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ON THE HOLONOMY GROUP OF A NORMAL COMPLEX ALMOST CONTACT MANIFOLD

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In a recent paper [4] Ishihara and Konishi have studied fiberings (with 1-dimensional fibre) of manifolds with an (real) almost contact 3-structure and obtained in the base space of a new kind of structure, called a (normal) complex almost contact structure. This structure is a complex contact structure. In the present paper, we use the curvature properties developed in their paper to prove the following.

THEOREM. *The holonomy group of a normal complex almost contact manifold of complex dimension* $2m+1$, $m>0$, *is the unitary group* $U(2m+1)$.

1. Preliminary. Definitions and proofs of statements in this section may be found in [1] through [4]. Let (M, g, F) be a Kählerian manifold with Riemannian metric g and complex structure F. $A=\{O, O', \cdots\}$ be an open covering of M consisting of coordinate neighborhoods. Suppose that there are in each $O\in A$ two covariant vector fields u, v and two tensor fields G, H of type one-one satisfying

$$(1.1)\qquad \begin{cases} u(X)=g(U, X), \quad v(X)=g(V, X) \qquad \forall X; \\ G^2=H^2=-I+u\otimes U+v\otimes V, \quad HG=-GH=F+u\otimes V-v\otimes U; \\ GF=-FG=H, \quad HF=-FH=-G; \\ GU=GV=HU=HV=0, \quad u\circ G=v\circ G=u\circ H=v\circ H=0; \\ FU=-V, \quad FV=U; \\ \|U\|=\|V\|=1, \quad g(U, V)=0; \\ g(GX, Y)=-g(GY, X), \quad g(HX, Y)=-g(HY, X), \qquad \forall X, Y \end{cases}$$

and for the corresponding tensor fields u', v', G' and H' defined in O' by (1.1) the relations

$$(1.2)\qquad \begin{cases} u'=au-bv, \quad v'=bu+av \\ G'=aG-bH, \quad H'=bG+aH \end{cases}$$

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hold in $O\cap O'$, where a and b are certain functions in $O\cap O'$ with $a^2+b^2=1$. Then the set $\{(O, u, v, G, H)|O\in A\}$ is called a *complex almost contact structure in* (M, g, F). In such a case, the manifold M is necessarily of odd complex dimension, say $2m+1$ $(m>0)$.

Let $\tilde{M}$ and M be two differentiable manifolds of dimension r and n, with $s=r-n>0$. Assume that there is a differentiable mapping $\pi: \tilde{M}\to M$ which is onto and of maximum rank n everywhere. Then for each point $p\in M$, $\pi^{-1}(p)$ is an s-dimensional submanifold of $\tilde{M}$, which is called a fibre. Moreover each fibre is assumed to be connected. In such a case, $(\tilde{M}, M, \pi)$ is called a *fibred space*, $\tilde{M}$ the *total space*, M the *base space* and $\pi: \tilde{M}\to M$ the *projection*.

Let $\tilde{M}$ be a fibred space with Sasakian structure $(\varphi, \alpha, \xi, \tilde{g})$ and assumed to admit a Sasakian 3-structure (ξ, η, ζ) where real $\dim \tilde{M}=4m+3$ $(m>0)$ and the base space M is of dimension $4m+2$. Denote by $\pi: \tilde{M}\to M$ the projection. Then M admits a Kählerian structure (g, F) with odd complex dimension $2m+1$ where g is the projection of $\tilde{g}$ and F the projection of φ defined by $FX=\pi^*\varphi X^L$, X^L being the horizontal lift of X. Take a coordinate neighborhood O in M and a local cross-section τ of $\tilde{M}$ over O. Let us define local 1-forms u, v and local one-one tensor fields G, H in O by

$$(1.3)\qquad \begin{cases} u(X)\circ\pi=\beta(\tau^*X), & v(X)\circ\pi=\gamma(\tau^*X), \\ GX=\pi^*\psi^H(\tau^*X), & H(X)=\pi^*\theta^H(\tau^*X) \end{cases}$$

for any vector field X on O. Here β, γ, ψ, θ are associated with the Sasakian 3-structure, which are defined as part of the following:

