INTEGRAL FORMULAS AND THEIR APPLICATIONS
IN QUATERNIONIC KÄHLERIAN MANIFOLDS

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Recently, quaternionic Kählerian manifolds have been studied by several authors (Alekseevskii [1], [2], Gray [3], Ishihara [4], [5], Ishihara and Konishi [6], Krainse [7] and Wolf [11]). On the other hand, Yano [8], [9] and Yano and Bochner [10] established some integral formulas in compact Kählerian manifolds and, using these integral formulas, obtained interesting results concerning Killing and analytic vectors in compact Kählerian manifolds. In the present note, we establish some integral formulas in compact quaternionic Kählerian manifolds and, using these integral formulas, prove some theorems concerning Killing vectors and vector fields preserving the quaternionic structure, which will be called infinitesimal Q-transformations.

In § 1, we recall definitions and some properties of quaternionic Kählerian manifolds. In § 2, we define infinitesimal Q-transformations in quaternionic Kählerian manifolds and give some properties of infinitesimal Q-transformations. § 3 is devoted to establish some integral formulas in compact quaternionic Kählerian manifolds for later use. In § 4, using integral formulas established in § 3, prove some theorems concerning Killing vectors and infinitesimal Q-transformations in compact quaternionic Kählerian manifolds.

Manifolds, mappings, tensor fields and other geometric objects we discuss are assumed to be differentiable and of class C^∞. The indices $h, \imath, \jmath, k, l, r, s, t$ run over the range $\{1, 2, \cdots, n\}$, $(n=4m, m \geq 1)$ and the summation convension will be used with respect to this system of indices.

§ 1. Quaternion Kählerian manifolds.

Let M be a differentiable manifold of dimension n and assume that there is a subbundle V of the tensor bundle of type $(1, 1)$ over M such that V satisfies the following condition:

(a) In any coordinate neighborhood U of M, there is a local basis $\{F, G, H\}$ of the bundle V, where F, G and H are tensor fields of type $(1, 1)$ in U, and satisfy

(1.1)
$$F^2 = -I, \qquad G^2 = -I, \qquad H^2 = -I,$$
$$GH = -HG = F, \qquad HF = -FH = G, \qquad FG = -GF = H,$$

I being the identity tensor field of type (1, 1) in M. Such a local basis $\{F, G, H\}$ of the bundle V is said to be *canonical* in U.

Thus the bundle V is 3-dimensional as a vector bundle. Such a bundle V is called an *almost quaternionic structure* and the pair (M, V) an *almost quaternionic manifold*. An almost quaternionic manifold is orientable and of dimension $n=4m$ $(m\geq 1)$ (see [5], for example).

For an almost quaternionic manifold (M, V), let $\{F, G, H\}$ and $\{F', G', H'\}$ be canonical local bases of V in U and in another coordinate neighborhood U' of M, respectively. Then we have in $U \cap U'$

$$F'=s_{11}F+s_{12}G+s_{13}H,$$

(1.2) $$G'=s_{21}F+s_{22}G+s_{23}H,$$

$$H'=s_{31}F+s_{32}G+s_{33}H,$$

where $S_{U,U'}=(s_{\gamma\beta})\in SO(3)$, $(\beta, \gamma=1, 2, 3)$, because $\{F, G, H\}$ and $\{F', G', H'\}$ satisfie (1, 1). Thus, if we put in U

(1.3) $$\varLambda=F\otimes F+G\otimes G+H\otimes H,$$

then, using (1.2), we easily see that $\varLambda$ determines in M a global tensor field of type (2, 2), which will be denoted also by $\varLambda$.

Next, let there be given an almost quaternionic structure V in a Riemannian manifold (M, g) and assume that, for any canonical local basis $\{F, G, H\}$ of V, all of F, G and H are almost Hermitian with respect to g. Moreover, we suppose that the set (M, g, V) satisfies the following condition:

(b) If ϕ is a cross-section of the bundle V, then $\varGamma_X\phi$ is also a cross-section of V for any vector field X in M, where $\varGamma$ denotes the Riemannian connection of the Riemannian manifold (M, g).

