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DEGREES OF MAPS AND HOMOTOPY TYPE

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§1. Let K be an oriented Poincaré complex and let μ_K be the oriented fundamental class of K. The degree of a map $f: K \to K$ is defined by the formula $f_*(\mu_K) = (\deg f)\mu_K$, $(\deg f \in \boldsymbol{Z})$. Let $[K, K]$ be the set of homotopy classes of maps from K to K. Then we get a correspondence $D: [K, K] \to \boldsymbol{Z}$, which is defined by $D(\{f\}) = \det f$. We denote by $D(K)$ the image of D. In the case where K is of the form $S^n \cup e^{n+k} \cup e^{2n+k}$, S. Sasao has got some results about relations between $D(K)$ and the homotopy type of K ([7]). In this note we shall investigate the case where $K = S^n \cup e^{2n} \cup e^{3n}$ $(n \geqq 3)$ and prove the following theorem:

THEOREM. *Suppose $K = S^n \cup e^{2n} \cup e^{3n}$ $(n \geqq 3)$ is a Poincaré complex. Then K is homotopy equivalent to $S^n \times S^{2n}$ if and only if $D(K)$ contains 2 and EK is reducible.*

§2. Let $K = S^n \cup e^{2n} \cup e^{3n}$ $(n \geqq 3)$ be a complex and x_i $(i = 1, 2, 3)$ be the oriented generators of $H^{in}(K; \boldsymbol{Z})$. Then the cohomology ring structure of $H(K; \boldsymbol{Z})$ is completely determined by two integers a, b such that

$$x_1^2 = a x_2 \quad \text{and} \quad x_1 x_2 = b x_3 .$$

If K is a Poincaré complex, we have $b = \pm 1$. Hence we can suppose $b = 1$ without the loss of generality.

Let $\alpha \in \pi_{2n-1}(S^n)$ be the homotopy class of the attaching map of e^{2n}. Then the following lemma is well known ([8]).

LEMMA 1. *The Hopf invariant $H(\alpha)$ of α is equal to $\pm a$.*

Let $f: K \to K$ be a map such that $f^*(x_1) = k x_1$ $(k \in \boldsymbol{Z})$. Then we have

$$(f^*(x_1))^3 = k^3 x_1^3 = k^3 a x_1 x_2 = k^3 a x_3$$

and

$$(f^*(x_1))^3 = f^*(x_1^3) = f^*(a x_1 x_2) = a(\deg f) x_3 .$$

Hence we obtain $a(k^3 - \deg f) = 0$. Thus if $D(K)$ contains 2, we have $a = 0$, i.e. $H(\alpha) = 0$.

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LEMMA 2. *Suppose K is a Poincare complex and $D(K)$ contains 2. Then K is of the form $(S^n \vee S^{2n}) \cup e^{3n}$.*

Proof. Let $f: K \to K$ be a map with $\deg f = 2$. And let $f^*(x_1) = ax_1$ and $f^*(x_2) = bx_2$. Then $f^*(x_3) = f^*(x_1 x_2) = abx_3$, and so it follows that $ab = 2$. On the other hand, let $S^n \cup e^{2n}$ be the $2n$-skelton of K. Then we have the following commutative diagram

$$\begin{array}{ccc} \pi_{2n}(S^n \cup e^{2n}, S^n) & \xrightarrow{f_*} & \pi_{2n}(S^n \cup e^{2n}, S^n) \\ \partial \downarrow & & \partial \downarrow \\ \pi_{2n-1}(S^n) & \xrightarrow{f_*} & \pi_{2n-1}(S^n) \end{array}$$

where $\pi_{2n}(S^n \cup e^{2n}, S^n)$ is free generated by σ and $\partial\sigma = \alpha$. Since $b\alpha = \partial(b\sigma) = \partial f_* \sigma = f_* \partial \sigma = f_* \alpha = a \iota_n \circ \alpha$ and α is a suspension element because of $H(\alpha) = 0$, we have $2\alpha = \alpha$, i. e. $\alpha = 0$. Thus the proof of lemma is completed.

Let $\beta \in \pi_{3n-1}(S^n \vee S^{2n})$ be the homotopy class of the attaching map for the cell e^{3n} of a Poinearé complex $K = (S^n \vee S^{2n}) \cup e^{3n}$. Since $\pi_{3n-1}(S^n \vee S^{2n})$ is isomorphic to the direct sum

$$\pi_{3n-1}(S^n) + \pi_{3n-1}(S^{2n}) + [\pi_n(S^n), \pi_{2n}(S^{2n})],$$

β has an expression $\iota_n \circ \beta_1 + \iota_{2n} \circ \beta_2 + m[\iota_n, \iota_{2n}]$ where $\beta_1 \in \pi_{3n-1}(S^n)$, $\beta_2 \in \pi_{3n-1}(S^{2n})$, $m \in \boldsymbol{Z}$ and ι_n, ι_{2n} denote the homotopy classes of inclusions S^n, $S^{2n} \to S^n \vee S^{2n}$ respectively. Then we may suppose $m = 1$ since K is a Poincaré complex ([5]).

