K. YANO
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ON CONTACT CONFORMAL CONNECTIONS

To Morio Obata on his fiftieth birthday

By Kentaro Yano

§ 1. Introduction.

Let M be an n-dimensional Riemannian manifold with metric tensor g_{ji} ($n \geqq 3$). The change of metric

$$\bar{g}_{ji}=e^{2p}g_{ji}\,, \tag{1.1}$$

where p is a certain scalar function, does not change the angle between two vectors at a point of M and is called a conformal change of metric.

Corresponding to the conformal change (1.1) of metric, we have a change of Christoffel symbols, that is, a change of Riemannian connection

$$\overline{\left\{ {h \atop j\ i} \right\}}=\left\{ {h \atop j\ i} \right\}+\delta_j^h p_i+\delta_i^h p_j-g_{ji}p^h\,, \tag{1.2}$$

where p_i is the gradient of p and $p^h=p_t g^{th}$, g^{th} being contravariant components of the metric tensor. If we denote by D_k the operator of covariant differentiation with respect to $\overline{\left\{ {h \atop j\ i} \right\}}$, we have of course

$$D_k(e^{2p}g_{ji})=0\,. \tag{1.3}$$

Computing the curvature tensor $\bar{K}_{kji}{}^h$ of $\overline{\left\{ {h \atop j\ i} \right\}}$, we find

$$\bar{K}_{kji}{}^h=K_{kji}{}^h+\delta_k^h p_{ji}-\delta_j^h p_{ki}+p_k{}^h g_{ji}-p_j{}^h g_{ki}\,, \tag{1.4}$$

$K_{kji}{}^h$ being the curvature tensor of $\left\{ {h \atop j\ i} \right\}$, where

$$p_{ji}=\nabla_j p_i-p_j p_i+\frac{1}{2}p_t p^t g_{ji} \tag{1.5}$$

and $p_k{}^h=p_{kt}g^{th}$, ∇_j denoting the operator of covariant differentiation with respect to $\left\{ {h \atop j\ i} \right\}$.

If there exists in M a scalar function p such that the curvature tensor $\bar{K}_{kji}{}^h$ vanishes, then the Riemannian manifold M with the metric tensor g_{ji} is

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said to be conformally flat. In this case, we have from (1.4)

$$K_{kji}{}^h+\delta_k^h p_{ji}-\delta_j^h p_{ki}+p_k{}^h g_{ji}-p_j{}^h g_{ki}=0\,, \tag{1.6}$$

from which we have

$$p_{ji}=C_{ji}\,, \tag{1.7}$$

where

$$C_{ji}=-\frac{1}{n-2}K_{ji}+\frac{1}{2(n-1)(n-2)}Kg_{ji}\,, \tag{1.8}$$

K_{ji} and K denoting the Ricci tensor and the scalar curvature of M respectively. Substituting (1.7) into (1.6), we find

$$C_{kji}{}^h=0\,, \tag{1.9}$$

where

$$C_{kji}{}^h=K_{kji}{}^h+\delta_k^h C_{ji}-\delta_j^h C_{ki}+C_k{}^h g_{ji}-C_j{}^h g_{ki} \tag{1.10}$$

is the Weyl conformal curvature tensor and $C_k{}^h=C_{kt}g^{th}$. Thus a necessary condition for M to be conformally flat is that the Weyl conformal curvature tensor of M vanishes.

In a previous paper [3], the present author studied a complex analogue of the above and proved

THEOREM A. *In a Kaehlerian manifold with Hermitian metric tensor g_{ji} and almost complex structure tensor $F_i{}^h$, the affine connection D with components Γ_{ji}^h which satisfies*

$$D_k(e^{2p}g_{ji})=0\,,\qquad D_k(e^{2p}F_{ji})=0$$

and

$$\Gamma_{ji}^h-\Gamma_{ij}^h=-2F_{ji}q^h\,,$$

where p is a scalar function, q^h a vector field and $F_{ji}=F_j{}^t g_{ti}$, is given by

$$\Gamma_{ji}^h=\left\{ {h \atop j\ i} \right\}+\delta_j^h p_i+\delta_i^h p_j-g_{ji}p^h+F_j{}^h q_i+F_i{}^h q_j-F_{ji}q^h\,, \tag{1.11}$$

where p_i is the gradient of p and

$$p^h=p_t g^{th}\,,\qquad q_i=-p_t F_i{}^t\,,\qquad q^h=q_t g^{th}\,.$$

We have called such an affine connection a *complex conformal connection* in a Kaehlerian manifold.