Such a set (M, g, V) is called a *quaternionic Kählerian manifold* and the set (g, V) a *quaternionic Kählerian structure* in M. The condition (b) is equivalent to the following condition:

(b)' For a canonical local basis $\{F, G, H\}$ of V in U,

$$\varGamma_X F= \qquad\qquad r(X)G-q(X)H,$$

(1.4) $$\varGamma_X G=-r(X)F \qquad\qquad +p(X)H,$$

$$\varGamma_X H= \quad q(X)F-p(X)G$$

for any vector field X in M, where p, q and r are certain local 1-forms in U (see [5]). Thus, using (1.4), we easily find

(1.5) $$\varGamma\varLambda=0.$$

Here, we can easily verify that the condition (1.5) is equivalent to the condition (b)'.

It is known that any quaternionic Kählerian manifold is an Einstein space,

i. e., that the Ricci tensor S of (M, g) has the form

$$(1.6) \qquad\qquad S = \frac{k}{4m} g,$$

k being the scalar curvature of (M, g), which is a constant if M is connected, where dim $M = 4m$ (see [1] and [5]).

We denote by $K_{kji}{}^h$ components of the curvature tensor of (M, g) and put $K_{kjih} = K_{kji}{}^s g_{hs}$, where g_{ji} are components of g. We define g^{ji} by $(g^{ji}) = (g_{ji})^{-1}$ and put $F^{ih} = g^{is} F_s{}^h$, $G^{ih} = g^{is} G_s{}^h$, $H^{ih} = g^{is} H_s{}^h$, which are all skew-symmetric. Using formulas (2.9) and (2.11) given in [5] and taking account of (1.6) given in the present paper, we obtain, if $m > 1$,

$$(1.7) \qquad K_{ksih} F_j{}^s F^{ih} = K_{ksih} G_j{}^s G^{ih} = K_{ksih} H_j{}^s H^{ih} = -\frac{k}{2(m+2)} g_{kj}.$$

Next, using formulas (2.9) and (2.12) given in [5] and taking account of (1.6) given in the present paper, we find, if $m = 1$,

$$(1.8) \qquad K_{ksih}(F_j{}^s F^{ih} + G_j{}^s G^{ih} + H_j{}^s H^{is}) = -\frac{k}{6} g_{kj}.$$

§2. Infinitesimal transformations.

Let (M, V) be an almost quaternionic manifold. If a transformation $f: M \to M$ leaves the bundle V invariant, then f is called a *Q-transformation* of (M, V). Let $\{F, G, H\}$ be a canonical local basis of V in a coordinate neighborhood V of M. A transformation $f: M \to M$ is a Q-transformation of (M, V) if and only if $\{f*F, f*G, f*H\}$ is a canonical local basis of V in $f(U)$, where $f*F$ denotes the tensor field induced by f from F and so on. Thus, a Q-transformation f preserves the tensor field $\varLambda$ defined by (1.3) globally in M. Conversely, using (1.1), we can easily prove that a transformation $f: M \to M$ is a Q-transformation of (M, V) if f preserves the tensor field $\varLambda$ invariant. Summing up, we have

PROPOSITION 2.1. *A transformation $f: M \to M$ is a Q-transformation of an almost quaternionic manifold (M, V) if and only if f preserves the tensor field $\varLambda$ invariant.*

A vector field X in M is called an *infinitesimal Q-transformation* of (M, V), if $\exp tX$ ($|t| < \varepsilon$, ε being a certain positive number) is a Q-transformation of (M, V). Thus we have directly, from Proposition 2.1,