LEMMA 3. *If $n \geqq 3$, then the kernel of the suspension $E: \pi_{3n-1}(S^n) \to \pi_{3n}(S^{n+1})$ is equal to $[\iota_n, \iota_n] \circ \pi_{3n-1}(S^{2n-1}) = [\iota_n, \pi_{2n}(S^n)]$.*

Proof. By Theorem 1.2 and Corollary 1.10 of [4], we have $E^{-1}(0) = [\iota_n, \iota_n] \circ \pi_{3n-1}(S^{2n-1})$. By Theorem 7.1 of [1], for any $\gamma \in \pi_{2n}(S^n)$,

$$[\iota_n, \gamma] = [\iota_n, \iota_n] \circ E^{n-1}(\gamma) + (-1)^{n+1}[\iota_n, [\iota_n, \iota_n]] \circ E^{n-1}(H(\gamma)).$$

Then from $H(\gamma) \in \pi_{2n}(S^{2n-1}) \approx Z_2$ and $3[\iota_n, [\iota_n, \iota_n]] = 0$, it follows that

$$[\iota_n, \pi_{2n}(S^n)] = [\iota_n, \iota_n] \circ E^{n-1}(\pi_{2n}(S^n)).$$

On the other hand $\pi_{2n+1}(S^{n+1})$ is generated by $E(\pi_{2n}(S^n))$ and the Whitehead product $[\iota_{n+1}, \iota_{n+1}]$, so that we obtain $E^{n-1}(\pi_{2n}(S^n)) = \pi_{3n-1}(S^{2n-1})$. Thus the proof is completed.

§ 3. **Proof of Theorem:** Suppose K is homotopy equivalent to $S^n \times S^{2n}$. Then it is clear that $D(K) = D(S^n \times S^{2n}) = Z$. Furthermore, the reducibility of EK is homotopy invariant and so EK is reducible.

Conversely suppose $D(K)$ contains 2 and EK is reducible. By Lemma 2, K is homotopy equivalent to a complex $(S^n \vee S^{2n}) \cup e^{3n}$ and also $E(\beta) = 0$ is equivalent to the reducibility of EK. Hence it follows that $E(\beta_1) = 0$ and $E(\beta_2) = 0$

in the expression as in § 2. Thus we have $\beta_2=0$ since $E: \pi_{3n-1}(S^{2n})\rightarrow\pi_{3n}(S^{2n+1})$ is an isomorphism. And moreover, by Lemma 3, we have $\beta_1=[\iota_n, \gamma]$ for some element $\gamma\in\pi_{2n}(S^n)$. Then let $f: S^n\vee S^{2n}\rightarrow S^n\vee S^{2n}$ be a map such that $\{f|S^n\}=\iota_n$, $\{f|S^{2n}\}=\gamma+\iota_{2n}$. Let τ be the generator of $\pi_{3n}(S^n\times S^{2n}, S^n\vee S^{2n})$ such that $\partial\tau=[\iota_n, \iota_{2n}]$. Then $f_*\partial\tau=f_*[\iota_n, \iota_{2n}]=[\iota_n, \gamma+\iota_{2n}]$, which is the homotopy class of the attaching map for e^{3n} of K. Clearly, an extension of f, $S^n\times S^{2n}\rightarrow K$ such that $\deg f=1$ is a homotopy equivalence. Thus the proof is completed.

Remark. Typical examples of Poincaré complexes of the form $S^n\cup e^{2n}\cup e^{3n}$ are the total space of S^n-orthogonal bundles over S^{2n}. Let E be the total space of a S^n-orthogonal bundle over S^{2n}. Then, by Lemma 2 and some computations, we have

(1) If $D(E)$ contains 2, the bundle has a cross section.

(2) If the bundle has two independent cross sections, then $D(E)=Z$.