THEOREM B. *If, in a real n-dimensional Kaehlerian manifold ($n\geqq 4$), there exists a scalar function p such that the complex conformal connection* (1.11) *is of zero curvature, then the Bochner curvature tensor of the manifold*

$$B_{kji}{}^h = K_{kji}{}^h + \delta_k^h L_{ji} - \delta_j^h L_{ki} + L_k{}^h g_{ji} - L_j{}^h g_{ki} + F_k{}^h M_{ji} - F_j{}^h M_{ki} + M_k{}^h F_{ji} - M_j{}^h F_{ki} - 2(M_{kj} F_i{}^h + F_{kj} M_i{}^h) \tag{1.12}$$

vanishes, where

$$L_{ji} = -\frac{1}{n+4} K_{ji} + \frac{1}{2(n+2)(n+4)} K g_{ji}, \qquad L_k{}^h = L_{kt} g^{th},$$

$$M_{ji} = -L_{jt} F_i{}^t, \qquad M_k{}^h = M_{kt} g^{th}.$$

The main purpose of the present paper is to find a contact analogue of the above.

In § 2, we state some of fundamental formulas in Sasakian manifolds to fix our notations and in § 3 we study a curvature tensor of a Sasakian manifold which corresponds to the Bochner curvature tensor in a Kaehlerian manifold.

In § 4 we introduce what we call contact conformal connections and in § 5 we study the condition for a Sasakian manifold to admit a contact conformal connection whose curvature tensor vanishes.

§ 2. Sasakian manifolds.

Let M be a $(2m+1)$-dimensional differentiable manifold of class C^∞ covered by a system of coordinate neighborhoods $\{U; x^h\}$ in which there are given a tensor field $\varphi_i{}^h$ of type (1, 1), a vector field ξ^h and a 1-form η_i satisfying

$$\varphi_j{}^i \varphi_i{}^h = -\delta_j^h + \eta_j \xi^h, \quad \varphi_i{}^h \xi^i = 0, \quad \eta_i \varphi_j{}^i = 0, \quad \eta_i \xi^i = 1, \tag{2.1}$$

where here and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, 2m+1\}$. Such a set of a tensor field φ of type (1, 1), a vector field ξ and a 1-form η is called an *almost contact structure* and a manifold with an almost contact structure an *almost contact manifold*. (Sasaki [2]).

If the set (φ, ξ, η) satisfies

$$N_{ji}{}^h + (\partial_j \eta_i - \partial_i \eta_j) \xi^h = 0, \tag{2.2}$$

where

$$N_{ji}{}^h = \varphi_j{}^t \partial_t \varphi_i{}^h - \varphi_i{}^t \partial_t \varphi_j{}^h - (\partial_j \varphi_i{}^t - \partial_i \varphi_j{}^t) \varphi_t{}^h$$

is the Nijenhuis tensor formed with $\varphi_i{}^h$ and $\partial_j = \partial / \partial x^j$, then the almost contact structure is said to be *normal* and the manifold is called a *normal almost contact manifold.*

If, in an almost contact manifold, there is given a Riemannian metric g_{ji} such that

$$g_{ts} \varphi_j{}^t \varphi_i{}^s = g_{ji} - \eta_j \eta_i, \qquad \eta_i = g_{ih} \xi^h, \tag{2.3}$$

then the almost contact structure is said to be *metric* and the manifold is called an *almost contact metric manifold.*

Comparing the first equations of (2.1) and (2.3), we see that $\varphi_{ji}=\varphi_j{}^t g_{ti}$ is skew-symmetric.

Since, in an almost contact metric manifold, we have the second equation of (2.3), we shall write η^h instead of ξ^h in the sequel.

If an almost contact metric manifold satisfies

$$\varphi_{ji}=\frac{1}{2}(\partial_j\eta_i-\partial_i\eta_j)\,, \tag{2.4}$$

then the almost contact metric structure is called a *contact structure*. A manifold with a normal contact structure is called a *Sasakian manifold*.

It is well known that in a Sasakian manifold, we have

$$\nabla_i\eta^h=\varphi_i{}^h \tag{2.5}$$

and

$$\nabla_j\varphi_i{}^h=-g_{ji}\eta^h+\delta_j^h\eta_i\,, \tag{2.6}$$

where ∇_j denotes the operator of covariant differentiation with respect to g_{ji}.

If we denote by $\mathcal{L}$ the operator of Lie derivation with respect to the vector field η^h, we have

$$\mathcal{L}g_{ji}=\nabla_j\eta_i+\nabla_i\eta_j=\varphi_{ji}+\varphi_{ij}$$

and consequently

$$\mathcal{L}g_{ji}=0\,, \tag{2.7}$$

which shows that the vector field η^h is a Killing vector field. From (2.7) we find, using formulas on Lie derivatives,

$$\mathcal{L}\left\{ {h \atop j\ i} \right\}=\nabla_j\nabla_i\eta^h+K_{kji}{}^h\eta^k=0\,, \tag{2.8}$$