$$\alpha(\tilde{X})=\tilde{g}(\xi, \tilde{X}), \quad \beta(\tilde{X})=\tilde{g}(\eta, \tilde{X}), \quad \gamma(\tilde{X})=\tilde{g}(\zeta, \tilde{X});$$

$$2\tilde{g}(\varphi\tilde{X}, \tilde{Y})=d\alpha(\tilde{X}, \tilde{Y}), \quad 2\tilde{g}(\psi\tilde{X}, \tilde{Y})=d\beta(\tilde{X}, \tilde{Y}), \quad 2\tilde{g}(\theta\tilde{X}, \tilde{Y})=d\gamma(\tilde{X}, \tilde{Y}),$$

$$\psi^H=\psi+\alpha\otimes\zeta-\gamma\otimes\xi, \qquad \theta^H=\theta+\beta\otimes\xi-\alpha\otimes\eta.$$

Then it is known that u, v, G, H defined in O by (1.3) satisfy (1.1). Thus M has a complex almost contact structure. Take coordinate neighborhoods $\{O, x^k\}$ $(1\leqq i, j, k, \cdots, \leqq 4m+2)$ of M and $\{\tilde{O}, y^A\}$ $(1\leqq A, B, C, \cdots, \leqq 4m+3)$ of $\tilde{M}$ in such a way that $\pi(\tilde{O})=O$. The projection $\pi: \tilde{M}\to M$ is expressed by $x^h=x^h(y^1, \cdots, y^{4m+3})$. Then in [4] the following relations are derived:

$$(1.4)\qquad \begin{cases} \nabla_j u_i=G_{ij}+\sigma_j V_i, & \nabla_j v_i=H_{ji}-\sigma_j U_i \\ (\sigma_j)_p=2(\xi_A\partial_j\tau^A)_{\tau(p)}; \end{cases}$$

$$(1.5)\qquad \begin{cases} \nabla_j G_i{}^h=\delta_j{}^h u_i-g_{ji}u^h-\varphi_j{}^h v_i+\varphi_{ji}v^h+\sigma_j H_i{}^h, \\ \nabla_j H_i{}^h=\delta_j{}^h v_i-g_{ji}v^h+\varphi_j{}^h u_i-\varphi_{ji}u^h-\sigma_j G_i{}^h, \end{cases}$$

∇ being the Riemannian connection of (M, g, F). When a complex almost contact structure satisfies (1.4) and (1.5) with a certain local 1-form $\sigma=\sigma_i dx^i$, it is said to be *normal*. Thus if a fibred space with Sasakian structure $(\varphi, \alpha, \xi, \tilde{g})$

admits a Sasakian 3-structure (ξ, η, ζ), then the base space M is a Kählerian manifold admitting a complex almost contact structure which is normal.

2. Curvature properties.

In this section let (M, g, F) be a Kählerian manifold of complex dimension $2m+1$ with complex almost contact structure $\{(O, u, v, G, H)\}$ which is normal. Using (1.4) and Ricci formulas we have

$$-K_{kji}{}^{s}u_s=u_j g_{ki}-u_k g_{ji}+v_j F_{ki}-v_k F_{ji}-2v_i F_{kj}+\Omega_{kj}v_i\,,$$

$$-K_{kji}{}^{s}v_s=v_j g_{ki}-v_k g_{ji}-u_j F_{ki}+u_k F_{ji}+2u_i F_{kj}-\Omega_{kj}u_i$$

where K is the curvature tensor of (M, g, F), $K_{kji}{}^{h}$ are its components. $F_{ji}=F_j{}^h g_{hi}$ and

$$\Omega_{ji}=\partial_j\sigma_i-\partial_i\sigma_j\,,$$

In § 5 of [4] it is shown that

$$\Omega_{ji}=4F_{ji}\,.$$

Thus we obtain the following two relations:

$$K(X, Y)U=\langle U, Y\rangle X-\langle U, X\rangle Y+\langle V, Y\rangle FX+2\langle FX, Y\rangle V-\langle V, X\rangle FY\,, \tag{2.1}$$

$$K(X, Y)V=\langle V, Y\rangle X-\langle V, X\rangle Y-\langle U, Y\rangle FX-2\langle FX, Y\rangle U+\langle U, X\rangle FY \tag{2.2}$$