PROPOSITION 2.2. *A vector field X in an almost quaternionic manifold is an infinitesimal Q-transformation if and only if $\mathcal{L}_X \varLambda = 0$, where $\mathcal{L}_X$ denotes the Lie derivation with respect to X.*

The condition $\mathcal{L}_X \varLambda = 0$ is equivalent to the condition that we have in each coordinate neighborhood U

$$\mathcal{L}_X F= \qquad \gamma G-\beta H,$$

(2.1)
$$\mathcal{L}_X G=-\gamma F \qquad +\alpha H,$$

$$\mathcal{L}_X H= \quad \beta F-\alpha G$$

for any canonical local basis $\{F, G, H\}$ of V in U, where α, β and γ are certain functions in U.

Let (M, g, V) be a quaternionic Kählerian manifold. If a transformation $f: M \rightarrow M$ is a Q-transformation of (M, V) and at the same time an isometry of (M, g), then f is called an *automorphism* of (M, g, V). If, for a vector field X in M, $\exp tX$ ($|t| < \varepsilon$, ε being a certain positive number) is an automorphism of (M, g, V), then X is called an *infinitesimal automorphism* of (M, g, V). Thus, from Proposition 2.2, we see that a vector field X in M is an infinitesimal automorphism of (M, g, V), if and only if $\mathcal{L}_X \Lambda = 0$ and $\mathcal{L}_X g = 0$. We have proved in [6] that, for a fibre-preserving Killing vector field $\tilde{X}$ in a fibred Riemannian manifold with Sasakian 3-structure, its projection $X = p\tilde{X}$, which is a vector filed in the base space, is an infinitesimal automorphism of the quaternionic Kählerian structure induced in the base space.

Let X be an infinitesimal Q-transformation. Then we have, from (1.4) and (2.1),

$$\mathcal{L}_X F - \nabla_X F = \zeta G - \eta H$$

in U for any vector field X in M, where η and ζ are certain functions in U. This equation is equivalent to

$$-\nabla_s X^h F_i{}^s + \nabla_i X^t F_t{}^h = \zeta G_i{}^h - \eta H_i{}^h,$$

from which, transvecting $G_h{}^i$ and $H_h{}^i$, we have respectively

$$\zeta = \frac{1}{4m} H_t{}^s \nabla_s X^t, \qquad \eta = \frac{1}{4m} G_t{}^s \nabla_s X^t.$$

Therefore we have $\mathcal{L}_X F - \nabla_X F = 0$ if and only if $G_t{}^s \nabla_s X^t = H_t{}^s \nabla_s X^t = 0$. Thus we see that

$$\mathcal{L}_X F - \nabla_X F = 0, \qquad \mathcal{L}_X G - \nabla_X G = 0, \qquad \mathcal{L}_X H - \nabla_X H = 0$$

hold if and only if $F_t{}^s \nabla_s X^t = G_t{}^s \nabla_s X^t = H_t{}^s \nabla_s X^t = 0$. Thus we have

LEMMA 2.3. *For an infinitesimal Q-transformation X in a quaternionic Kählerian manifold,*

(2.2)
$$F_t{}^s \nabla_s X^t = G_t{}^s \nabla_s X^t = H_t{}^s \nabla_s X^t = 0$$

hold if and only if

(2.3)
$$\mathcal{L}_X \phi - \nabla_X \phi = 0$$

for any cross-section ϕ of the bundle V.

§3. Some formulas.

In the present section, we give some formulas containing a vector field X in a quaternionic Kählerian manifold (M, g, V).