$$\begin{aligned}\mathcal{L}K_{kji}{}^h=&\eta^t\nabla_tK_{kji}{}^h-K_{kji}{}^t\nabla_t\eta^h+K_{tji}{}^h\nabla_k\eta^t+K_{kti}{}^h\nabla_j\eta^t+K_{kjt}{}^h\nabla_i\eta^t\\ =&0\,,\end{aligned} \tag{2.9}$$

$$\mathcal{L}K_{ji}=\eta^t\nabla_tK_{ji}+K_{ti}\nabla_j\eta^t+K_{jt}\nabla_i\eta^t=0 \tag{2.10}$$

and

$$\mathcal{L}K=\eta^t\nabla_tK=0\,, \tag{2.11}$$

where $\left\{ {h \atop j\ i} \right\}$, $K_{kji}{}^h$, K_{ji} and K are Christoffel symbols, the curvature tensor, the Ricci tensor and the scalar curvature of M respectively.

Now from equations (2.5), (2.6) and the Ricci identity

$$\nabla_k\nabla_j\eta^h-\nabla_j\nabla_k\eta^h=K_{kjt}{}^h\eta^t\,,$$

we find

$$K_{kjt}{}^h\eta^t=\delta_k^h\eta_j-\delta_j^h\eta_k \tag{2.12}$$

or

(2.13) $$K_{kji}{}^{t}\eta_t=\eta_k g_{ji}-\eta_j g_{ki}\,,$$

from which, by contraction,

(2.14) $$K_{jt}\eta^t=2m\eta_j\,.$$

From equations (2.5), (2.6) and the Ricci identity

$$\nabla_k\nabla_j\varphi_i{}^h-\nabla_j\nabla_k\varphi_i{}^h=K_{kjt}{}^h\varphi_i{}^t-K_{kji}{}^t\varphi_t{}^h\,,$$

we find

(2.15) $$K_{kjt}{}^h\varphi_i{}^t-K_{kji}{}^t\varphi_t{}^h=-\varphi_k{}^h g_{ji}+\varphi_j{}^h g_{ki}-\delta_k^h\varphi_{ji}+\delta_j^h\varphi_{ki}\,,$$

from which, by contraction,

(2.16) $$K_{jt}\varphi_i{}^t+K_{tjis}\varphi^{ts}=-(2m-1)\varphi_{ji}\,,$$

where $\varphi^{ts}=g^{ti}\varphi_i{}^s$, g^{ti} being contravariant components of the metric tensor. Since

$$K_{tjis}\varphi^{ts}=K_{sijt}\varphi^{ts}=-K_{tijs}\varphi^{ts}\,,$$

we have from (2.16)

(2.17) $$K_{jt}\varphi_i{}^t+K_{it}\varphi_j{}^t=0\,.$$

Since

$$K_{tjis}\varphi^{ts}=\frac{1}{2}(K_{tjis}-K_{sjit})\varphi^{ts}$$
$$=-\frac{1}{2}K_{tsji}\varphi^{ts}\,,$$

we also have from (2.16)

(2.18) $$K_{tsji}\varphi^{ts}=2K_{jt}\varphi_i{}^t+2(2m-1)\varphi_{ji}\,.$$

From (2.10) and (2.17), we find

(2.19) $$\eta^t\nabla_t K_{ji}=0\,.$$

§ 3. Contact Bochner curvature tensor.

In previous papers [4, 5], we have defined the contact Bochner curvature tensor by

(3.1) $$\begin{aligned}B_{kji}{}^h=&K_{kji}{}^h+(\delta_k^h-\eta_k\eta^h)L_{ji}-(\delta_j^h-\eta_j\eta^h)L_{ki}+L_k{}^h(g_{ji}-\eta_j\eta_i)\\&-L_j{}^h(g_{ki}-\eta_k\eta_i)+\varphi_k{}^h M_{ji}-\varphi_j{}^h M_{ki}+M_k{}^h\varphi_{ji}-M_j{}^h\varphi_{ki}\\&-2(M_{kj}\varphi_i{}^h+\varphi_{kj}M_i{}^h)+(\varphi_k{}^h\varphi_{ji}-\varphi_j{}^h\varphi_{ki}-2\varphi_{kj}\varphi_i{}^h)\,,\end{aligned}$$

where

(3.2) $$L_{ji}=-\frac{1}{2(m+2)}[K_{ji}+(L+3)g_{ji}-(L-1)\eta_j\eta_i],\quad L_k{}^h=L_{kt}g^{th}\,,$$

(3.3) $$M_{ji}=-L_{jt}\varphi_i{}^t\,,\qquad M_k{}^h=M_{kt}g^{th}$$

and consequently

(3.4) $$M_{ji}=\frac{1}{2(m+2)}[K_{jt}\varphi_i{}^t-(L+3)\varphi_{ji}]$$

and

(3.5) $$L=g^{ji}L_{ji}.$$

The definition (3.2) of L_{ji} shows that L_{ji} is symmetric and (2.17) and (3.4) show that M_{ji} is skew-symmetric. From (3.2) and (3.5), we find

