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ON NON-PARAMETRIC SURFACES IN THREE DIMENSIONAL SPHERES

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0. Introduction.

Let D be a bounded domain with boundary ∂D in the Euclidean 2-plane E^2. We denote by $C^2(D)$ the set of real-valued functions of class C^2 on D. For a function $u \in C^2(D)$ we consider the non-parametric surface M in the Euclidean 3-space E^3 defined by

$$\tilde{u}(x)=(x_1, x_2, u(x)) \in E^3, \qquad x=(x_1, x_2) \in D. \tag{0.1}$$

Now we take the unit normal vector field η on M as follows:

$$\eta=\frac{1}{\sqrt{1+|\nabla u|^2}}(-p, -q, 1),$$

where $p=\partial u/\partial x_1$, $q=\partial u/\partial x_2$ and $|\nabla u|^2=p^2+q^2$. Then the mean curvature H of M with respect to η is expressed as

$$H(x)=\frac{1}{2} \operatorname{div} W(x) \qquad \text{at each point } x \in D,$$

where $W=\frac{1}{\sqrt{1+|\nabla u|^2}}(p, q)$. It can be rewritten as follows:

$$(1+q^2)r-2pqs+(1+p^2)t=2H(1+|\nabla u|^2)^{3/2}, \tag{0.2}$$

where $r=\partial^2 u/\partial x_1^2$, $s=\partial^2 u/\partial x_1 \partial x_2$, $t=\partial^2 u/\partial x_2^2$.

Conversely, let H be a given continuous real-valued function on D. If $u \in C^2(D)$ is a solution of the equation (0.2), then for this u the mean curvature of the surface in E^3 defined by (0.1) is equal to H.

Now, we assume that the boundary ∂D of D is smooth. Let $\mathcal{A}$ and $\mathcal{L}$ be the area of D and the length of ∂D respectively. The following theorem was proved by R. Finn [3].

THEOREM. *For a function $u \in C^2(D)$ and a positive constant H_0 suppose that the mean curvature H of the non-parametric surface in E^3 defined by* (0.1) *satisfies the inequality $|H(x)| \geqq H_0$ for all $x \in D$. Then we have $\mathcal{A}/\mathcal{L} \leqq 1/2H_0$. In particular, if D is the disk of radius R, then $RH_0 \leqq 1$.*

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It is interesting that H_0 is restricted by the geometrical quantity of D. From a viewpoint of the theory of differential equation the second assertion of the above theorem implies the following:

Let H_0 be a positive constant and H a continuous real-valued function on D. Assume that $H(x) \geqq H_0$ for all $x \in D$. If the equation (0.2) has a solution, then D can not contain the disk of radius $1/H_0$.

The second assertion of the above theorem was also proved by E. Heinz [4]. S. S. Chern extended the results of E. Heinz to higher dimensional Euclidean spaces [1].

The purpose of this paper is to study non-parametric surfaces in $S^3(a)$ from the viewpoint stated above, where $S^3(a)$ denotes the Euclidean 3-sphere of radius a. In Section 1, we show that the mean curvature of a non-parametric surface in $S^3(a)$ can be expressed by the divergence form (1.9). From this we get the same result as that of R. Finn stated above.

Rewriting the equation (1.9), we have the quasi-linear elliptic partial differential equation of second order (2.3). It is complicated in comparison with the equation (0.2). In fact, let $u \in C^2(D)$ be any solution of the equation (0.2). Then, for example, we have the following:

(1) For any constant c, $u+c$ is also a solution of the equation (0.2).

(2) For any solution v of the equation (0.2) which agrees with u on the boundary of D equals u throughout D.

But the above properties do not always hold for the equation (2.3) because its coefficients contain the unknown function u as a variable.

In Section 2, we study the partial differential inequality (2.5). It is obtained from some geometrical condition which is connected with the mean curvature of non-parametric surfaces in $S^3(a)$. We prove that the minimum principle holds for a solution of the inequality (2.5). From this result we can conclude that the position of non-parametric surfaces with boundary in $S^3(a)$ is restricted by its mean curvature and the position of its boundary. In Section 3 we study a smilar problem as in Section 2.

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1. The mean curvature of non-parametric surfaces in $S^3(a)$.

Let D be a bounded domain with boundary ∂D in the Euclidean 2-plane E^2. We denote by $\bar{D}$ the closure of D. $C^2(D)$ denotes the set of real-valued functions of class C^2 on D.

In the following, let a and k be positive constants satisfying

$$a^2 > b^2 + k^2 , \tag{1.1}$$

where $b = \max_{x \in \bar{D}} |x|$, $x = (x_1, x_2) \in E^2$ and $|x|^2 = x_1{}^2 + x_2{}^2$. Let $S^3(a)$ be the 3-dimensional sphere of radius a in the Euclidean 4-space E^4.

