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KŌDAI MATH. SEM. REP.
28 (1976), 19—27

A NOTE ON SASAKIAN MANIFOLDS WITH VANISHING C-BOCHNER CURVATURE TENSOR

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In [3], Ryan proved

THEOREM A. *Let M be a compact conformally flat Riemannian manifold with constant scalar curvature. If the Ricci tensor is positive semi-definite, then the simply connected Riemannian covering of M is one of*

$$S^n(c), \qquad R\times S^{n-1}(c) \qquad or \qquad E^n,$$

the real space forms of curvature c being denoted by $S^n(c)$ or E^n depending on whether c is positive or zero.

In 1974, Yano and Ishihara [6] proved the following theorem corresponding to Theorem A due to Ryan, replacing the vanishing of the Weyl conformal curvature tensor in a Riemannian manifold by that of the Bochner curvature tensor in a Kaehlerian manifold.

THEOREM B. *Let M be a Kaehlerian manifold of real dimension n with constant scalar curvature whose Bochner curvature tensor vanishes and whose Ricci tensor is positive semi-definite. If M is compact, then the universal covering manifold is a complex projective space $CP^{n/2}$ or a complex space $C^{n/2}$.*

The purpose of the present paper is to prove the following theorem corresponding to Theorems A and B, replacing the vanishing of the Weyl conformal curvature tensor or Bochner curvature tensor by that of C-Bochner curvature tensor (See [1]) in a Sasakian manifold.

THEOREM. *Let M^n be a Sasakian manifold of dimension n with constant scalar curvature whose C-Bochner curvature tensor vanishes. If Ricci tensor is positive semi-definite, then M^n is locally C-Fubinian.*

§ 1. Introduction.

Recently, in an n-dimensional Sasakian manifold M^n, Matsumoto and Chūman [1] introduced the C-Bochner curvature tensor $B_{kji}{}^h$ defined by

Received June 1, 1975.

$$B_{kji}{}^h = K_{kji}{}^h + \frac{1}{n+3}(K_{ki}\delta_j{}^h - K_{ji}\delta_k{}^h + g_{ki}K_j{}^h - g_{ji}K_k{}^h \tag{1.1}$$
$$+ S_{ki}\phi_j{}^h - S_{ji}\phi_k{}^h + \phi_{ki}S_j{}^h - \phi_{ji}S_k{}^h + 2S_{kj}\phi_i{}^h + 2\phi_{kj}S_i{}^h$$
$$- K_{ki}\eta_j\eta^h + K_{ji}\eta_k\eta^h - \eta_k\eta_i K_j{}^h + \eta_j\eta_i K_k{}^h)$$
$$- \frac{k+n-1}{n+3}(\phi_{ki}\phi_j{}^h - \phi_{ji}\phi_k{}^h + 2\phi_{kj}\phi_i{}^h)$$
$$- \frac{k-4}{n+4}(g_{ki}\delta_j{}^h - g_{ji}\delta_k{}^h)$$
$$+ \frac{k}{n+3}(g_{ki}\eta_j\eta^h + \eta_k\eta_i\delta_j{}^h - g_{ji}\eta_k\eta^h - \eta_j\eta_i\delta_k{}^h),$$

where $\phi_j{}^i$ is the structure tensor, η^i the structure vector, g_{ji} the positive definite metric tensor, $\eta_j = g_{ji}\eta^i$, $K_{kji}{}^h$ the curvature tensor, K_{ji} the Ricci tensor, K the scalar curvature, $S_{kj} = \phi_k{}^s K_{sj}$, $S_k{}^i = S_{kj}g^{ji}$ and $k = \dfrac{K+n-1}{n+1}$.

They proved the following theorems.

