$

$$\text{c) } (z, w)\longmapsto (e^{2\pi i/3}z, -w).$$

Then $G:=\langle\varphi\rangle=\{\varphi^0, \cdots, \varphi^5\}$ is an automorphism group on X_g of order 6.

Every $k>1$ can be written in the form $k=6\kappa+3\varepsilon+\delta$ where κ, ε, $\delta\geqq 0$, $\varepsilon=0, 1$ and

$$\text{a)}\quad \delta=0, 1, 2 \qquad \text{b)}\quad \delta=0, 1, 2 \qquad \text{c)}\quad \delta=0, 2, 4\,.$$

Since G operates discontinuously on X_g there certainly are distinct points $P_1, \cdots, P_\kappa$ on X_g such that

$$\mathcal{P}_\kappa:=\{\varphi^m(P_j)\,|\,j=1, \cdots, \kappa\,;\, m=0, \cdots, 5\}$$

contains 6κ mutually distinct points. In each case let Q_1 contain the three points. of X_g lying over $z=1, e^{2\pi i/3}, e^{4\pi i/3}$. In the cases a) and b) let $\mathcal{R}_1$ contain the point corresponding to $z=0$ and in case c) let $\mathcal{R}_4$ contain the four points of X_g corresponding to $z=0, \infty$. Finally let $\mathcal{R}_2$ contain the two points of X_g lying over

$$\text{a)}\quad z=0, \infty \qquad \text{b)}\quad z=\infty \qquad \text{c)}\quad z=\infty$$

and define $\mathcal{P}_0$, Q_0, $\mathcal{R}_0$ as the empty set. Then in all three cases $\mathcal{P}_\kappa\cup Q_\varepsilon\cup\mathcal{R}_\delta$ is a set of $k=6\kappa+3\varepsilon+\delta$ distinct points of X_g which are permuted by the automorphism group G. This concludes the proof of Proposition 1.

Since the surfaces X_g constructed in the above proof are all hyperelliptic we have actually proved: For every $g\geqq 2$, $k\geqq 0$ there is a hyperelliptic Riemann surface X_g of genus g admitting an automorphism group G of order 6 permuting k suitably chosen points of X_g.

The following lower bound improves Proposition 1 for $g\geqq 72$.

PROPOSITION 2. *$N(g, k)>(g/2)^{1/2}$ for all g, $k\geqq 0$.*

Proof. This lower bound is trivial for $g=0, 1, 2$, hence we many suppose that $g>2$. For the proof it is sufficient to construct for any given genus $g>2$ a compact Riemann surface X_g which admits an automorphism φ of order $m>(g/2)^{1/2}$ with at least $m-1$ fixed points. Then we can find for all values of $k=\kappa m+\kappa'$, $0\leqq\kappa'\leqq m-1$, a set of k mutually distinct points of X_g (containing κ' fixed points and κ disjoint orbits of $\langle\varphi\rangle$) being permuted by φ and hence by the group $\langle\varphi\rangle$.

To construct X_g let Y_π be a compact Riemann surface of genus π lying over the Riemann sphere and slit Y_π along r disjoint segments over the real axis that contain no branch points. Take m copies of the slit surface Y_π and join them in the usual cyclic manner along the slits to give a model of X_g. The corresponding cyclic permutation of the m copies of Y_π yields an automorphism φ of X_g of order m with $2r$ fixed points (the end points of the slits). Using the Riemann-Hurwitz formula [3] one calculates the genus g of X_g:

$$g=m\pi+(r-1)(m-1)\,. \tag{3}$$

To conclude the proof of Proposition 2 one must find for given $g>2$ integers π, r, m satisfying (3) and the additional requirements

(4) $$\pi \geqq 0, \quad 2r \geqq m-1, \quad m > (g/2)^{1/2}.$$

Now every $g>2$ can be written in the form

(5) $$g=2h^2+\mu h+\nu, \quad h \geqq 1, \quad g<2(h+1)^2, \quad 0 \leqq \mu \leqq 5, \quad 0 \leqq \nu \leqq h-1.$$

Given g in the form (5),

(6) $$\pi=\nu, \quad m=h+1, \quad r=2h+\mu+1-\nu$$

is a solution of (3) fulfilling the conditions (4). This concludes the proof of Proposition 2.

I do not know whether Proposition 2 is in any sense exact. For special cases one easily obtains better bounds. For example using the hyperelliptic Riemann surface X_g defined by the algebraic equation

$$w^2=z^{2g+2}-1$$

and the group $G=\langle \varphi_1, \varphi_2 \rangle$ of order $8(g+1)$ generated by the automorphisms

$$\varphi_1: (z, w) \longmapsto (e^{\pi i/g+1}z, w)$$

$$\varphi_2: (z, w) \longmapsto (1/z, iw/z^{g+1})$$

we get

$$N(g, \nu(2g+2)) \geqq 8(g+1)$$

$$N(g, \nu(2g+2)+4) \geqq 8(g+1), \qquad \nu \geqq 0,$$

These bounds are exact for infinitely many g since by the results of Accola [1] and Maclachlan [6] $N(g, k) \leqq N(g)=8(g+1)$ for an infinite family of g's.

Many similar estimates can be found using other Riemann surfaces and automorphism groups, but the ones I came across either do by no means cover all values of g and k or do not improve the bounds given in Proposition 1 and 2.

References

[1] Accola, R. D. M.: On the number of automorphisms of a closed Riemann surface. Trans. Amer. Math. Soc., **131** (1968), 398-407.

[2] Heins, M.: On the number of 1-1 directly conformal maps which a multiply-connected plane region of finite connectivity p ($p>2$) admits onto itself. Bull. Amer. Math. Soc., **52** (1946), 454-457.

[3] Hurwitz, A.: Über algebraische Gebilde mit eindeutigen Transformationen in sich. Math. Ann., **41** (1892), 403-442.

[4] Kato, T.: On the number of automorphisms of a compact bordered Riemann surface. Kodai Math. Sem. Rep., **24** (1972), 224-233.

[5] Macbeath, A. M.: On a theorem of Hurwitz. Proc. Glasgow Math. Assoc. **5** (1961), 90-96.

[6] MACLACHLAN, C.: A bound for the number of automorphisms of a compact Riemann surface. J. London Math. Soc., **44** (1969), 265-272.

[7] OIKAWA, K.: Notes on Conformal Mappings of a Riemann Surface onto itself. Kodai Math. Sem. Rep., **8** (1956), 23-30, 115-116.

[8] TSUJI, R.: On conformal mapping of a hyperelliptic Riemann surface onto itself. Kodai Math. Sem. Rep., **10** (1958), 127-136.

INSTITUT FÜR MATHEMATIK
TECHNISCHE UNIVERSITÄT HANNOVER
D-3 HANNOVER
WELFENGARTEN 1
BUNDESREPUBLIK DEUTSCHLAND