(3.6) $$L=-\frac{K+2(3m+2)}{4(m+1)}.$$

Transvecting (3.2) with η^i and using (2.14), we find

(3.7) $$L_{ji}\eta^i=-\eta_j.$$

From the first equation of (3.3), we find

(3.8) $$M_{ji}\eta^i=0$$

and

$$M_{jt}\varphi_i{}^t=-L_{js}\varphi_t{}^s\varphi_i{}^t$$
$$=-L_{js}(-\delta_i^s+\eta_i\eta^s),$$

from which, using (3.7),

(3.9) $$M_{jt}\varphi_i{}^t=L_{ji}+\eta_j\eta_i.$$

Now, from the definition (3.1) of the contact Bochner curvature tensor, we easily see that

(3.10) $$B_{kji}{}^h=-B_{jki}{}^h,$$

(3.11) $$B_{kji}{}^h+B_{jik}{}^h+B_{ikj}{}^h=0,$$

(3.12) $$B_{tji}{}^t=0,$$

(3.13) $$B_{kjih}=-B_{jkih},\qquad B_{kjih}=-B_{kjhi},$$

(3.14) $$B_{kjih}=B_{ihkj},$$

where $B_{kjih}=B_{kji}{}^tg_{th}$,

(3.15) $$B_{kji}{}^t\eta_t=0,$$

(3.16) $$B_{kjt}{}^h\varphi_i{}^t-B_{kji}{}^t\varphi_t{}^h=0.$$

(3.17) $$B_{kjts}\varphi^{ts}=0.$$

We also can verify by a straightforward computation that the contact Bochner curvature tensor satisfies

$$(3.18)\qquad \nabla_t B_{kji}{}^t = -2m\Big[\nabla_k L_{ji} - \nabla_j L_{ki} + \eta_k(M_{ji}+\varphi_{ji}) - \eta_j(M_{ki}+\varphi_{ki}) - 2(M_{kj}+\varphi_{kj})\eta_i - \frac{1}{2(m+2)}(\varphi_k{}^t\varphi_{ji} - \varphi_j{}^t\varphi_{ki} - 2\varphi_{kj}\varphi_i{}^t)(\nabla_t L)\Big].$$

(See Matsumoto and Chūman [1], Yano [5]).

§ 4. Contact conformal connections.

We consider an affine connection D in a Sasakian manifold M and denote by Γ_{ji}^h the components of the affine connection and by D_j the operator of covariant differentiation with respect to Γ_{ji}^h.

We assume that the affine connection D satisfies

$$(4.1)\qquad D_k(e^{2p}g_{ji}) = 2e^{2p}p_k\eta_j\eta_i$$

and the torsion tensor of D satisfies

$$(4.2)\qquad \Gamma_{ji}^h - \Gamma_{ij}^h = -2\varphi_{ji}u^h,$$

where p is a certain scalar function, $p_i=\partial_i p$ and u^h is a certain vector field.

From (4.1) we have

$$(4.3)\qquad 2e^{2p}p_k g_{ji} + e^{2p}\partial_k g_{ji} - \Gamma_{kj}^t e^{2p} g_{ti} - \Gamma_{ki}^t e^{2p} g_{jt} = 2e^{2p}p_k\eta_j\eta_i.$$

We can solve (4.2) and (4.3) with respect to Γ_{ji}^h and obtain

$$(4.4)\qquad \Gamma_{ji}^h = \left\{ {h \atop j\ i} \right\} + (\delta_j^h - \eta_j\eta^h)p_i + (\delta_i^h - \eta_i\eta^h)p_j - (g_{ji}-\eta_j\eta_i)p^h + \varphi_j{}^h u_i + \varphi_i{}^h u_j - \varphi_{ji}u^h,$$

where

$$p^h = p_t g^{th}, \qquad u^h = u_t g^{th}.$$

Using (4.4) we compute the covariant derivative $D_j\varphi_i{}^h$ of $\varphi_i{}^h$ with respect to Γ_{ji}^h and obtain

$$(4.5)\qquad D_j\varphi_i{}^h = (\delta_j^h - \eta_j\eta^h)(u_i - q_i + \eta_i) - (g_{ji}-\eta_j\eta_i)(u^h - q^h + \eta^h) + \varphi_j{}^h(u_t\varphi_i{}^t - p_i) + \varphi_{ji}(\varphi_t{}^h u^t + p^h),$$

where

$$q_i = -p_t\varphi_i{}^t, \qquad q^h = q_t g^{th}.$$

We now assume that the affine connection D also satisfies

$$(4.6)\qquad D_j\varphi_i{}^h = 0.$$

Then we have

$$(\delta_j^h-\eta_j\eta^h)(u_i-q_i+\eta_i)-(g_{ji}-\eta_j\eta_i)(u^h-q^h+\eta^h)$$
$$+\varphi_j{}^h(u_t\varphi_i{}^t-p_i)+\varphi_{ji}(\varphi_t{}^h u^t+p^h)=0\,,$$