For a function $u \in C^2(D)$ satisfying $|u(x)| \leqq k$ for all $x \in D$, we consider the non-parametric surface M in $S^3(a)$ defined by

$$\tilde{u}(x)=(x_1, x_2, u(x), U(x))\in S^3(a),\ x=(x_1, x_2)\in D, \tag{1.2}$$

where $U(x)=\sqrt{a^2-|x|^2-u(x)^2}$. We put

$$X_1=(1, 0, p, U_1), \qquad X_2=(0, 1, q, U_2), \tag{1.3}$$

where $p=\partial u/\partial x_1$, $q=\partial u/\partial x_2$ and $U_i=\partial U/\partial x_i$, $i=1, 2$. Then X_1 and X_2 are linearly independent tangent vector fields on M. We can take the unit normal vector field η on M in $S^3(a)$ as follows:

We put $\eta=(\eta_1, \eta_2, \eta_3, \eta_4)$. Then each component of η is given by

$$\begin{aligned}
\eta_1&=-\{a^2p+(u-\nabla u\cdot x)x_1\}/a\sqrt{g}, \\
\eta_2&=-\{a^2q+(u-\nabla u\cdot x)x_2\}/a\sqrt{g}, \\
\eta_3&=\{a^2-(u-\nabla u\cdot x)u\}/a\sqrt{g}, \\
\eta_4&=-(u-\nabla u\cdot x)U/a\sqrt{g},
\end{aligned} \tag{1.4}$$

where $g=a^2(1+|\nabla u|^2)-(u-\nabla u\cdot x)^2>0$, $\nabla u=(p, q)$ and $\nabla u\cdot x=px_1+qx_2$.

Now, we put $N=-(1/a)\tilde{u}(x)$, $x\in D$. Then we have

$$N\cdot X_i=N\cdot\eta=\frac{\partial N}{\partial x_i}\cdot\eta=0, \qquad i=1, 2, \tag{1.5}$$

where the dot denotes the inner product in E^4. For a moment we denote by D the Riemannian connection on $S^3(a)$ defined by the standard Riemannian metric of $S^3(a)$. Then, at each point of M we have

$$\frac{\partial\eta}{\partial x_i}=D_{X_i}\eta+h(X_i, \eta)N, \qquad i=1, 2.$$

By (1.5) we have

$$h(X_i, \eta)=\frac{\partial\eta}{\partial x_i}\cdot N=-\eta\cdot\frac{\partial N}{\partial x_i}=0, \qquad i=1, 2.$$

Hence we have

$$\frac{\partial\eta}{\partial x_i}=D_{X_i}\eta, \qquad i=1, 2.$$

By the Weingarten's formula $D_{X_1}\eta$ and $D_{X_2}\eta$ are expressed as

$$D_{X_i}\eta=a_{i1}X_1+a_{i2}X_2, \qquad i=1, 2, \tag{1.7}$$

where a_{ij}, $i, j=1, 2$, are continuous functions on D. By (1.3), (1.4), (1.6) and (1.7) we have

$$a_{11}=\frac{\partial\eta_1}{\partial x_1}, \qquad a_{22}=\frac{\partial\eta_2}{\partial x_2}. \tag{1.8}$$

Let H be the mean curvature of M with respect to the direction η. Then, by (1.4), (1.7) and (1.8) we have

$$H=-\frac{1}{2}(a_{11}+a_{22})=\frac{1}{2}\operatorname{div} W, \tag{1.9}$$

where

$$W=(\{a^2p+(u-\nabla u\cdot x)x_1\}/a\sqrt{g},\ \{a^2q+(u-\nabla u\cdot x)x_2\}/a\sqrt{g}).$$

In what follows and in the following sections, we always understand that the mean curvature of non-parametric surfaces in $S^3(a)$ defined by (1.2) is derived from η given by (1.4).

We shall prove the following theorem.

THEOREM 1.1. *Let D be a bounded domain in E^2 with boundary ∂D which consists of finitely many non-intersecting closed Jordan curves of class C^2. For a function $u\in C^2(D)$ satisfying $|u(x)|\leqq k$ for all $x\in D$, let M be the non-parametric surface in $S^3(a)$ defined by* (1.2) *and H the mean curvature of M. For a positive constant H_0, suppose that H satisfies the inequality $|H(x)|\geqq H_0$ for all $x\in D$. Then we have*

$$\mathcal{A}/\mathcal{L}\leqq 1/2H_0,$$

where $\mathcal{A}$ and $\mathcal{L}$ denote the area of D and the length of ∂D respectively.

Proof. For a positive number ε, we put $D_\varepsilon=\{x\in D;\ d(x,\partial D)>\varepsilon\}$ where $d(x,\partial D)$ denotes the distance from x to ∂D. Then, by taking a sufficiently small positive number δ, we can assume that the boundary ∂D_ε of D_ε is of class C^1 for any ε such that $0<\varepsilon<\delta$. Therefore we may assume that $\mathcal{A}_\varepsilon$ and $\mathcal{L}_\varepsilon$ converge to $\mathcal{A}$ and $\mathcal{L}$ respectively as $\varepsilon\to 0$, where $\mathcal{A}_\varepsilon$ and $\mathcal{L}_\varepsilon$ denote the area of D_ε and the length of ∂D_ε respectively. Without loss of generality, we can assume that $H(x)\geqq H_0$ for all $x\in D$. For a ε such that $0<\varepsilon<\delta$, let n_ε be the outward unit normal vector field of ∂D_ε. By the divergence formula and (1.9), we have

$$\iint_{D_\varepsilon} 2H\,dx_1\wedge dx_2=\iint_{D_\varepsilon}\operatorname{div} W\,dx_1\wedge dx_2=\int_{\partial D_\varepsilon} W\cdot n_\varepsilon ds<\mathcal{L}_\varepsilon.$$