Consider a vector field X in (M, g, V) and denote by X^h components of X and put $X_\iota = g_{\iota s} X^s$ in a coordinate neighborhood U of M. First, we obtain in U

$$(3.1) \qquad \mathcal{L}_X \Lambda_{k}{}^{\jmath}{}_{\iota}{}^{h} = \nabla_k X^s \Lambda_{s}{}^{\jmath}{}_{\iota}{}^{h} - \nabla_\iota X^\jmath \Lambda_{k}{}^{t}{}_{\iota}{}^{h} + \nabla_\iota X^s \Lambda_{k}{}^{\jmath}{}_{s}{}^{h} - \nabla_\iota X^h \Lambda_{k}{}^{\jmath}{}_{\iota}{}^{t} ,$$

where

$$(3.2) \qquad \Lambda_{k}{}^{\jmath}{}_{\iota}{}^{h} = F_{k}{}^{\jmath} F_{\iota}{}^{h} + G_{k}{}^{\jmath} G_{\iota}{}^{h} + H_{k}{}^{\jmath} H_{\iota}{}^{h}$$

are components of Λ in U, since we have $\nabla \Lambda = 0$ because of (1.4) (see [9]).

We now assume that X is an infinitesimal Q-transformation of (M, g, V). Then, using (3.1), we have, by means of Proposition 2.2,

$$\nabla_k X^s \Lambda_{s}{}^{\jmath}{}_{\iota}{}^{h} - \nabla_\iota X^\jmath \Lambda_{k}{}^{t}{}_{\iota}{}^{h} + \nabla_\iota X^s \Lambda_{k}{}^{\jmath}{}_{s}{}^{h} - \nabla_\iota X^h \Lambda_{k}{}^{\jmath}{}_{\iota}{}^{t} = 0 ,$$

from which, differentiating covariantly,

$$(3.3) \qquad (\nabla_\iota \nabla_k X^s) \Lambda_{s}{}^{\jmath}{}_{\iota}{}^{h} - (\nabla_\iota \nabla_\iota X^\jmath) \Lambda_{k}{}^{t}{}_{\iota}{}^{h} + (\nabla_\iota \nabla_\iota X^s) \Lambda_{k}{}^{\jmath}{}_{s}{}^{h} - (\nabla_\iota \nabla_\iota X^h) \Lambda_{k}{}^{\jmath}{}_{\iota}{}^{t} = 0 ,$$

where we have used $\nabla \Lambda = 0$. Transvecting (3.3) with F^{lk} and using (1.1) and (3.2), we obtain

$$(F^{lk} \nabla_\iota \nabla_k X^s) \Lambda_{s}{}^{\jmath}{}_{\iota}{}^{h} + (\nabla^l \nabla_\iota X^\jmath) F_{\iota}{}^{h} + (\nabla_\iota \nabla_\iota X^\jmath)(H^{lt} G_{\iota}{}^{h} - G^{lt} H_{\iota}{}^{h})$$

$$- (\nabla^\jmath \nabla_\iota X^s) F_{s}{}^{h} + (\nabla_\iota \nabla_\iota X^s)(-H^{l\jmath} G_{s}{}^{h} + G^{l\jmath} H_{s}{}^{h})$$

$$+ (\nabla^\jmath \nabla_\iota X^h) F_{\iota}{}^{t} - (\nabla_\iota \nabla_\iota X^h)(-H^{l\jmath} G_{\iota}{}^{t} + G^{l\jmath} H_{\iota}{}^{t}) = 0 ,$$

where we have put $\nabla^h = g^{hs} \nabla_s$. Next, transvecting the equation above with $-F_h{}^\iota$, we obtain

$$(3.4) \qquad 2m F^{lh} F_{t}{}^{\jmath} \nabla_\iota \nabla_h X^t + 2m \nabla^l \nabla_\iota X^\jmath + (G^{l\jmath} G_{t}{}^{\iota} + H^{l\jmath} H_{t}{}^{\iota}) \nabla_\iota \nabla_\iota X^t = 0 ,$$

where $\dim M = 4m$.