from which by contraction with respect to h and j

$$2m(u_i-q_i+\eta_i)+2(u_i-q_i-\eta_i\eta_t u^t)=0\,, \tag{4.7}$$

from which, by transvection with η^i

$$2m(u_i\eta^i+1)=0$$

and consequently substituting $u_t\eta^t=-1$ into (4.7) we find

$$2(m-1)(u_i-q_i+\eta_i)=0$$

and consequently

$$u_i=q_i-\eta_i\,. \tag{4.8}$$

Thus (4.4) takes the form

$$\Gamma_{ji}^h=\left\{ {h \atop j\ i} \right\}+(\delta_j^h-\eta_j\eta^h)p_i+(\delta_i^h-\eta_i\eta^h)p_j-(g_{ji}-\eta_j\eta_i)p^h$$
$$+\varphi_j{}^h(q_i-\eta_i)+\varphi_i{}^h(q_j-\eta_j)-\varphi_{ji}(q^h-\eta^h)\,. \tag{4.9}$$

Using (4.9) we now compute the covariant derivative $D_j\eta^h$ of η^h with respect to the affine connection D and find

$$D_j\eta^h=(\delta_j^h-\eta_j\eta^h)p_i\eta^i\,,$$

from which we see that

$$D_j\eta^h=0\,, \tag{4.10}$$

if and only if $p_i\eta^i=0$, that is, if and only if

$$\mathcal{L}p=0\,. \tag{4.11}$$

Computing $D_j\eta_i$, we find

$$D_j\eta_i=(g_{ji}-\eta_j\eta_i)p^t\eta_t\,, \tag{4.12}$$

from which we see that $D_j\eta_i=0$ if and only if (4.11) is satisfied.

Thus we have

PROPOSIOION 4.1. *In a Sasakian manifold with structure tensors $(\varphi_i{}^h, \eta_i, g_{ji})$, the affine connection D which satisfies*

$$D_k(e^{2p}g_{ji})=2e^{2p}p_k\eta_j\eta_i\,,\qquad D_j\varphi_i{}^h=0\,,\qquad D_j\eta^h=0$$

and whose torsion tensor satisfies

$$\Gamma_{ji}^h-\Gamma_{ij}^h=-2\varphi_{ji}u^h\,,$$

where p is a scalar function and u^h a vector field, is given by

$$\Gamma^h_{ji}=\left\{ {h \atop j\ i} \right\}+(\delta^h_j-\eta_j\eta^h)p_i+(\delta^h_i-\eta_i\eta^h)p_j-(g_{ji}-\eta_j\eta_i)p^h$$
$$+\varphi_j{}^h(q_i-\eta_i)+\varphi_i{}^h(q_j-\eta_j)-\varphi_{ji}(q^h-\eta^h),$$

where

$$p_i=\partial_i p, \quad p^h=p_t g^{th}, \quad q_i=-p_t\varphi_i{}^t, \quad q^h=q_t g^{th}$$

and p satisfies

$$\mathcal{L}p=0.$$

We call such an affine connection a *contact conformal connection.* Since a contact conformal connection satisfies

$$D_k(e^{2p}g_{ji})=2e^{2p}p_k\eta_j\eta_i \quad \text{and} \quad D_k\eta_j=0,$$

we have

$$D_k\{e^{2p}(g_{ji}-\eta_j\eta_i)\}=D_k(e^{2p}g_{ji})-(D_ke^{2p})\eta_j\eta_i$$
$$=0.$$

Thus we have

PROPOSITION 4.2. *A contact conformal connection in a Sasakian manifold satisfies*

$$D_k\{e^{2p}(g_{ji}-\eta_j\eta_i)\}=0. \tag{4.13}$$

§ 5. Curvature tensor of a contact conformal connection.

We consider a contact conformal connection

$$\Gamma^h_{ji}=\left\{ {h \atop j\ i} \right\}+(\delta^h_j-\eta_i\eta^h)p_i+(\delta^h_i-\eta_i\eta^h)p_j-(g_{ji}-\eta_j\eta_i)p^h \tag{5.1}$$
$$+\varphi_j{}^h(q_i-\eta_i)+\varphi_i{}^h(q_j-\eta_j)-\varphi_{ji}(q^h-\eta^h),$$

where

$$p_i=\partial_i p, \quad p^h=p_t g^{th}, \quad q_i=-p_t\varphi_i{}^t, \quad q^h=q_t g^{th}, \tag{5.2}$$

p being a scalar function such that

$$\mathcal{L}p=p_i\eta^i=0 \tag{5.3}$$

in a Sasakian manifold with structure tensors $(\varphi_i{}^h, \eta_i, g_{ji})$.