On the other hand, we have

$$\iint_{D_\varepsilon} 2H\,dx_1\wedge dx_2\geqq 2H_0\mathcal{A}_\varepsilon,$$

because $H(x)\geqq H_0$ for all $x\in D$. From the above inequalities we have

$$2H_0\mathcal{A}_\varepsilon<\mathcal{L}_\varepsilon.$$

Thus, letting $\varepsilon\to 0$ in the last inequality, we obtain $\mathcal{A}/\mathcal{L}\leqq 1/2H_0$.

COROLLARY 1.1. *In Theorem* 1.1, *suppose that D is the disk of radius R. Then we have $RH_0\leqq 1$.*

COROLLARY 1.2. *Under the same condition as in Corollary* 1.1, *suppose that the Gaussian curvature K of M satisfies the inequality $K\geqq K_0$ for a positive constant K_0 such that $K_0>a^{-2}$. Then we have*

$$R\cdot\sqrt{K_0-a^{-2}}\leqq 1\,.$$

Proof. By the equation of Gauss, we have

$$K(x)=a^{-2}+\lambda_1\cdot\lambda_2\,,$$

where λ_1 and λ_2 are eigenvalues of the second fundamental form of M in $S^3(a)$ at a point $\tilde{u}(x)\in M$, $x\in D$. Since $\lambda_1\cdot\lambda_2\leqq((\lambda_1+\lambda_2)/2)^2=H(x)^2$ and $K(x)-a^{-2}\geqq K_0-a^{-2}>0$, we have

$$|H(x)|\geqq\sqrt{K_0-a^{-2}}\qquad\text{for all}\quad x\in D\,.$$

Therefore, from Corollary 1.1 we obtain $R\cdot\sqrt{K_0-a^{-2}}\leqq 1$.

2. Non-parametric surfaces with boundary and the minimum principle.

Throughout this section, let D be a bounded domain with boundary ∂D in the Euclidean 2-plane E^2 and $C^{0,2}(\bar{D}, D)$ the set of continuous real-valued functions on $\bar{D}$ which are of class C^2 in D, where $\bar{D}=D\cup\partial D$. Moreover, in the following, let a and k be positive constants such that

$$a^2>b^2+k^2\,, \tag{2.1}$$

where $b=\max_{x\in\bar{D}}|x|$, $x=(x_1, x_2)\in E^2$ and $|x|^2=x_1{}^2+x_2{}^2$.

For a function $u\in C^{0,2}(\bar{D}, D)$ satisfying $|u(x)|\leqq k$ for all $x\in\bar{D}$, we consider the non-parametric surface M with boundary in $S^3(a)$ defined by

$$\tilde{u}(x)=(x_1, x_2, u(x), \sqrt{a^2-|x|^2-u(x)^2})\in S^3(a)\,,\qquad x\in\bar{D}\,, \tag{2.2}$$

where $S^3(a)$ denotes the 3-dimensional sphere of radius a in the Euclidean 4-space E^4. Let η be the unit normal vector field on M in $S^3(a)$ which is given by (1.4). Then, by (1.9), the mean curvature H of M is expressed as

$$H(x)=\frac{1}{2}\left\{\frac{\partial}{\partial x_1}\left(\frac{a^2p+(u-\nabla u\cdot x)x_1}{a\sqrt{g}}\right)+\frac{\partial}{\partial x_2}\left(\frac{a^2q+(u-\nabla u\cdot x)x_2}{a\sqrt{g}}\right)\right\}.$$

We can rewrite it as

$$\sum_{i,j=1}^{2}A_{ij}(x, u, \nabla u)u_{ij}=A(x, u, \nabla u, H)\,, \tag{2.3}$$

where $u\in C^{0,2}(\bar{D}, D)$, $|u(x)|\leqq k$ for all $x\in\bar{D}$, $u_{ij}=\partial^2u/\partial x_i\partial x_j$, $i, j=1, 2$, and

$$\begin{aligned} &A_{11}(x, u, \nabla u)=a^2(1+q^2)-|x|^2q^2-x_1{}^2-u^2+2qux_2\,,\\ &A_{12}(x, u, \nabla u)=-\{a^2pq-|x|^2pq+x_1x_2+u(px_2+qx_1)\}\,,\\ &A_{21}(x, u, \nabla u)=A_{12}(x, u, \nabla u)\,,\\ &A_{22}(x, u, \nabla u)=a^2(1+p^2)-|x|^2p^2-x_2{}^2-u^2+2pux_1\,,\\ &A(x, u, \nabla u, H)=\frac{2}{a^2}g\{aH\sqrt{g}-(u-\nabla u\cdot x)\}\,, \end{aligned} \tag{2.4}$$

$$g=a^2(1+|\nabla u|^2)-(u-\nabla u\cdot x)^2,\quad |x|^2=x_1{}^2+x_2{}^2,$$

$$\nabla u=\left(\frac{\partial u}{\partial x_1},\ \frac{\partial u}{\partial x_2}\right)=(p,\ q),\qquad \nabla u\cdot x=px_1+qx_2\,.$$

Conversely, let H be a given continuous real-valued function on D. If $u\in C^{0,2}(\bar{D},\ D)$ is a solution of the equation (2.3), then for this u the mean curvature of the surface in $S^3(a)$ defined by (2.2) equals H.