for any two tangent vectors X, Y to M. It readily follows from (1.1) that

$$K(U, V)U=-4V\,,\qquad K(U, V)V=4U\,. \tag{2.3}$$

Let U, $V=-FU$, X_1, $X_{1^*}=FX_1$, X_2, $X_{2^*}=FX_2$, $\cdots$, X_{2m}, $X_{2m^*}=FX_{2m}$ be an orthonormal frame in the (real) tangent space at $p\in M$. Then from (2.1) we have

$$\begin{cases} K(X_i, U)U=X_i\,, & K(X_{i^*}, U)U=X_{i^*}\,, \\ K(X_i, V)U=X_{i^*}\,, & K(X_{i^*}, V)U=-X_i\,, \end{cases} \tag{2.4}$$

From (2.2) we have

$$\begin{cases} K(X_i, U)V=-X_{i^*}\,, & K(X_{i^*}, U)V=X_i\,, \\ K(X_i, V)V=X_i\,, & K(X_{i^*}, V)V=X_{i^*}\,. \end{cases} \tag{2.5}$$

The first Bianchi identity and (2.4), (2.5) yield

$$\begin{cases} K(U, V)X_i=2X_{i^*}\,, \\ K(U, V)X_{i^*}=-2X_i\,. \end{cases} \tag{2.6}$$

Each $K(X, Y)$ for $X, Y\in T_p(M)$ is an element in the Lie algebra of $U(2m+1)$. The real matrix $K(X, Y)$ with respect to the basis $\{U, V, X_1, X_{1^*}, \cdots, X_{2m}, X_{2m^*}\}$ of $T_p(M)$ is a skew symmetric matrix. Since $FK(X, Y)=K(X, Y)F$, $K(X, Y)$ takes the following form:

$$(2.7)\qquad \begin{bmatrix} B_{00}, & B_{01}, & \cdots\cdots\cdots, & B_{0,2m} \\ -B_{01}, & B_{11}, & \cdots\cdots\cdots, & B_{1,2m} \\ \vdots & & & \vdots \\ -B_{0,2m}, & -B_{1,2m}, & \cdots, & B_{2m,2m} \end{bmatrix}$$

where each B_{ij} is a 2×2 matrix such that

$$(2.7)'\qquad \begin{aligned} &B_{ii}=\begin{bmatrix} 0 & a_i \\ -a_i & 0 \end{bmatrix}, \quad 0\leqq i\leqq 2m\,; \\ &B_{0i}=\begin{bmatrix} b_i & c_i \\ c_i & -b_i \end{bmatrix}, \quad B_{ij}=\begin{bmatrix} d_{ij} & e_{ij} \\ -e_{ij} & d_{ij} \end{bmatrix}, \quad 1\leqq i,\ j\leqq 2m\,. \end{aligned}$$

3. Proof of the theorem.

Let L be the Lie algebra generated by $\{K(X, Y)\,;\, X, Y\in T_p(M)\}$. The Lie algebra of the unitary group $U(2m+1)$ is a vector space of (real) dimension $(2m+1)^2$. L is a subalgebra of the Lie algebra of $U(2m+1)$. In order to prove the theorem we have only to prove that L is of dimension $(2m+1)^2$. On the other hand the set of all matrices of the form (2.7) clearly is a vector space of dimension $(2m+1)^2$. We thus have only to prove that L contains all matrices of the form (2.7). This would be done if we show that L contains all the following matrices (3.1) through (3.5):

$$(3.1)\qquad \begin{bmatrix} 0 & \cdots\cdots\cdots & 0 \\ \vdots & \ddots & \\ & E & \\ & & \ddots \\ 0 & \cdots\cdots\cdots & 0 \end{bmatrix}, \quad E=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

E is at (i, i) block, $0\leqq i\leqq 2m$ and other blocks are zero matrices;

$$(3.2)\qquad \begin{bmatrix} 0 \cdots & C & \cdots 0 \\ \vdots & & \\ -C & 0 & \\ \vdots & & \\ 0 & & \end{bmatrix}, \quad C=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

C is at $(0, i)$ block, $1\leqq i\leqq 2m$ and other blocks except C and $-C$ are zero matrices;

$$(3.3)\qquad \begin{bmatrix} 0 \cdots & D & \cdots 0 \\ \vdots & & \\ -D & 0 & \\ \vdots & & \\ 0 & & \end{bmatrix}, \quad D=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