From (5.2) and (5.3) we see that

$$p_t\varphi_i{}^t=-q_i, \quad q_t\varphi_i{}^t=p_i, \quad \varphi_t{}^hp^t=q^h, \quad \varphi_t{}^hq^t=-p^h, \tag{5.4}$$

$$p_i\eta^i=0, \quad q_i\eta^i=0, \quad p_iq^i=0 \tag{5.5}$$

and

$$p_tp^t=q_tq^t. \tag{5.6}$$

We now compute the curvature tensor of Γ^h_{ji}:

(5.7) $$R_{kji}{}^{h}=\partial_{k}\Gamma_{ji}^{h}-\partial_{j}\Gamma_{ki}^{h}+\Gamma_{kt}^{h}\Gamma_{ji}^{t}-\Gamma_{jt}^{h}\Gamma_{ki}^{t}.$$

By a straightforward computation, we find

(5.8) $$\begin{aligned}R_{kji}{}^{h}=K_{kji}{}^{h}&-(\delta_{k}^{h}-\eta_{k}\eta^{h})p_{ji}+(\delta_{j}^{h}-\eta_{j}\eta^{h})p_{ki}-p_{k}{}^{h}(g_{ji}-\eta_{j}\eta_{i})\\&+p_{j}{}^{h}(g_{ki}-\eta_{k}\eta_{i})-\varphi_{k}{}^{h}q_{ji}+\varphi_{j}{}^{h}q_{ki}-q_{k}{}^{h}\varphi_{ji}+q_{j}{}^{h}\varphi_{ki}\\&+(\nabla_{k}q_{j}-\nabla_{j}q_{k})\varphi_{i}{}^{h}+2\varphi_{kj}(q_{i}p^{h}-p_{i}q^{h})\\&+(\varphi_{k}{}^{h}\varphi_{ji}-\varphi_{j}{}^{h}\varphi_{ki}-2\varphi_{kj}\varphi_{i}{}^{h}),\end{aligned}$$

where

(5.9) $$p_{ji}=\nabla_{j}p_{i}-p_{j}p_{i}+(q_{j}-\eta_{j})(q_{i}-\eta_{i})+\frac{1}{2}p_{t}p^{t}(g_{ji}-\eta_{j}\eta_{i}),$$

(5.10) $$q_{ji}=\nabla_{j}q_{i}-p_{j}(q_{i}-\eta_{i})-p_{i}(q_{j}-\eta_{j})+\frac{1}{2}p_{t}p^{t}\varphi_{ji}.$$

Since $p_{i}=\partial_{i}p$, we have from (5.9)

(5.11) $$p_{ji}=p_{ij}.$$

Transvecting (5.9) with η^{j} and noting that

$$\begin{aligned}\eta^{j}\nabla_{j}p_{i}=\eta^{j}\nabla_{i}p_{j}&=-(\nabla_{i}\eta^{j})p_{j}\\&=-\varphi_{i}{}^{j}p_{j}=q_{i},\end{aligned}$$

we have

(5.12) $$\eta^{j}p_{ji}=\eta_{i}.$$

Also we compute $p_{jt}\varphi_{i}{}^{t}$ using (5.9) and find

(5.13) $$q_{ji}=-p_{jt}\varphi_{i}{}^{t}$$

and consequently

(5.14) $$\eta^{j}q_{ji}=0,\qquad q_{ji}\eta^{i}=0.$$

We compute $q_{js}\varphi_{i}{}^{s}$ using (5.10) and find

$$q_{js}\varphi_{i}{}^{s}=p_{ji}-\eta_{j}\eta_{i},$$

that is,

(5.15) $$p_{ji}=q_{jt}\varphi_{i}{}^{t}+\eta_{j}\eta_{i},$$

from which, p_{ji} being symmetric,

$$q_{jt}\varphi_{i}{}^{t}-q_{it}\varphi_{j}{}^{t}=0,$$

from which we have, using (5.14),

(5.16) $$q_{ts}\varphi_{j}{}^{t}\varphi_{i}{}^{s}=-q_{ij}.$$

We now assume that the curvature tensor of the contact conformal connection vanishes:

(5.17) $$R_{kji}{}^{h}=0.$$

Then from (5.8), we have

(5.18) $$K_{kji}{}^{h}=(\delta_k^h-\eta_k\eta^h)p_{ji}-(\delta_j^h-\eta_j\eta^h)p_{ki}+p_k{}^h(g_{ji}-\eta_j\eta_i)$$
$$-p_j{}^h(g_{ki}-\eta_k\eta_i)+\varphi_k{}^h q_{ji}-\varphi_j{}^h q_{ki}+q_k{}^h\varphi_{ji}-q_j{}^h\varphi_{ki}$$
$$+\alpha_{kj}\varphi_i{}^h+\varphi_{kj}\beta_i{}^h-(\varphi_k{}^h\varphi_{ji}-\varphi_j{}^h\varphi_{ki}-2\varphi_{kj}\varphi_i{}^h),$$

where we have put

(5.19) $$\alpha_{kj}=-(\nabla_k q_j-\nabla_j q_k),$$

(5.20) $$\beta_i{}^h=2(p_i q^h-q_i p^h)$$

and consequently, for $\beta_{ih}=\beta_i{}^t g_{th}$, we have

(5.21) $$\beta_{ih}=2(p_i q_h-q_i p_h).$$

We see that α_{kj} and β_{ih} are both skew-symmetric and satisfy

(5.22) $$\alpha_{kj}\eta^j=0$$

and

(5.23) $$\eta^i\beta_{ih}=0$$

respectively.

We also compute $\alpha=\varphi^{kj}\alpha_{kj}$ and $\beta=\varphi^{ih}\beta_{ih}$ and obtain

(5.24) $$\alpha=\varphi^{kj}\alpha_{kj}=-2\nabla_t p^t$$

and

(5.25) $$\beta=\varphi^{ih}\beta_{ih}=4p_t p^t$$

respectively, from which

(5.26) $$\alpha-\beta=-2(\nabla_t p^t+2p_t p^t).$$

Now equation (5.18) can be written in the covariant form

(5.27) $$K_{kjih}=(g_{kh}-\eta_k\eta_h)p_{ji}-(g_{jh}-\eta_j\eta_h)p_{ki}+p_{kh}(g_{ji}-\eta_j\eta_i)$$
$$-p_{jh}(g_{ki}-\eta_k\eta_i)+\varphi_{kh}q_{ji}-\varphi_{jh}q_{ki}+q_{kh}\varphi_{ji}-q_{jh}\varphi_{ki}$$
$$+\alpha_{kj}\varphi_{ih}+\varphi_{kj}\beta_{ih}-(\varphi_{kh}\varphi_{ji}-\varphi_{jh}\varphi_{ki}-2\varphi_{kj}\varphi_{ih}).$$

Substituting this into

$$K_{kjih}-K_{ihkj}=0,$$

we find

(5.28) $$\varphi_{kh}(q_{ji}+q_{ij})-\varphi_{jh}(q_{ki}+q_{ik})+(q_{kh}+q_{hk})\varphi_{ji}$$
$$-(q_{jh}+q_{hj})\varphi_{ki}+(\alpha_{kj}-\beta_{kj})\varphi_{ih}-\varphi_{kj}(\alpha_{ih}-\beta_{ih})=0,$$

from which, transvecting with φ^{kh}, we find

$$(2m-2)(q_{ji}+q_{ij})=0\,.$$

Thus if $2m+1>3$, we have

$$q_{ji}+q_{ij}=0\,, \tag{5.29}$$

which shows that q_{ji} is skew-symmetric, and consequently we have from (5.16)

$$q_{ts}\varphi_j{}^t\varphi_i{}^s=q_{ji}\,. \tag{5.30}$$

From (5.28) and (5.29) we find

$$(\alpha_{kj}-\beta_{kj})\varphi_{ih}-\varphi_{kj}(\alpha_{ih}-\beta_{ih})=0\,,$$

from which, transvecting with φ^{kj},

$$\alpha_{ih}-\beta_{ih}=\frac{1}{2m}(\alpha-\beta)\varphi_{ih}$$

and consequently using (5.26),

$$\alpha_{ih}-\beta_{ih}=-\frac{1}{m}(\nabla_t p^t+2p_t p^t)\varphi_{ih}\,. \tag{5.31}$$

On the other hand, from the definition (5.10) of q_{ji} and the skew-symmetry of q_{ji}, we find

$$2q_{ji}=\nabla_j q_i-\nabla_i q_j+p_t p^t\varphi_{ji}\,.$$

Thus from the definition (5.19) of α_{ji}, we have

$$\alpha_{ji}=-2q_{ji}+p_t p^t\varphi_{ji}\,. \tag{5.32}$$

Equations (5.31) and (5.32) give

$$\beta_{ji}=-2q_{ji}+\frac{1}{m}[\nabla_t p^t+(m+2)p_t p^t]\varphi_{ji}\,. \tag{5.33}$$

Since we have from (5.9)

$$p_t{}^t=\nabla_t p^t+mp_t p^t+1\,, \tag{5.34}$$

we can write (5.33) in the form

$$\beta_{ji}=-2q_{ji}+\frac{1}{m}(p_t{}^t+2p_t p^t-1)\varphi_{ji}\,. \tag{5.35}$$