Now, we set

$$Q_m^k=\{(x_1,\ x_2,\ x_3,\ x_4)\in S^3(a)\,;\ m\leqq x_3\leqq k,\ x_4>0\}$$

for a constant m such that $0<m<k$.

THEOREM 2.1. *Let H_0 be a constant such that $0<H_0<k/a\sqrt{a^2-k^2}$. For a function $u\in C^{0,2}(\bar{D},\ D)$ satisfying the inequality*

$$m_0:=a^2H_0/\sqrt{a^2H_0{}^2+1}\leqq u(x)\leqq k\qquad \textit{for all}\quad x\in\bar{D}\,,$$

let M be the surface with boundary in $S^3(a)$ defined by (2.2) *and H the mean curvature of M in $S^3(a)$. Suppose that H satisfies the inequality $H(x)\leqq H_0$ for all $x\in D$. Let m_1 be a constant such that $m_0<m_1<k$. If $\tilde{u}(\partial D)\subset Q_{m_1}^k$, then $\tilde{u}(\bar{D})\subset Q_{m_1}^k$.*

Remark. We note that $k/a\sqrt{a^2-k^2}$ equals the mean curvature of the small 2-sphere in $S^3(a)$ which is the intersection of $S^3(a)$ and the hyperplane in E^4 defined by $x_3=k$.

Now, let H' be a given continuous function on D. For this H', we define the operator $L_{H'}$ on $C^{0,2}(\bar{D},\ D)$ by

$$L_{H'}(v)=\sum_{i,j=1}^{2}A_{ij}(x,\ v,\ \nabla v)v_{ij}-A(x,\ v,\ \nabla v,\ H')\,,$$

where $v\in C^{0,2}(\bar{D},\ D)$, $|v(x)|\leqq k$ for all $x\in\bar{D}$, $v_{ij}=\partial^2 v/\partial x_i\partial x_j$, $i,\ j=1,\ 2$, and $A_{ij}(x,\ v,\ \nabla v)$, $i,\ j=1,\ 2$, and $A(x,\ v,\ \nabla v,\ H')$ are given in (2.4).

Under the hypotheses of Theorem 2.1, we have $L_H(u)=0$ and

$$L_{H_0}(u)=L_{H_0}(u)-L_H(u)=\frac{2}{a}g\sqrt{g}\,(H-H_0)\leqq 0\,.$$

In what follows, we shall consider the following partial differential inequality on $\bar{D}$:

$$\sum_{i,j=1}^{2}A_{ij}(x,\ v,\ \nabla v)v_{ij}\leqq A(x,\ v,\ \nabla v,\ H_0)\,, \tag{2.5}$$

where $v\in C^{0,2}(\bar{D},\ D)$, $|v(x)|\leqq k$ for all $x\in\bar{D}$ and H_0 is a constant such that $0<H_0<k/a\sqrt{a^2-k^2}$ and $A_{ij}(x,\ v,\ \nabla v)$, $i,\ j=1,\ 2$, and $A(x,\ v,\ \nabla v,\ H_0)$ are given in (2.4).

Theorem 2.1 follows immediately from the following theorem.

THEOREM 2.2. *Suppose that $u\in C^{0,2}(\bar{D},\ D)$ is a solution of the inequality* (2.5) *satisfying*

(2.6) $$m_0 \leqq u(x) \leqq k \quad \text{for all} \quad x \in \bar{D},$$

where $m_0 = a^2 H_0 / \sqrt{a^2 H_2^0 + 1}$. *Let* m_1 *be a constant such that* $m_0 < m_1 < k$. *If* $u \geqq m_1$ *on* ∂D, *then* $u \geqq m_1$ *in* D.

We first prove some lemmas. From (2.4) we have

(2.7) $$\begin{aligned} &A_{11}(x, u, 0) = a^2 - x_1^2 - u^2, \quad A_{12}(x, u, 0) = -x_1 x_2 = A_{21}(x, u, 0), \\ &A_{22}(x, u, 0) = a^2 - x_2^2 - u^2, \quad A(x, u, 0, H_0) = \frac{2}{a^2}(a^2 - u^2)(aH_0\sqrt{a^2 - u^2} - u). \end{aligned}$$

By (2.1) and (2.7), we have

LEMMA 2.1. *For all* $x \in D$, *the* 2×2 *matrix* $\tilde{A}(x) := (A_{ij}(x, u(x), 0))$ *is positive definite.*

LEMMA 2.2. *For all* $x \in D$, *we have*

(1) $$|A_{11}(x, u, \nabla u) - A_{11}(x, u, 0)| \leqq a^2(|q|^2 + 2|q|),$$

(2) $$|A_{12}(x, u, \nabla u) - A_{12}(x, u, 0)| \leqq a^2(|p||q| + |p| + |q|),$$

(3) $$|A_{22}(x, u, \nabla u) - A_{22}(x, u, 0)| \leqq a^2(|p|^2 + 2|p|).$$

Proof. We note that $|x|$ and $|u| := \sup_{x \in D} |u(x)|$ are smaller than a.