THEOREM C. *Let M^n be a compact Sasakian space M^n of dimension n ($n \geqq 5$) with vanishing C-Bochner curvature tensor of constant scalar curvature. Suppose that M^n satisfies one of the following conditions·*

i) *$\theta > -2$, where θ denotes the smallest eigenvalue of the Ricci tensor,*

ii) *$K_{\lambda\mu} + K_{\lambda\mu}{}^* > -\dfrac{3(2-\delta_{\lambda\mu})}{n-2}$, (especially $\sum_{\mu}(K_{\lambda\mu} + K_{\lambda\mu}{}^*) > -3$),*

ii) *M^n is μ-holomorphically pinched with $\mu > \dfrac{n-3}{2(n-1)}$.*

Then M^n is locally C-Fubinian. (A locally C-Fubinian manifold was defined in [4].)

THEOREM D. *If a Sasakian space M^n with vanishing C-Bochner curvature tensor is a C-Einstein space, then M^n is locally C-Fubinian. (A C-Einstein space was defined in* [2].)

In § 2, we shall recall fundamental properties of a Sasakian space with vanishing C-Bochner curvature tensor and in § 3 prove that the Laplacian $\Delta(Z_{ji}Z^{ji})$ of the tensor Z_{ji} defined by

$$Z_{ji} = K_{ji} - \Big(\frac{K}{n-1} - 1\Big)g_{ji} + \Big(\frac{K}{n-1} - n\Big)\eta_j\eta_i \tag{1.2}$$

is zero in a Sasakian manifold with constant scalar curvature whose C-Bochner curvature tensor vanishes.

In the last § 4 we prove the main theorem stated as before by using Theorem D and the Laplacian $\Delta(L_{ji}L^{ji})$ of the tensor L_{ji} defined by

$$L_{ji} = K_{ji} + \Big(\frac{K}{n-1} - n\Big)\eta_j\eta_i,$$

where $L^{ji}=L_{st}g^{sj}g^{ti}$.

§2. Properties of a Sasakian manifold with vanishing C-Bochner curvature tensor.

Let M^n be an n-dimensional Sasakian manifold ($n\geqq 3$). If we denote by ∇_i the operator of covariant differentiation with respect to the Riemannian connection of M^n, then the following relations hold:

$$\begin{aligned}
&S_{ji}=-S_{ij}, \qquad \nabla_k S_j{}^k=\frac{1}{2}\phi_j{}^k\nabla_k K+(K-n+1)\eta_j,\\
&\nabla_k S_{ji}=\eta_j K_{ik}-(n-1)g_{jk}\eta_i+\phi_j{}^t\nabla_k K_{ti},\\
(2.1)\quad &\phi_j{}^t\nabla_t S_{ik}=-\eta_i S_{kj}+(n-1)\phi_{ij}\eta_k+\phi_j{}^r\phi_i{}^s\nabla_r K_{sk},\\
&\nabla_k K_{ji}-\nabla_j K_{ki}=-\phi_i{}^r\nabla_r S_{kj}-2S_{kj}\eta_i+(n-1)(\phi_{ki}\eta_j-\phi_{ji}\eta_k+2\phi_{kj}\eta_i),
\end{aligned}$$

because the differential form $S=(1/2)S_{ji}dx^j\wedge dx^i$ is closed and $K_{ji}\eta^i=(n-1)\eta_j$ (See [2]).

Differentiating (1.1) covariantly and using (2.1), we have

$$\begin{aligned}
(2.2)\qquad &(n+3)\nabla_t B_{kji}{}^t\\
&=(n+2)(\nabla_k K_{ji}-\nabla_j K_{ki})-\phi_k{}^r\phi_j{}^s(\nabla_r K_{si}-\nabla_s K_{ri})+2\phi_i{}^s\phi_k{}^r\nabla_s K_{rj}\\
&\quad+\eta^r(\eta_k\nabla_r K_{ji}-\eta_j\nabla_r K_{ki})-(n+2)\eta_k S_{ji}+n\eta_j S_{ki}+2(n+1)\eta_i S_{kj}\\
&\quad+\frac{1}{n+1}(g_{ki}\eta_j-g_{ji}\eta_k)\eta^r\nabla_r K\\
&\quad+\frac{n-1}{2(n+1)}\{(g_{ki}-\eta_k\eta_i)\nabla_j K-(g_{ji}-\eta_j\eta_i)\nabla_k K\\
&\qquad\qquad+(\phi_{ki}\phi_j{}^r-\phi_{ji}\phi_k{}^r+2\phi_{kj}\phi_i{}^r)\nabla_r K\}\\
&\quad+(n+1)\{(n+2)\eta_k\phi_{ji}-n\eta_j\phi_{ki}-2(n+1)\eta_i\phi_{kj}\}.
\end{aligned}$$