§ 4. **Addendum:** If $M=S^n\cup e^{2n}\cup e^{3n}$ has the same homotopy type as $S^n\times S^{2n}$, it is clear that S^n is a retraction of M. In general the converse is not true. However, in some cases the converse is true. For example we have;

THEOREM. *Let M be a smooth closed manifold up to homotopy. If $n\equiv 3, 5, 6, 7 \bmod 8$, M has the same homotopy type as $S^n\times S^{2n}$ if and only if S^n is a retract of M.*

For the proof we need the following lemma. Let ν be the characteristic element of the normal bundle of an embedding of S^n into M such that $H_n(S^n)\rightarrow H_n(M)$ is an isomorphism, and $p: S^n\cup e^{2n}\rightarrow S^{2n}$ be the pinching map.

LEMMA. *$p_*(\tau)=\pm J\nu$ where τ denotes the attaching map for e^{3n} and J is the J-homomorphism $\cdot$ $\pi_{n-1}(SO(2n))\rightarrow\pi_{3n-1}(S^{2n})$.*

Proof. Let $S\nu$, $D\nu$ be the associated sphere bundle, disk bundle. Then $S\nu$ is of the form $(S^n\vee S^{2n-1})\cup e^{3n-1}$ up to homotopy and the attaching map for e^{3n} is $J\bar{\nu}+[\iota_n, \iota_{2n-1}]$ where $i_*\bar{\nu}=\nu$, $i_*: \pi_{n-1}(SO(2n-1))\rightarrow\pi_{n-1}(SO(2n))$ is an isomorphism induced by the canonical inclusion. The suspension of $S\nu$, $ES\nu$ is of the form $S^{n+1}\vee(S^{2n}\cup e^{3n})$. $D\nu=S^n\rightarrow T\nu=S^{2n}\cup e^{3n}\rightarrow ES\nu=S^{n+1}\vee(S^{2n}\cup e^{3n})\rightarrow ED\nu=S^{n+1}$ is a cofibration sequence where $T\nu$ is the Thom space of ν. Then there is an inclusion $S^{n+1}=ED\nu\rightarrow ES\nu$ induced by a cross section. It is easily verified that $T\nu$ is homotopy equivalent to $ES\nu/S^{n+1}=S^{2n}\cup e^{3n}$ where the attaching map for e^{3n} is $E(J\bar{\nu}+[\iota_n, \iota_{2n-1}])=EJ\bar{\nu}=-Ji_*\bar{\nu}=-J\nu$.

Let $i: S^n\rightarrow M$ be the embedding and $j: M\rightarrow T\nu$ be projection. In the following diagram it is verified that maps except j^{2n} are isomorphisms, by Thom isomorphism, Poincare duality. Hence j^{2n} is also.

$$\begin{array}{ccc} H^n(S^n)\otimes H^{2n}(T\nu) & \longrightarrow & H^{3n}(T\nu) \\ \uparrow i^n \quad \downarrow j^{2n} & & \downarrow j^{3n} \\ H^n(M)\otimes H^{2n}(M) & \longrightarrow & H^{3n}(M)\,. \end{array}$$

We may consider j as $q \circ p$ where $p: M \to M/S^n$ is the canonical projection and $q: M/S^n \to T\nu$ is a map induced by j. From the above argument it follows that q is a homotopy equivalence.

$$
\begin{array}{ccccc}
\pi_{3n}(M, S^n \cup e^{2n}) & \xrightarrow{p_*} & \pi_{3n}(M/S^n, S^{2n}) & \xrightarrow{q_*} & \pi_{3n}(T_\nu, S^{2n}) \\
\downarrow \partial & & \downarrow \partial & \swarrow \partial & \\
\pi_{3n-1}(S^n \cup e^{2n}) & \xrightarrow{p_*} & \pi_{3n-1}(S^{2n}) & &
\end{array}
$$

Let σ be the generator of $\pi_{3n}(M, S^n \cup e^{2n})$ such that $\partial\sigma = \tau$. Then $j_*\sigma$ is a generator of $\pi_{3n}(T\nu, S^{2n})$ and so $p_*(\tau) = \partial(j_*\sigma) = \pm J\nu$.

Now we proceed to the proof of the theorem. Let $r: M \to S^n$ be a retraction. Obviously we may suppose that M has a form $S^n \vee S^{2n} \cup e^{3n}$, and $\tau = [\iota_n, \iota_{2n}] + \iota_n \circ \beta_1 + \iota_{2n} \circ \beta_2$. Since $p_*(\tau) = \beta_2$, we get $\beta_2 = 0$ by the lemma and $n \equiv 3, 5, 6, 7 \bmod 8$. Let γ be the map $r | S^{2n}$. Then we have $(r | S^n \vee S^{2n})_*(\tau) = [\iota_n, \gamma] + \beta_1$. Hence we have $[\iota_n, \gamma] \in [\iota_n, \pi_{2n}(S^n)]$ and the proof is completed by Lemma 3 in § 2.

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