On the other hand, taking account of (1.7), we have

$$(3.5) \qquad 2m F^{lk} F_{t}{}^{\jmath} \nabla_\iota \nabla_k X^t = m F^{lk} F_{t}{}^{\jmath}(\nabla_\iota \nabla_k X^t - \nabla_k \nabla_\iota X^t)$$

$$= m F^{lk} F_{t}{}^{\jmath} K_{lks}{}^{t} X^s$$

$$= \frac{mk}{2(m+2)} X^\jmath ,$$

where we have used the identity $\nabla_\iota \nabla_k X^t - \nabla_k \nabla_\iota X^t = K_{lks}{}^{t} X^s$. Substituting (3.5) into (3.4), we find, if $m > 1$,

$$(3.6) \qquad \frac{mk}{2(m+2)} X^\jmath + 2m \nabla^l \nabla_\iota X^\jmath + (G^{l\jmath} G_{t}{}^{\iota} + H^{l\jmath} H_{t}{}^{\iota}) \nabla_\iota \nabla_\iota X^t = 0 .$$

Similarly, we have, if $m>1$,

(3.7) $-\dfrac{mk}{2(m+2)}X^j+2m\nabla^l\nabla_l X^j+(H^{lj}H_t{}^i+F^{lj}F_t{}^i)\nabla_l\nabla_i X^t=0 ,$

(3.8) $-\dfrac{mk}{2(m+2)}X^j+2m\nabla^l\nabla_l X^j+(F^{lj}F_t{}^i+G^{lj}G_t{}^i)\nabla_l\nabla_i X^t =0 .$

Summing up (3.6), (3.7) and (3.8), we find, if $m>1$,

(3.9) $3m\Big[\dfrac{k}{4(m+2)}X^j+\nabla^l\nabla_l X^j\Big]+\Lambda^{lj}{}_t{}^i\nabla_l\nabla_i X^t=0 ,$

where $\Lambda^{lj}{}_t{}^i=g^{ls}\Lambda_s{}^j{}_t{}^i$. However, using (1.8) instead of (1.7), we can easily verify that (3.9) is still established even if $m=1$. Thus we have, from (3.9),

LEMMA 3.1. *A necessary condition for a vector field X to be an infinitesimal Q-transformations in a quaternionic Kählerian manifold is that*

(3.10) $3m\Big[\nabla^l\nabla_l X^j+\dfrac{k}{4(m+2)}X^j\Big]-\Lambda^{kjih}\nabla_k\nabla_i X_h=0 ,$

where $\Lambda^{kjih}=g^{kt}g^{is}\Lambda_t{}^j{}_s{}^h$ and $\dim M=4m$.

Given a tensor field T of arbitrary type, we denote by $\|T\|$ its length. Then, taking account of (3.1), we have by a straightforward computation

(3.11) $\|\mathcal{L}_X\Lambda\|^2=16m[3\|\nabla X\|^2-\Lambda^{kjih}(\nabla_k X_i)(\nabla_j X_h)]$

                        $-16[(F_i{}^h\nabla_h X^i)^2+(G_i{}^h\nabla_h X^i)^2+(H_i{}^h\nabla_h X^i)^2]$

for any vector field X in M.

On the other hand, we get

$\nabla_j[3(\nabla^j X^i)X_i-\Lambda^{kjih}(\nabla_k X_i)X_h]$

$=3\|\nabla X\|^2-\Lambda^{kjih}(\nabla_k X_i)(\nabla_j X_h)+3(\nabla^j\nabla_j X^i)X_i+\Lambda^{kjih}(\nabla_k\nabla_j X_i)X_h$

$=3\|\nabla X\|^2-\Lambda^{kjih}(\nabla_k X_i)(\nabla_j X_h)+3(\nabla^j\nabla_j X^i)X_i$

$\qquad\qquad +\dfrac{1}{2}\Lambda^{kjih}(\nabla_k\nabla_j X_i-\nabla_j\nabla_k X_i)X_h$