Transvecting (2.2) with $\phi_l{}^k\phi_m{}^j$ and adding the resulting equation to (2.2), we obtain

$$\begin{aligned}
\nabla_t B_{lmi}{}^t+\phi_l{}^k\phi_m{}^j\nabla_t B_{kji}{}^t&=(\nabla_l K_{mi}-\nabla_m K_{li})-\phi_l{}^k\phi_m{}^j(\nabla_k K_{ji}-\nabla_j K_{ki})\\
&\quad+(n-1)(\eta_l\phi_{mi}-\eta_m\phi_{li})-\eta_l S_{mi}+\eta_m S_{li}\\
&\quad+\frac{1}{2(n+3)}(g_{li}\eta_m-g_{mi}\eta_l)\eta^t\nabla_t K.
\end{aligned}$$

On the other hand, using (2.1), we have

$$\phi_l{}^k\phi_m{}^j\nabla_t B_{kji}{}^t=-\nabla_t B_{mli}{}^t,$$

from which,

$$\nabla_k K_{ji} - \nabla_j K_{ki} - \phi_k{}^r \phi_j{}^s (\nabla_r K_{si} - \nabla_s K_{ri}) - \eta_k S_{ji} + \eta_j S_{ki}$$
$$+ \frac{1}{2(n+3)} (g_{ki}\eta_j - g_{ji}\eta_k)\eta^r \nabla_r K + (n-1)(\eta_k \phi_{ji} - \eta_j \phi_{ki}) = 0 .$$

Contracting the last equation with η^k and $\eta^k g^{ji}$, we find respectively

$$\eta^t \nabla_t K = 0 , \qquad \eta^t \nabla_t K_{ji} = 0 , \tag{2.3}$$

from which,

$$\frac{n+3}{n-1} \nabla_t B_{kji}{}^t = \nabla_k K_{ji} - \nabla_j K_{ki} - \eta_k \{S_{ji} - (n-1)\phi_{ji}\}$$
$$+ \eta_j \{S_{ki} - (n-1)\phi_{ki}\} + 2\eta_i \{S_{kj} - (n-1)\phi_{kj}\}$$
$$+ \frac{1}{2(n+1)} \{(g_{ki} - \eta_k \eta_i)\delta_j{}^t - (g_{ji} - \eta_j \eta_i)\delta_k{}^t$$
$$+ \phi_{ki}\phi_j{}^t - \phi_{ji}\phi_k{}^t + 2\phi_{kj}\phi_i{}^t\} \nabla_t K .$$

Thus, in a Sasakian manifold with vanishing C-Bochner curvature tensor, we get

$$\begin{aligned} \nabla_k K_{ji} - \nabla_j K_{ki} &= \eta_k \{S_{ji} - (n-1)\phi_{ji}\} - \eta_j \{S_{ki} - (n-1)\phi_{ki}\} - 2\eta_i \{S_{kj} - (n-1)\phi_{kj}\} \\ &- \frac{1}{2(n+1)} \{(g_{ki} - \eta_k \eta_i)\delta_j{}^t - (g_{ji} - \eta_j \eta_i)\delta_k{}^t + \phi_{ki}\phi_j{}^t - \phi_{ji}\phi_k{}^t + 2\phi_{kj}\phi_i{}^t\} \nabla_t K , \end{aligned} \tag{2.4}$$

$$\begin{aligned} \nabla_k S_{ji} = \eta_j K_{ki} - \eta_i K_{kj} + \frac{1}{2(n+1)} \{&\phi_{jk}\delta_i{}^t - \phi_{ik}\delta_j{}^t + 2\phi_{ji}\delta_k{}^t \\ &+ (g_{ik} - \eta_i \eta_k)\phi_j{}^t - (g_{jk} - \eta_j \eta_k)\phi_i{}^t\} \nabla_t K , \end{aligned} \tag{2.5}$$

which have already proved in [1].