(1): $$\begin{aligned} &|A_{11}(x, u, \nabla u) - A_{11}(x, u, 0)| \\ &= |(a^2 - |x|^2)q^2 + 2qux_2| \leqq a^2|q|^2 + 2|q||u||x_2| \\ &\leqq a^2(|q|^2 + 2|q|). \end{aligned}$$

(2): $$\begin{aligned} &|A_{12}(x, u, \nabla u) - A_{12}(x, u, 0)| \\ &= |(a^2 - |x|^2)pq + u(px_2 + qx_1)| \leqq a^2|p||q| + |u|(|p||x_2| + |q||x_1|) \\ &\leqq a^2(|p||q| + |p| + |q|). \end{aligned}$$

By the same way as in (1), we can prove (3).

LEMMA 2.3. *For all* $x \in D$, *we have*

$$\begin{aligned} &|A(x, u, \nabla u, H_0) - A(x, u, 0, H_0)| \\ &\qquad \leqq 4a(a^4 G H_0 + 1)(|p| + |q|) + \frac{2}{a^2}(aGH_0 P_1 + P_2), \end{aligned}$$

where $G = \{g\sqrt{g} + (a^2 - u^2)\sqrt{a^2 - u^2}\}^{-1}$ *and* P_1, P_2 *are polynomials of* $|p|$ *and* $|q|$ *such that the degree of each term is greater than* 1 *and the coefficient of each term is a function of* a.

Proof.

$$|A(x, u, \nabla u, H_0)-A(x, u, 0, H_0)|$$
$$=\frac{2}{a^2}|aH_0\{g\sqrt{g}-(a^2-u^2)\sqrt{a^2-u^2}\}+(a^2-u^2)u-g(u-\nabla u\cdot x)|$$
$$\leqq\frac{2}{a}H_0|g\sqrt{g}-(a^2-u^2)\sqrt{a^2-u^2}|+\frac{2}{a^2}|g(u-\nabla u\cdot x)-(a^2-u^2)u|$$
$$=\frac{2}{a}H_0|g^3-(a^2-u^2)^3|G+\frac{2}{a^2}|g(u-\nabla u\cdot x)-(a^2-u^2)u|,$$

where $G=\{g\sqrt{g}+(a^2-u^2)\sqrt{a^2-u^2}\}^{-1}$. By a direct calculation, we have

$$g^3-(a^2-u^2)^3=6(a^4u-2a^2u^3+u^5)(\nabla u\cdot x)+P,$$

where P is a polynomial of p and q such that the degree of each term is greater than 1 and the coefficient of each term is a function of a and u. Now, we have

$$0<6(a^4u-2a^2u^3+u^5)\leqq\frac{96}{25\sqrt{5}}a^5<2a^5$$

and

$$|\nabla u\cdot x|=|px_1+qx_2|\leqq a(|p|+|q|).$$

Thus, from the above inequalities, we obtain

$$|g^3-(a^2-u^2)^3|\leqq 2a^6(|p|+|q|)+P_1, \tag{2.9}$$

where P_1 is a polynomial of $|p|$ and $|q|$ such that the degree of each term is greater than 1 and the coefficient of each term is a function of a. On the other hand,

$$|g(u-\nabla u\cdot x)-(a^2-u^2)u|$$
$$=|\{a^2(1+|\nabla u|^2)-(u-\nabla u\cdot x)^2\}(u-\nabla u\cdot x)-(a^2-u^2)u|$$
$$=|(3u^2-a^2)(\nabla u\cdot x)+a^2u|\nabla u|^2-a^2|\nabla u|^2(\nabla u\cdot x)-3u(\nabla u\cdot x)^2+(\nabla u\cdot x)^3|$$
$$\leqq|3u^2-a^2||\nabla u\cdot x|+|a^2u|\nabla u|^2-a^2|\nabla u|^2(\nabla u\cdot x)-3u(\nabla u\cdot x)^2+(\nabla u\cdot x)^3|.$$

Since $|3u^2-a^2|<2a^2$ and $|\nabla u\cdot x|\leqq a(|p|+|q|)$, we have

$$|g(u-\nabla u\cdot x)-(a^2-u^2)u|\leqq 2a^3(|p|+|q|)+P_2, \tag{2.10}$$

where P_2 is a polynomial of $|p|$ and $|q|$ such that the degree of each term is greater than 1 and the coefficient of each term is a function of a. Hence, from (2.8), (2.9) and (2.10) we have

$$|A(x, u, \nabla u, H_0)-A(x, u, 0, H_0)|$$
$$\leqq\frac{2}{a}H_0\{2a^6(|p|+|q|)+P_1\}G+\frac{2}{a^2}\{2a^3(|p|+|q|)+P_2\}$$
$$=4a(a^4H_0G+1)(|p|+|q|)+\frac{2}{a^2}(aH_0GP_1+P_2).$$

Now, we shall prove Theorem 2.2.