$=3\|\nabla X\|^2-\Lambda^{kjih}(\nabla_k X_i)(\nabla_j X_h)+3(\nabla^j\nabla_j X^i)X_i-\dfrac{1}{2}\Lambda^{kjih}K_{kji}{}^s X_s X_h ,$

from which, using (1.7) and (1.8),

(3.12) $\nabla_j[3(\nabla^j X^i)X_i-\Lambda^{kjih}(\nabla_k X_i)X_h]$

                $=[3\|\nabla X\|^2-\Lambda^{kjih}(\nabla_k X_i)(\nabla_j X_h)]$

                        $+3\Big[(\nabla^j\nabla_j X^i)X_i+\dfrac{k}{4(m+2)}\|X\|^2\Big] .$

We now have, using (3.11) and (3.12),

$$(3.13) \qquad 3m\Big[\nabla^j\nabla_j X^i + \frac{k}{4(m+2)} X^i\Big]X_i + \frac{1}{16}\|\mathscr{L}_X \Lambda\|^2$$
$$+ [(F_i{}^h\nabla_h X^i)^2 + (G_i{}^h\nabla_h X^i)^2 + (H_i{}^h\nabla_h X^i)^2]$$
$$= m\nabla_j[3(\nabla^j X^i)X_i - \Lambda^{kjih}(\nabla_k X_i)X_h]$$

for any vector field X in M.

Next, we have an identity

$$\nabla_k(\Lambda^{kjih} X_j \nabla_i X_h) = \Lambda^{kjih}(\nabla_k X_j)(\nabla_i X_h) + \Lambda^{kjih}(\nabla_k \nabla_i X_h)X_j,$$

from which,

$$(3.14) \qquad (F_i{}^h\nabla_h X^i)^2 + (G_i{}^h\nabla_h X^i)^2 + (H_i{}^h\nabla_h X^i)^2$$
$$= \Lambda^{kjih}(\nabla_k X_j)(\nabla_i X_h)$$
$$= -\Lambda^{kjih}(\nabla_k \nabla_i X_h)X_j + \nabla_k(\Lambda^{kjih} X_j \nabla_i X_h).$$

Substituting (3.14) into (3.13), we obtain

$$(3.15) \qquad 3m\Big[\nabla^j\nabla_j X^i + \frac{k}{4(m+2)} X^i\Big]X_i - \Lambda^{kjih}(\nabla_k \nabla_i X_h)X_j + \frac{1}{16}\|\mathscr{L}_X \Lambda\|^2$$
$$= \nabla_k[3m(\nabla^k X^i)X_i - \Lambda^{jkih}(\nabla_j X_i)X_h - \Lambda^{kjih} X_j \nabla_i X_h]$$

for any vector field X in M.

We now assume that M is compact. Then, integrating (3.13), we have an integral formula

$$(3.16) \qquad \int_M \Big[3m\Big[\nabla^j\nabla_j X^i + \frac{k}{4(m+2)} X^i\Big]X_i + \frac{1}{16}\|\mathscr{L}_X \Lambda\|^2$$
$$+ [(F_i{}^h\nabla_h X^i)^2 + (G_i{}^h\nabla_h X^i)^2 + (H_i{}^h\nabla_h X^i)^2]\Big]d\sigma = 0$$

for any vector field X in M, where $d\sigma$ denotes the volume element of (M, g). Next, integrating (3.15), we obtain another integral formula

$$(3.17) \qquad \int_M \Big[\Big\{3m\Big[\nabla^i\nabla_i X^i + \frac{k}{4(m+2)} X^i\Big]$$
$$- \Lambda^{kjih}(\nabla_k \nabla_j X_h)\Big\}X_i + \frac{1}{16}\|\mathscr{L}_X \Lambda\|^2\Big]d\sigma = 0$$

for any vector field X in M.

§4. Some theorems.