In the rest of the section, we are going to compute $\nabla_k K_{ji}$ by using (2.3), (2.4) and (2.5). Differentiating covariantly $S_{ji} = \phi_j{}^t K_{ti}$ gives

$$\nabla_k S_{ji} = (\eta_j \delta_k{}^t - \eta^t g_{kj}) K_{ti} + \phi_j{}^t \nabla_k K_{ti} = \eta_j K_{ki} - (n-1)\eta_i g_{kj} + \phi_j{}^t \nabla_k K_{ti} ,$$

which together with (2.5) implies

$$\begin{aligned} \phi_j{}^t \nabla_k K_{ti} = (n-1)\eta_i g_{kj} - \eta_i K_{kj} + \frac{1}{2(n+1)} \{&\phi_{jk}\delta_i{}^t - \phi_{ik}\delta_j{}^t \\ &+ 2\phi_{ji}\delta_k{}^t + (g_{ik} - \eta_i \eta_k)\phi_j{}^t - (g_{jk} - \eta_j \eta_k)\phi_i{}^t\} \nabla_t K . \end{aligned} \tag{2.6}$$

Transvecting (2.6) with $\phi_l{}^j$ and using (2.3) and (2.4) give

$$\begin{aligned} \nabla_k K_{ji} = &- \eta_j \{S_{ki} - (n-1)\phi_{ki}\} - \eta_i \{S_{kj} - (n-1)\phi_{kj}\} \\ &- \frac{1}{2(n+1)} \{(-g_{jk} + \eta_j \eta_k)\delta_i{}^t - \phi_{ik}\phi_j{}^t + 2(-g_{ji} + \eta_j \eta_i)\delta_k{}^t \\ &- (g_{ik} - \eta_i \eta_k)\delta_j{}^t - \phi_{jk}\phi_i{}^t\} \nabla_t K , \end{aligned} \tag{2.7}$$

which together with (1.2) implies

$$(2.8)\qquad \nabla_k Z_{ji} = -\eta_j\Big\{S_{ki} - \Big(\frac{K}{n-1}-1\Big)\phi_{ki}\Big\} - \eta_i\Big\{S_{kj} - \Big(\frac{K}{n-1}-1\Big)\phi_{kj}\Big\}$$
$$-\frac{1}{n-1}(\nabla_k K)g_{ji} + \frac{1}{n-1}(\nabla_k K)\eta_j\eta_i$$
$$-\Big\{\frac{1}{2(n+1)}(-g_{kj}+\eta_k\eta_j)\delta_i^t - \phi_{ik}\phi_j^t$$
$$+2(-g_{ji}+\eta_j\eta_i)\delta_k^t - (g_{ik}-\eta_i\eta_k)\delta_j^t - \phi_{jk}\phi_i^t\Big\}\nabla_t K.$$

§ 3. Laplacian $\Delta(Z_{ji}Z^{ji})$.

In order to calculate the Laplacian

$$(3.1)\qquad \frac{1}{2}\Delta(Z_{ji}Z^{ji}) = g^{kj}(\nabla_k\nabla_j Z_{ih})Z^{ih} + (\nabla_k Z_{ji})(\nabla^k Z^{ji}),$$

where the tensor Z_{ji} defined by (1.2) and $Z^{ji}=Z_{st}g^{sj}g^{ti}$, in a Sasakian manifold with vanishing C-Bochner curvature tensor, we first consider the term $g^{kj}(\nabla_k\nabla_j Z_{ih})Z^{ih}$. Using (2.5) and (2.8), we obtain