D is at $(0, i)$ block, $1\leqq i\leqq 2m$ and other blocks except D and $-D$ are zero matrices;

$$(3.4)\qquad \begin{bmatrix} 0 & \cdots\cdots\cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots\cdots I \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots -I \cdots\cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots\cdots\cdots & 0 \end{bmatrix}, \qquad I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

I is at $(\imath, \jmath)$ block, $1 \leqq \imath < \jmath \leqq 2m$ and other blocks except I and $-I$ are zero matrices;

$$(3.5)\qquad \begin{bmatrix} 0 & \cdots\cdots\cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots\cdots E \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots E \cdots\cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots\cdots\cdots & 0 \end{bmatrix},$$

E is given in (3.1), at $(\imath, \jmath)$ and $(\jmath, i)$ block, $1 \leqq \imath < j \leqq 2m$ and other blocks except E are zero matrices.

By (2.3) and (2.6),

$$A_0 \overset{\text{def.}}{=} \frac{1}{2} K(U, V) = \begin{bmatrix} -2E & & & \\ & E & & 0 \\ & & \ddots & \\ 0 & & & E \end{bmatrix} \in L,$$

where the (0, 0) block is $-2E$, all other diagonal blocks are E and the rest are zero matrices.

By the first relation of (2.4) and (2.5),

$$A_\imath \overset{\text{def.}}{=} \frac{1}{2} K(X_\imath, U) = \begin{bmatrix} 0 \cdots & -D & \cdots 0 \\ \vdots & & \\ D & U^{(i)}_{jk} & \\ \vdots & & \\ 0 & & \end{bmatrix} \in L$$

where D is at $(i, 0)$ block, $1 \leqq \imath \leqq 2m$, other blocks in the first row and column are zero matrices and

$$(3.6)\qquad U^{(i)}_{jk} = -U^{(i)}_{kj} = \begin{bmatrix} a^{(i)}_{jk} & -b^{(i)}_{jk} \\ b^{(i)}_{jk} & a^{(i)}_{jk} \end{bmatrix}, \qquad 1 \leqq \jmath,\ k \leqq 2m,\ a^{(i)}_{jj} = 0.$$

Every two matrices of the form (3.6) are commutative. We then obtain the following bracket products in L:

$$C_\imath \overset{\text{def.}}{=} [A_0, A_\imath] = \begin{bmatrix} 0 \cdots & C & \cdots 0 \\ \vdots & & \\ -C & 0 & \\ \vdots & & \\ 0 & & \end{bmatrix} \in L$$

which is (3.2);

$$D_i \overset{\text{def.}}{=} [A_0, C_i] = \begin{bmatrix} 0 \cdots D \cdots 0 \\ \vdots & & \\ -D & 0 & \\ \vdots & & \\ 0 & & \end{bmatrix} \in L$$

which is (3.3);

$$-[D_i, D_j] = \begin{bmatrix} 0 & \cdots\cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots\cdots I \cdot\cdot & 0 \\ \vdots & & \\ 0 & \cdot -I \cdots\cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots\cdots & 0 \end{bmatrix} \in L$$

which is (3.4);

$$-[C_i, D_j] = \begin{bmatrix} 0 & \cdots\cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots\cdots E \cdots & 0 \\ \vdots & & \\ 0 & \cdots E \cdots\cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots\cdots & 0 \end{bmatrix} \in L$$

which is (3.5);

$$G_i \overset{\text{def.}}{=} \frac{1}{2}[D_i, C_i] = \begin{bmatrix} E & \cdots\cdots & 0 \\ \vdots & 0 & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & E & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots\cdots & 0 \end{bmatrix} \in L,$$

where E is at $(0, 0)$ and (i, i) $(1 \leqq i \leqq 2m)$ blocks and the other blocks are zero matrices.

Since $A_0 \in L$, $G_i \in L$ $(i=1, 2, \cdots, 2m)$ we have that

$$\begin{bmatrix} 0 & \cdots\cdots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & E & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots\cdots & 0 \end{bmatrix} \in L$$

which is (3.1). The proof is thus complete.

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