Proof of Theorem 2.2. Suppose for contradiction that there exists a point $x \in D$ such that $u(x) < m_1$. Since u has the minimum value on $\bar{D}$, there exists a point $x_0 \in D$ such that $u(x_0) \leqq u(x)$ for all $x \in D$. Then, of course $u(x_0) < m_1$. We put

$$m_2 = \frac{1}{2}(u(x_0) + m_1). \tag{2.11}$$

Put $D' = \{x \in D\,;\ u(x) < m_2\}$, and let D_0 be the connected component of D' containing x_0. Then, $\bar{D}_0 \subset D$ and $u(x) = m_2$ for all $x \in \partial D_0 := \bar{D}_0 - D_0$. We put

$$K = \sup_{x \in \bar{D}_0} \{ |u_{ij}(x)|\,;\ i, j = 1, 2\}. \tag{2.12}$$

By (2.1) and (2.6) there exists a positive constant d such that

$$a^2 - |x|^2 - u(x)^2 \geqq d^2 \qquad \text{for all} \quad x \in \bar{D}_0. \tag{2.13}$$

From Lemma 2.1, we see that there exists a positive constant λ such that

$$\sum_{i,j=1}^{2} A_{ij}(x, u(x), 0) X_i X_j \geqq \lambda (X_1^2 + X_2^2) \tag{2.14}$$

for any non-zero vector $X = (X_1, X_2)$ and all $x \in \bar{D}_0$. We put

$$\xi(x) = \exp(C(x_1 + x_2)), \qquad x \in \bar{D}, \tag{2.15}$$

where C is a constant such that

$$C > \frac{4a}{\lambda} \{aK + (a^4 H_0 / 2d^3 + 1)\}. \tag{2.16}$$

For a positive ε, we consider the function w_ε on $\bar{D}$ defined by

$$w_\varepsilon(x) = u(x) - \varepsilon \cdot \xi(x), \qquad x \in \bar{D}. \tag{2.17}$$

LEMMA 2.4. *For any positive* δ, *we can take a number* ε *with the following properties:*

(1) $0 < \varepsilon < \delta$;

(2) w_ε *takes its minimum value on* $\bar{D}_0$ *at a point of* D_0.

In fact, suppose that for some $\delta > 0$ the assertion of the above lemma is not true. Then, for any ε such that $0 < \varepsilon < \delta$, w_ε takes its minimum value on $\bar{D}_0$ at a point of $\partial D_0 := \bar{D}_0 - D_0$. Therefore, we have

$$w_\varepsilon(x) > w_\varepsilon(y_\varepsilon) \qquad \text{for all} \quad x \in D_0, \tag{2.18}$$

where $y_\varepsilon \in \partial D_0$ and $w_\varepsilon(y_\varepsilon) = \min(w_\varepsilon | \bar{D}_0)$. Put $\xi_0 = \max(\xi | \bar{D}_0)$. Then, we have

$$w_\varepsilon(y_\varepsilon) = u(y_\varepsilon) - \varepsilon \cdot \xi(y_\varepsilon) \geqq m_2 - \varepsilon \cdot \xi_0. \tag{2.19}$$

From (2.18) and (2.19), at $x_0 \in D_0$ we have

$$w_\varepsilon(x_0)=u(x_0)-\varepsilon\cdot\xi(x_0)>m_2-\varepsilon\cdot\xi_0 .$$

Hence, we obtain

$$u(x_0)-m_2>\varepsilon(\xi(x_0)-\xi_0) .$$

Since the above inequality holds for any ε such that $0<\varepsilon<\delta$, we get $u(x_0)\geqq m_2$, which contradicts (2.11). Thus the assertion of Lemma 2.4 holds.

By virtue of Lemma 2.4, we can conclude the following:

LEMMA 2.5. *There exists a monotone decreasing sequence* $\{\varepsilon_n\}$, $n=1, 2, \cdots$, *with the following properties:*

(1) $\varepsilon_n>0$, $n=1, 2, \cdots$, $\lim_{n\to\infty}\varepsilon_n=0$;

(2) *For each* ε_n, *the function* w_{ε_n} *defined by* (2.17) *takes its minimum value on* $\bar{D}_0$ *at a point of* D_0.

In what follows, let $\{\varepsilon_n\}$, $n=1, 2, \cdots$, be a sequence with properties (1), (2) stated in Lemma 2.5. For simplicity we put $w_{\varepsilon_n}=w_n$. Let x_n be a point of D_0 which gives the minimum value of w_n on $\bar{D}_0$. By taking a subsequence if necessary, we may assume that $\{x_n\}$, $n=1, 2, \cdots$, converges to a point $y\in\bar{D}_0$.