$$(3.2)\qquad \nabla_k\nabla_j Z_{ih}$$
$$= -\phi_{ki}\Big\{S_{jh} - \Big(\frac{K}{n-1}-1\Big)\phi_{jh}\Big\} - \eta_i\nabla_k\Big\{S_{jh} - \Big(\frac{K}{n-1}-1\Big)\phi_{jh}\Big\}$$
$$-\phi_{kh}\Big\{S_{ji} - \Big(\frac{K}{n-1}-1\Big)\eta_{ji}\Big\} - \eta_h\nabla_k\Big\{S_{ji} - \Big(\frac{K}{n-1}-1\Big)\phi_{ji}\Big\}$$
$$-\frac{1}{n-1}(\nabla_k\nabla_j K)g_{ih} + \frac{1}{n-1}(\nabla_k\nabla_j K)\eta_i\eta_h + \frac{1}{n-1}(\nabla_j K)(\phi_{ki}\eta_h + \eta_i\phi_{kh})$$
$$-\frac{1}{2(n+1)}\{(\phi_{kj}\eta_i + \eta_j\phi_{ki})\delta_h^t - (\eta_h g_{kj} - \eta_j g_{kh})\phi_i^t$$
$$-\phi_{hj}(\eta_i\delta_k^t - \eta^t g_{ki}) + 2(\phi_{ki}\eta_h + \eta_i\phi_{kh})\delta_j^t$$
$$+(\phi_{kh}\eta_j + \eta_h\phi_{kj})\delta_i^t - (\eta_i g_{kj} - \eta_j g_{ki})\phi_h^t - \phi_{ij}(\eta_h\delta_k^t - \eta^t g_{hk})\}\nabla_t K$$
$$-\frac{1}{2(n+1)}\{-g_{ji}+\eta_j\eta_i)\delta_h^t - \phi_{hj}\phi_i^t + 2(-g_{ih}+\eta_i\eta_h)\delta_j^t$$
$$-(g_{hj}-\eta_h\eta_j)\delta_i^t - \phi_{ij}\phi_h^t\}\nabla_k\nabla_t K.$$

Transvecting (3.2) with $g^{kj}Z^{ih}$ and making use of $Z_{ji}\eta^i=0$ and $Z_i^i=0$, we can easily verify

$$(3.3)\qquad g^{kj}(\nabla_k\nabla_j Z_{ih})Z^{ih} = -2\phi_{si}S^s{}_h Z^{ih} + \frac{1}{n+1}\{Z_k^t + \phi_{hk}\phi_i^t Z^{ih}\}\nabla^k\nabla_t K.$$

On the other hand, taking account of the skew-symmetry of $S_{ji}=\phi_j^t K_{ti}$, we have

$$\phi_{si}S^s{}_h Z^{ih}=K_{ih}Z^{ih}$$
$$=K_{ih}K^{ih}-\Big(\frac{K}{n-1}-1\Big)K+(n-1)\Big(\frac{K}{n-1}-n\Big)$$

and

$$\phi_{hk}\phi_i{}^t Z^{ih}=Z_k{}^t .$$

Substituting the last two equations into (3.3) implies

$$g^{kj}(\nabla_k\nabla_j Z_{ih})Z^{ih}=-2K_{ih}K^{ih}+2\Big(\frac{K}{n-1}-1\Big)K-2(n-1)\Big(\frac{K}{n-1}-n\Big)$$
$$+\frac{2}{n+1}\{\nabla_k(Z^{kt}\nabla_t K)-(\nabla_k Z_t{}^k)\nabla^t K\} . \tag{3.4}$$