Now substituting (5.27) into

$$K_{kjih}+K_{jikh}+K_{ikjh}=0\,,$$

we find

$$\begin{aligned}&2(\varphi_{kh}q_{ji}+\varphi_{jh}q_{ik}+\varphi_{ih}q_{kj}+q_{kh}\varphi_{ji}+q_{jh}\varphi_{ik}+q_{ih}\varphi_{kj})\\&\quad+(\alpha_{kj}\varphi_{ih}+\alpha_{ji}\varphi_{kh}+\alpha_{ik}\varphi_{jh})+(\varphi_{kj}\beta_{ih}+\varphi_{ji}\beta_{kh}+\varphi_{ik}\beta_{jh})=0\,.\end{aligned} \tag{5.36}$$

Substituting (5.32) and (5.35) into this equation, we find

$$(\varphi_{kj}\varphi_{ih}+\varphi_{ji}\varphi_{kh}+\varphi_{ik}\varphi_{jh})[p_t{}^t+(m+2)p_tp^t-1]=0\,,$$

from which

$$p_t{}^t+(m+2)p_tp^t-1=0\,. \tag{5.37}$$

Thus equation (5.35) can be written as

$$\beta_{ji}=-2q_{ji}-p_tp^t\varphi_{ji}\,. \tag{5.38}$$

Now, from (5.18), contracting with respect to h and k and using

$$\alpha_{tj}\varphi_i{}^t=2p_{ji}-2\eta_j\eta_i-p_tp^t(g_{ji}-\eta_j\eta_i)$$

obtained from (5.32) and

$$\beta_{tj}\varphi_i{}^t=2p_{ji}-2\eta_j\eta_i+p_tp^t(g_{ji}-\eta_j\eta_i)$$

obtained from (5.38), we find

$$K_{ji}=2(m+2)p_{ji}+(p_t{}^t-3)g_{ji}-(p_t{}^t+1)\eta_j\eta_i\,, \tag{5.39}$$

from which, transvecting with g^{ji},

$$K=4(m+1)p_t{}^t-6m-4\,, \tag{5.40}$$

and consequently

$$p_t{}^t=\frac{K+2(3m+2)}{4(m+1)}\,,$$

that is,

$$p_t{}^t=-L\,. \tag{5.41}$$

Substituting (5.41) into (5.39), we find

$$K_{ji}=2(m+2)p_{ji}-(L+3)g_{ji}+(L-1)\eta_j\eta_i\,,$$

from which

$$p_{ji}=\frac{1}{2(m+2)}[K_{ji}+(L+3)g_{ji}-(L-1)\eta_j\eta_i]\,,$$

that is,

$$p_{ji}=-L_{ji}\,, \tag{5.42}$$

from which

$$q_{ji}=-M_{ji}\,. \tag{5.43}$$

On the other hand, from (5.37) and (5.41), we have

$$-L+(m+2)p_tp^t-1=0\,,$$

from which

$$p_tp^t=\frac{1}{m+2}(L+1)\,. \tag{5.44}$$

Substituting (5.42) and (5.43) into (5.32) and (5.38), we find

(5.45) $$\alpha_{ji}=2M_{ji}+\frac{L+1}{m+2}\varphi_{ji}$$

and

(5.46) $$\beta_{ji}=2M_{ji}-\frac{L+1}{m+2}\varphi_{ji}$$

respectively.

Substituting (5.42), (5.43), (5.45) and (5.46) into (5.18), we find

(5.47) $$B_{kji}{}^{h}=0\,.$$

Thus we have

THEOREM 5.1. *If, in a* $(2m+1)$*-dimensional Sasakian manifold* $(2m+1>3)$, *there exists a scalar function* p *such that the contact conformal connection*

$$\Gamma_{ji}^{h}=\left\{ {h \atop j\ i} \right\}+(\delta_j^h-\eta_j\eta^h)p_i+(\delta_i^h-\eta_i\eta^h)p_j-(g_{ji}-\eta_j\eta_i)p^h$$
$$+\varphi_j{}^h(q_i-\eta_i)+\varphi_i{}^h(q_j-\eta_j)-\varphi_{ji}(q^h-\eta^h)\,,$$

where $p_i=\partial_i p$, $p^h=p_t g^{th}$, $q_i=-p_t\varphi_i{}^t$, $q^h=q_t g^{th}$, *is of zero curvature, then the contact Bochner curvature tensor of the manifold vanishes.*

BIBLIOGRAPHY

[1] M. MATSUMOTO AND G. CHŪMAN, On the C-Bochner curvature tensor, TRU. Mathematics, **5** (1969), 21-30.

[2] S. SASAKI, Almost contact manifolds, Lecture Notes, **I**, (1965), Tôhoku University.

[3] K. YANO, On complex conformal connections, Kōdai Math. Sem. Rep., **26** (1975), 137-151.

[4] K. YANO, Anti-invariant submanifolds of a Sasakian manifold with vanishing contact Bochner curvature tensor, to appear.

[5] K. YANO, Differential geometry of anti-invariant submanifolds of a Sasakian manifold, Bollettino U. M. I., (4) **12** (1975), 279-296.

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