Now, we rewrite the inequality (2.5) as

$$\sum_{i,j=1}^{2}(A_{ij}(x, u, \nabla u)-A_{ij}(x, u, 0))u_{ij}+\sum_{i,j=1}^{2}A_{ij}(x, u, 0)u_{ij} \leqq A(x, u, \nabla u, H_0) . \tag{2.20}$$

Then, by Lemma 2.2 and (2.12), on $\bar{D}_0$ we have

$$\begin{aligned}\sum_{i,j=1}^{2}&(A_{ij}(x, u, \nabla u)-A_{ij}(x, u, 0))u_{ij}\\ &\geqq -K\Big(\sum_{i,j=1}^{2}|A_{ij}(x, u, \nabla u)-A_{ij}(x, u, 0)|\Big)\\ &\geqq -a^2K(|p|^2+|q|^2+2|p|\cdot|q|+4(|p|+|q|))\\ &=-a^2K(|p|+|q|)(|p|+|q|+4) .\end{aligned} \tag{2.21}$$

Since $u(x)=w_n(x)+\varepsilon_n\cdot\xi(x)$ for each $x\in\bar{D}$, by (2.20) and (2.21), on $\bar{D}_0$ we have

$$\sum_{i,j=1}^{2}A_{ij}(x, u, 0)(w_{nij}+\varepsilon_n\cdot\xi_{ij})-a^2K(|p|+|q|)(|p|+|q|+4) \leqq A(x, u, \nabla u, H_0) , \tag{2.22}$$

where $w_{nij}=\partial^2 w_n/\partial x_i\partial x_j$, $\xi_{ij}=\partial^2\xi/\partial x_i\partial x_j$.

In the following, we shall estimate the inequality (2.22) at x_n. We put $\xi(x_n)=\xi_n$ and $u(x_n)=u_n$. Then from (2.15) we have

$$\frac{\partial\xi}{\partial x_1}(x_n)=\frac{\partial\xi}{\partial x_2}(x_n)=C\cdot\xi_n \quad\text{and}\quad \xi_{ij}(x_n)=C^2\cdot\xi_n ,\quad i, j=1, 2 . \tag{2.23}$$

Since w_n takes its minimum value on $\bar{D}_0$ at $x_n\in D_0$, $(\partial\omega_n/\partial x_1)(x_n)=(\partial\omega_n/\partial x_2)(x_n)$

$=0$. Thus, from (2.17) we have

$$p(x_n)=q(x_n)=\varepsilon_n\cdot C\cdot\xi_n\,. \tag{2.24}$$

Furthermore, we see that the 2×2 matrix $W_n:=(w_{nij}(x_n))$ is positive semi-definite at x_n, $n=1, 2, \cdots$. Therefore, from this fact and Lemma 2.1, we see

$$\sum_{i,j=1}^{2} A_{ij}(x_n, u_n, 0)w_{nij}(x_n)\geqq 0\,. \tag{2.25}$$

By (2.14) and (2.23), we have

$$\sum_{i,j=1}^{2} A_{ij}(x_n, u_n, 0)\varepsilon_n\cdot\xi_{ij}(x_n)\geqq 2C^2\lambda\cdot\varepsilon_n\cdot\xi_n\,. \tag{2.26}$$

Thus, by (2.24), (2.25) and (2.26), at x_n we have

(2.27) the left-hand side of (2.22)

$$\geqq 2C^2\lambda\cdot\varepsilon_n\cdot\xi_n-4a^2K\cdot\varepsilon_n\cdot C\cdot\xi_n(\varepsilon_n\cdot C\cdot\xi_n+2)\,.$$

On the other hand, from Lemma 2.3 and (2.24), at x_n we have

(2.28) the right-hand side of (2.22)

$$\leqq A(x_n, u_n, 0, H_0)+8a(a^4H_0G(x_n)+1)(\varepsilon_n\cdot C\cdot\xi_n)$$
$$+\frac{2}{a^2}(aH_0G(x_n)\bar{P}_1+\bar{P}_2)(\varepsilon_n\cdot C\cdot\xi_n)\,,$$

where $\bar{P}_1$ and $\bar{P}_2$ are polynomials of $\varepsilon_n\cdot C\cdot\xi_n$ which have no constant terms, and the coefficient of each term is a function of a. From (2.6) and (2.7), we see

$$A(x_n, u_n, 0, H_0)\leqq 0\,. \tag{2.29}$$

Thus, by (2.27), (2.28) and (2.29), at x_n we have

$$2\varepsilon_n\cdot C\cdot\xi_n(C\lambda-2a^2K(\varepsilon_n\cdot C\cdot\xi_n+2))$$
$$\leqq 8a(a^4H_0G(x_n)+1)\cdot\varepsilon_n\cdot C\cdot\xi_n+\frac{2}{a^2}(aH_0G(x_n)\bar{P}_1+\bar{P}_2)\cdot\varepsilon_n\cdot C\cdot\xi_n\,.$$

Since $\varepsilon_n\cdot C\cdot\xi_n>0$, at x_n we have

$$C\lambda-2a^2K(\varepsilon_n\cdot C\cdot\xi_n+2) \tag{2.30}$$
$$\leqq 4a(a^4H_0G(x_n)+1)+\frac{1}{a^2}(aH_0G(x_n)\bar{P}_1+\bar{P}_2)\,.$$