Next we consider the second term in the right hand side of (3.1). Taking account of the definition (1.2) of Z_{ji}, we have by a strightford computation

$$(\nabla_k Z_{ji})(\nabla^k Z^{ji})=\Big\{\nabla_k K_{ji}-\frac{1}{n-1}(\nabla_k K)g_{ji}+\frac{1}{n-1}(\nabla_k K)\eta_j\eta_i$$
$$+\Big(\frac{K}{n-1}-n\Big)(\phi_{kj}\eta_i+\eta_j\phi_{ki})\Big\}\Big\{\nabla^k K^{ji}-\frac{1}{n-1}(\nabla^k K)g^{ji}$$
$$+\frac{1}{n-1}(\nabla^k K)\eta^j\eta^i+\Big(\frac{K}{n-1}-n\Big)(\phi^{kj}\eta^i+\eta^j\phi^{ki})\Big\} ,$$

which reduces to

$$(\nabla_k Z_{ji})(\nabla^k Z^{ji})=(\nabla_k K_{ji})(\nabla_k K^{ji})-\frac{1}{n-1}(\nabla^k K)(\nabla_k K)$$
$$+4\Big(\frac{K}{n-1}-n\Big)\{(-K+n(n-1)\}+2(n-1)\Big(\frac{K}{n-1}-n\Big)^2 \tag{3.5}$$

because of $\phi_{kj}\eta_i(\nabla^k K^{ji})=-K+n(n-1)$ which is a consequence of $K_{ji}\eta^i=(n-1)\eta_j$. On the other hand, we find from (2.7)

$$(\nabla_k K_{ji})(\nabla^k K^{ji})=\Big[\eta_j\{S_{ki}-(n-1)\phi_{ki}\}+\eta_i\{S_{kj}-(n-1)\phi_{kj}\}$$
$$+\frac{1}{2(n+1)}\{(-g_{jk}+\eta_j\eta_k)\delta_i{}^t-\phi_{ik}\phi_j{}^t+2(-g_{ji}+\eta_j\eta_i)\delta_k{}^t$$
$$-(g_{ik}-\eta_i\eta_k)\delta_j{}^t-\phi_{jk}\phi_i{}^t\}\nabla_t K\Big]\cdot\Big[\eta^j\{S^{ki}-(n-1)\phi^{ki}\}$$
$$+\eta^i\{S^{kj}-(n-1)\phi^{kj}\}+\frac{1}{2(n+1)}\{(-g^{jk}+\eta^j\eta^k)g^{is}-\phi^{ik}\phi^{js}$$
$$+2(-g^{ji}+\eta^j\eta^i)g^{ks}-(g^{ik}-\eta^i\eta^k)g^{js}-\phi^{jk}\phi^{is}\}\nabla_s K\Big]$$
$$=2K_{ji}K^{ji}-4(n-1)K+2n(n-1)^2+\frac{2}{n+1}(\nabla_t K)(\nabla^t K) , \tag{3.6}$$

where we have used (2.3), $S_{ji}S^{ji}=K_{ji}K^{ji}-(n-1)^2$ and $\phi_{ji}S^{ji}=K-(n-1)$. Contracting (2.8) with g^{kj} and using (2.3), we get

$$(\nabla_k Z^k{}_i)\nabla^i K=\frac{n-3}{2(n-1)}(\nabla_t K)(\nabla^t K). \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.5) and (3.4) respectively and substituting the resulting equations into (3.1), we obtain

$$\frac{1}{2}\Delta(Z_{ji}Z^{ji})=\frac{2}{n+1}\nabla_k(Z^{kt}\nabla_t K),$$

which implies

LEMMA 1. *In a Sasakian manifold with constant scalar curvature whose C-Bochner curvature tensor vanishes, we have* $\Delta(Z_{ji}Z^{ji})=0$.

§4. The proof of the theorem.