First, taking account of Lemma 3.1, we have, from (3.17),

THEOREM 4.1. *Let* (M, g, V) *be a compact quaternionic Kählerian manifold.*

Then a necessary and sufficient condition for a vector field X in M to be an infinitesimal Q-transformation is that

(4.1) $$3m\left[\nabla^l\nabla_l X^j+\frac{k}{4(m+2)}X^j\right]-\Lambda^{kjih}\nabla_k\nabla_i X_h=0,$$

where $\dim M=4m$.

Next, let X be a Killing vector in a quaternionic Kählerian manifold. Then, as is well known (see [9]), we have

(4.2) $$\nabla_k\nabla_j X^h+K_{ikj}{}^h X^i=0,$$

from which, using (1.6), (1.7) and (1.8),

(4.3) $$\nabla^l\nabla_l X^i+\frac{k}{4m}X^i=0,$$

(4.4) $$\Lambda^{kjih}\nabla_k\nabla_i X_h=-\frac{3k}{2(m+2)}X^j.$$

Using these equations, we easily see that X satisfies (4.1). Thus, from Theorem 4.1, we have

THEOREM 4.2. *In a compact quaternionic Kählerian manifold, a Killing vector is necessarily an automorphism.*

Substituting the identity

$$(\nabla^l\nabla_l X^i)X_i=\frac{1}{2}\nabla^l\nabla_l(X^iX_i)-(\nabla_k X_j)(\nabla^k X^j)$$

into (3.16) with $\mathcal{L}_X\Lambda=0$, we obtain an integral formula

(4.5) $$\int_M\left[-\frac{3mk}{4(m+2)}\|X\|^2+3m\|\nabla X\|^2\right.$$
$$\left.-[(F_i{}^h\nabla_h X^i)^2+(G_i{}^h\nabla_h X^i)^2+(H_i{}^h\nabla_h X^i)^2]\right]d\sigma=0$$

for any infinitesimal Q-transformation X in a compact quaternion Kählerian manifold (M, g, V). Thus, taking account of Lemma 2.3 and using (4.5), we have

THEOREM 4.3. *Let X be an infinitesimal Q-transformation in a compact quaternionic Kählerian manifold with scalar curvature k and assume that X satisfies $\mathcal{L}_X\phi-\nabla_X\phi=0$ for any cross-section ϕ of the bundle V. Then, if $k=0$, X is necessarily parallel. If $k<0$, X is necessarily zero.*

To give another proof of Theorem 4.3, we put

$$\Phi=F_{ih}dx^i\wedge dx^h, \qquad \Psi=G_{ih}dx^i\wedge dx^h, \qquad \Theta=H_{ih}dx^i\wedge dx^h,$$

in a coordinate neighborhood U of a quaternionic Kählerian manifold, where $F_{ih}=F_i{}^s g_{hs}$, $G_{ih}=G_i{}^s g_{hs}$ and $H_{ih}=H_i{}^s g_{hs}$ are all skewsymmetric. If we put in U

$$\Omega = \Phi \wedge \Phi + \Psi \wedge \Psi + \Theta \wedge \Theta \,,$$

then, using (1.2), we see that Ω determines a global 4-form in M, which is also denote by Ω. As a consequence of (1.4), we have $\nabla \Omega = 0$. Therefore Ω is a non-zero harmonic 4-form in M. Thus, assuming that M is compact, for any Killing vector X in M, we have $\mathcal{L}_X \Omega = 0$, from which, we have (2.1). Therefore, X is an infinitesimal Q-transformation. This proves Theorem 4.2.

Since any quaternionic Kählerian manifold is an Einstein space, its Ricci tensors is negative definite, if $k < 0$, and S is zero, if $k = 0$. Thus, in a compact quaternionic Kählerian manifold (M, g, V), there is no Killing vector other than zero if $k < 0$, and in (M, g, V) any Killing vector is parallel if $k = 0$ (see [3]). Therefore a compact quaternionic Kählerian manifold with vanishing scalar curvature is a torus if it admits a transitive group of isometries.

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