Since ξ is bounded on $\bar{D}_0$, by (1) of Lemma 2.5 we have $\lim_{n\to\infty}\bar{P}_i=0$, $i, j=1, 2$. Moreover, from (2.4) we have

$$\lim_{n\to\infty}G(x_n)=\lim_{n\to\infty}(g(x_n)^{3/2}+(a^2-u_n{}^2)^{3/2})^{-1}=\frac{1}{2}(a^2-u(y)^2)^{-3/2}$$

because $\lim_{n\to\infty}x_n=y\in\bar{D}_0$. By (2.13), we have $a^2-u(y)^2\geqq d^2>0$. Now, by letting

$n\to\infty$ in the inequality (2.30), we obtain

$$C\lambda-4a^2K\leqq 4a(a^4H_0/2d^3+1)\,,$$

which contradicts (2.16). This contradiction is due to our hypothesis that there exists a point $x\in D$ such that $u(x)<m_1$. Thus we complete the proof of Theorem 2.2.

In Theorem 2.1, if M is a minimal surface in $S^3(a)$, then we can put $H_0=0$. As a corollary of Theorem 2.1 we have

COROLLARY 2.1. *In Theorem* 2.1, *suppose that* M *is a minimal surface in* $S^3(a)$. *Let* m_1 *be a constant such that* $0<m_1<k$. *If* $\tilde{u}(\partial D)\subset Q^k_{m_1}$, *then* $\tilde{u}(\bar{D})\subset Q^k_{m_1}$.

Our proof in Theorem 2.2 was inspired from the results of R. Redheffer [5].

3. Non-parametric surfaces with boundary and the maximum principle.

In this section, as in Section 2, let D be a bounded domain with boundary ∂D in E^2 and $C^{0,2}(\bar{D}, D)$ the set of continuous real-valued functions on $\bar{D}$ which are of class C^2 on D, where $\bar{D}=D\cup\partial D$.

In the following, let a and k be positive constants such that

$$a^2>b^2+k^2\,, \tag{3.1}$$

where $b=\max_{x\in\bar{D}}|x|$, $x=(x_1, x_2)\in E^2$ and $|x|^2=x_1{}^2+x_2{}^2$.

We put

$$H_1=\frac{m_1}{a}(a^2-m_1{}^2)^{-1/2} \tag{3.2}$$

where m_1 is a constant such that $k\leqq m_1<a$.

Now, we consider the following partial differential inequality on $\bar{D}$:

$$\sum_{i,j=1}^{2} A_{ij}(x, u, \nabla u)u_{ij}\geqq A(x, u, \nabla u, H_1)\,, \tag{3.3}$$

where $u\in C^{0,2}(\bar{D}, D)$, $|u(x)|\leqq k$ for all $x\in\bar{D}$ and $A_{ij}(x, u, \nabla u)$, $i, j=1, 2$, and $A(x, u, \nabla u, H_1)$ are given in (2.4).

We note that Lemma 2.3 also holds for H_1. We can prove the following theorem by a similar argument as in proof of Theorem 2.2.

THEOREM 3.1. *Suppose that* $u\in C^{0,2}(\bar{D}, D)$ *is a solution of the inequality* (3.3) *satisfying* $0\leqq u(x)\leqq k$ *for all* $x\in\bar{D}$. *Let* m *be a constant such that* $0<m<k$. *If* $u\leqq m$ *on* ∂D, *then* $u\leqq m$ *in* D.

For a constant m such that $0<m<a$, we set

$$Q_0^m=\{(x_1, x_2, x_3, x_4)\in S^3(a)\,;\ 0\leqq x_3\leqq m,\ x_4>0\}\,.$$

THEOREM 3.2. *For a function $u\in C^{0,2}(\bar{D}, D)$ satisfying $0\leqq u(x)\leqq k$ for all $x\in\bar{D}$, let M be the surface with boundary in $S^3(a)$ defined by* (2.2) *and H the mean curvature of M in $S^3(a)$. Suppose that H satisfies the inequality $H(x)\geqq H_1$ for all $x\in D$, where H_1 is defined by* (3.2). *Let m be a constant such that $0<m<k$. If $\tilde{u}(\partial D)\subset Q_0^m$, then $\tilde{u}(\bar{D})\subset Q_0^m$.*

Proof. For a continuous function H' on D, we define the operator $L_{H'}$ on $C^{0,2}(\bar{D}, D)$ by

$$L_{H'}(v)=\sum_{i,j=1}^{2} A_{ij}(x, v, \nabla v)v_{ij}-A(x, v, \nabla v, H')\,,$$

where $v\in C^{0,2}(\bar{D}, D)$, $|v(x)|\leqq k$ for all $x\in\bar{D}$ and $A_{ij}(x, v, \nabla v)$, $i, j=1, 2$, and $A(x, v, \nabla v, H')$ are given in (2.4). Then, from the hypotheses of Theorem 3.2, we have $L_H(u)=0$ and

$$L_{H_1}(u)=L_{H_1}(u)-L_H(u)=\frac{2}{a}g\sqrt{g}(H-H_1)\geqq 0\,.$$

Since the inequality $L_{H_1}(u)\geqq 0$ is equivalent to (3.3), then we can apply Theorem 3.1 to it. Therefore, Theorem 3.2 is an immediate consequence of Theorem 3.1.

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