In this section we assume the scalar curvature K is constant and define a tensor field L_{ji} by

$$L_{ji}=K_{ji}+\left(\frac{K}{n-1}-n\right)\eta_j\eta_i. \tag{4.1}$$

We prepare some equalities to get the Laplacian $\Delta(L_{ji}L^{ji})$. From (4.1) we have

$$L=L_i{}^i=\frac{n}{n-1}K-n, \tag{4.2}$$

Substituting (4.2) into (4.1) implies

$$L_{ji}=K_{ji}+\left(\frac{L}{n}+1-n\right)\eta_j\eta_i, \tag{4.3}$$

and consequently

$$L_{ji}\eta^i=\frac{L}{n}\eta_j. \tag{4.4}$$

Using (2.4) and $K=$const. gives

$$\begin{aligned}(4.5)\qquad &\nabla_k L_{ji}-\nabla_j L_{ki}\\ &=\eta_k\left\{S_{ji}-\frac{L}{n}\phi_{ji}\right\}-\eta_j\left\{S_{ki}-\frac{L}{n}\phi_{ki}\right\}-2\eta_i\left\{S_{kj}-\frac{L}{n}\phi_{kj}\right\}.\end{aligned}$$

Moreover, from (4.1) and (4.4), it follows that

$$\begin{aligned}(4.6)\qquad &S_{ji}=\phi_j{}^t L_{ti},\\ &S_{ki}\phi_j{}^h L_h{}^k L^{ji}=-L_i{}^t L_{tj}L^{ij}+\frac{L^3}{n^3},\quad \phi_{ki}\phi_j{}^h L_h{}^k L^{ji}=-L_{ji}L^{ji}+\frac{L^2}{n^2}.\end{aligned}$$

Now, because of (1.1) and (4.6), the following equalities are easily verified:

$$(4.7)\quad K_{kji}{}^{h}L_{h}{}^{k}L^{ji}$$

$$=-\frac{1}{n+3}\Big[2K_{ki}L_{h}{}^{k}L^{hi}-2LK_{ji}L^{ji}+S_{ki}\phi_{j}{}^{h}L_{h}{}^{k}L^{ji}+\phi_{ki}S_{j}{}^{h}L_{h}{}^{k}L^{ji}$$

$$+2S_{kj}\phi_{i}{}^{h}L_{h}{}^{k}L^{ji}+2\phi_{kj}S_{i}{}^{h}L_{h}{}^{k}L^{ji}-\frac{2(n-1)}{n^{2}}L^{2}+\frac{2L}{n}K_{ji}L^{ji}\Big]$$

$$+\frac{k+n-1}{n+3}\Big[\phi_{ki}\phi_{j}{}^{h}L_{h}{}^{k}L^{ji}+2\phi_{kj}\phi_{i}{}^{h}L_{h}{}^{k}L^{ji}\Big]+\frac{k-4}{n+3}[L_{ji}L^{ji}-L^{2}]$$

$$-\frac{2k}{n+3}\Big[\frac{L^{2}}{n^{2}}-\frac{L^{2}}{n}\Big]$$

$$=-\frac{1}{n+3}\Big[-4L_{i}{}^{t}L_{th}L^{ih}+\frac{2(1-n)}{n}L\,L_{ji}L^{ji}+\frac{2(n+1)}{n^{3}}L^{3}-\frac{2(n-1)^{2}}{n^{2}}L^{2}\Big]$$

$$+\frac{3(k+n-1)}{n+3}\Big[-L_{ji}L^{ji}+\frac{L^{2}}{n^{2}}\Big]+\frac{k-4}{n+3}[L_{ji}L^{ji}-L^{2}]+\frac{2(n-1)}{n^{2}(n+3)}k\,L^{2}.$$

On the other hand, applying the Ricci formula to L_{ji} and using (1.1), (4.5) $\nabla_{k}L_{j}{}^{k}=0$, we get

$$g^{kj}(\nabla_{k}\nabla_{j}L_{ih})L^{ih}=L_{i}{}^{t}L_{tj}L^{ij}-K_{sih}{}^{t}L_{t}{}^{s}L^{ih}-3L_{ji}L^{ji}-\frac{L^{3}}{n^{3}}+\frac{4n-1}{n^{2}}L^{2}.$$

Substituting (4.7) into the last equation and making use of $k=\frac{n-1}{n(n+1)}L+\frac{2(n-1)}{n+1}$, we can easily verify

$$(4.8)\quad \frac{1}{2}\Delta(L_{ji}L^{ji})=\frac{n-1}{n+3}L_{i}{}^{t}L_{th}L^{ih}-\Big\{\frac{2(n-1)}{(n+1)(n+3)}L+\frac{4}{n+1}\Big\}L_{ji}L^{ji}$$

$$+\frac{(n-1)^{2}}{n^{2}(n+1)(n+3)}L^{3}+\frac{4}{n(n+1)}L^{2}+(\nabla_{k}L_{ji})(\nabla^{k}L^{ji}).$$

Next we compute $(\nabla_{k}L_{ji})(\nabla^{k}L^{ji})$ by using $\nabla_{k}L_{ji}=\nabla_{k}Z_{ji}$, (3.6) and (3.7). Substituting (3.6) into (3.5) and taking account of (4.2), (4.3) and K=const., we find

$$(\nabla_{k}L_{ji})(\nabla^{k}L^{ji})=2K_{ji}K^{ji}+4\Big(\frac{K}{n-1}-n\Big)\{-K+n(n-1)\}$$

$$+2(n-1)\Big(\frac{K}{n-1}-n\Big)^{2}-4(n-1)K+2n(n-1)^{2}$$

$$=2L_{ji}L^{ji}-\frac{2}{n}L^{2},$$

which together with (4.8) implies

$$\frac{1}{2}\Delta(L_{ji}L^{ji})=\frac{n-1}{n+3}L_{it}L_{h}{}^{t}L^{ih}-\Big\{\frac{2(n-1)}{(n+1)(n+3)}L-\frac{2(n-1)}{n+1}\Big\}L_{ji}L^{ji}$$

$$+\frac{(n-1)^{2}}{n^{2}(n+1)(n+3)}L^{3}-\frac{2(n-1)}{n(n+1)}L^{2}.$$

Taking account of $\Delta(L_{ji}L^{ji})=\Delta(Z_{ji}Z^{ji})$ and Lemma 1, the last equation gives

$$\frac{n-1}{n+3}L_{it}L_h{}^tL^{ih}-\left\{\frac{2(n-1)}{(n+1)(n+3)}L-\frac{2(n-1)}{n+1}\right\}L_{ji}L^{ji} + \frac{(n-1)^2}{n^2(n+1)(n+3)}L^3-\frac{2(n-1)}{n(n-1)}L^2=0. \tag{4.9}$$

The following lemma was proved by Yano and Ishihara (See [6]):

LEMMA 2. *In a Riemannian manifold of dimension n, for*

$$P=nL_t{}^sL_s{}^rL_r{}^t-\frac{2n-1}{n-1}L\,L_{ji}L^{ji}+\frac{1}{n-1}L^3,$$

we have

$$P=\frac{1}{n-1}\sum_i\sum_{j\neq k}\lambda_i(\lambda_i-\lambda_j)(\lambda_i-\lambda_k),$$

$\lambda_1\leqq\lambda_2\leqq\cdots\leqq\lambda_n$ *being eigenvalues of the tensor* L_{ji}. *Moreover, if* L_{ji} *is positive semi-definite, then* $P\geqq 0$.

Using P given in Lemma 2, we have from (4.9)

$$\frac{n-1}{n(n+3)}P+\frac{1}{n+1}\left\{\frac{3n-1}{n(n+3)}L+2(n-1)\right\}\left(L_{ji}-\frac{L}{n}g_{ji}\right)\left(L^{ji}-\frac{L}{n}g_{ji}\right)=0. \tag{4.10}$$

If Ricci tensor K_{ji} is positive semi-definite, then (4.10) give $L_{ji}=(L/n)g_{ji}$, that is, $Z_{ji}=0$ because the positive semi-difiniteness of K_{ji} implies that of L_{ji}. Thus, combining Theorem B and the above result $L_{ji}=(L/n)g_{ji}$, we have completely proved the theorem stated at the end of the first section.

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