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CERTAIN HYPERSURFACES IN THE EUCLIDEAN SPHERE

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§ 0. Introduction.

It has been proved by R. Osserman [6] that if the mean curvature vector of a surface S in the Euclidean space E^3 is always orthogonal to a fixed direction, then S is either a minimal surface, or else a locally cylindrical surface with its generator parallel to the fixed direction.

In this paper, we consider a unit sphere S^{n+1} in the Euclidean space E^{n+2}, and study about a hypersurface M^n in S^{n+1} whose mean curvature vector is always orthogonal to a fixed direction.

We first show that when M is complete, M must be a minimal hypersurface (Theorem I). The necessity of completeness will be discussed in § 2.

We next show that in the case $n=2$, M is either a minimal surface, or else a locally cylindrical surface in S^3, by the latter we mean some open piece of such a surface as is generated by a family of semi-great circles through a fixed pair of antipodal points of S^3 (Theorem II). This corresponds exactly to the result of R. Osserman.

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§ 1. A result in the complete case.

In this section, we prove the following:

THEOREM I. *Let M be an n-dimensional complete Riemannian manifold isometrically immersed in S^{n+1}. If the mean curvature vector of M is always orthogonal to a fixed direction, then M is a minimal hypersurface.*

Proof. As minimality is a local property, we may assume M to be orientable. Without loss of generality, consider S^{n+1} as the unit sphere in E^{n+2} with center at the origin, and let $f: M \rightarrow S^{n+1}$ be the immersion in the theorem. For $p \in M$, $x_{f(p)}$ denotes the position vector of $f(p) \in S^{n+1}$ in E^{n+2}, and $T_p(M)$ is the tangent space of M at p, usually identified with $f_*(T_p(M))$. We denote by $\langle\, ,\rangle$ the metric on E^{n+2}, S^{n+1} and M without distinction, and by D, $\tilde{\nabla}$ and ∇ the

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Riemannian connections on E^{n+2}, S^{n+1} and M, respectively. By orientability of M, choose a unit normal vector field ξ of M in S^{n+1}, and let A be the second fundamental tensor field of M satisfying for $X\in T(M)$:

$$AX=-\tilde{\nabla}_X\xi=-D_X\xi\,. \tag{1.1}$$

Then the mean curvature vector H and the scalar mean curvature h of M are given by

$$H=(\mathrm{Tr}\,A)\xi=h\xi\,,$$

where Tr denotes the trace.

Now by the assumption of the theorem, there exists a unit vector a in E^{n+2} such that

$$\langle H, a\rangle=h\langle\xi, a\rangle=0\,.$$

Therefore defining the open subset M' of M by $M':=\{p\in M\,|\,h(p)\neq 0\}$, we have

$$\langle\xi, a\rangle=0 \quad \text{on} \quad M'\,. \tag{1.2}$$

For the proof of the theorem, we assume $M'\neq\phi$ and lead to a contradiction.

First we define a tangent vector field Z on M by projecting the vector a onto each tangent space, that is, by

$$Z_p:=a-\langle x_{f(p)}, a\rangle x_{f(p)}-\langle\xi_{f(p)}, a\rangle\xi_{f(p)}\,.$$

In the sequel we work chiefly on M' so both (1.2) and

$$Z=a-\langle x, a\rangle x \quad \text{on} \quad M' \tag{1.3}$$

are to be remarked.

Differentiating (1.2) on M', we have for any $X\in T(M')$:

$$0=\langle D_X\xi, a\rangle=\langle -AX, a\rangle=-\langle AX, Z\rangle=-\langle AZ, X\rangle$$

by (1.1) and the symmetry of A, so that

$$AZ=0 \quad \text{on} \quad M'\,. \tag{1.4}$$

Moreover since

$$\begin{aligned}\nabla_X Z=\tilde{\nabla}_X Z-\langle AX, Z\rangle\xi=\tilde{\nabla}_X Z=D_X Z-\langle D_X Z, x\rangle x\\ =-\langle X, a\rangle x-\langle x, a\rangle X+\langle Z, X\rangle x=-\langle x, a\rangle X\,,\end{aligned}$$

putting $\beta(p)=\langle x_{f(p)}, a\rangle$ for $p\in M'$, we obtain

$$\nabla_X Z=-\beta X \quad \text{on} \quad M'\,. \tag{1.5}$$

Next, by the Codazzi's equation and (1.4) we have

$$\begin{aligned}(\nabla_Z A)X=(\nabla_X A)Z=\nabla_X(AZ)-A(\nabla_X Z)\\ =-A(-\beta X)=\beta AX\end{aligned}$$

for each $X\in T(M')$, that is,

$$\nabla_Z A=\beta A \quad \text{on} \quad M',$$

from which it follows immediately that

$$Z(h)=\beta h \quad \text{on} \quad M'. \tag{1.6}$$

Using this formula, we show Z never vanishes on M' as follows: if $Z_p=0$ at some $p\in M'$, then $a=\langle x_{f(p)}, a\rangle x_{f(p)}$ and so $|\beta(p)|=1$. Therefore (1.6) implies $h(p)=0$, a contradiction. On the other hand, as we get

$$\nabla_Z Z=-\beta Z \tag{1.7}$$

from (1.5), $Z/|Z|$ is a geodesic vector field on M' where $|Z|$ denotes the length of Z. Then fixing $p\in M'$, let $\gamma(s)$ be the infinitely extended geodesic of M through p tangent to Z_p, where s denotes the arc length with $\gamma(0)=p$. We define a function l on $\boldsymbol{R}$ by

$$l(s):=\langle\dot{\gamma}(s), Z_{\gamma(s)}\rangle \tag{1.8}$$

where $\dot{\gamma}(s)$ is the velocity vector of $\gamma(s)$. Let (a, b) be the maximal interval containing zero for which $\gamma((a, b))$ lies in the connected component of M' containing p.

Now differentiating (1.8) along γ, we have by (1.5)

$$\frac{dl}{ds}(s)=\langle\dot{\gamma}(s), \nabla_{\dot{\gamma}(s)}Z_{\gamma(s)}\rangle=\langle\dot{\gamma}(s), -\beta(\gamma(s))\dot{\gamma}(s)\rangle=-\beta(\gamma(s)), \tag{1.9}$$

and thus

$$\begin{aligned}\frac{d^2l}{ds^2}(s)&=-\frac{d}{ds}\langle x_{\gamma(s)}, a\rangle\\ &=-\langle\dot{\gamma}(s), Z_{\gamma(s)}\rangle=-l(s) \qquad \text{for} \quad s\in(a, b).\end{aligned}$$

Hence $l(s)$ must be expressed as

$$l(s)=c_1\cos s+c_2\sin s \qquad \text{for} \quad s\in(a, b), \tag{1.10}$$

where

$$c_1=l(0)=\langle\dot{\gamma}(0), Z_p\rangle=|Z_p|$$

and

$$c_2=\frac{dl}{ds}(0)=-\beta(p).$$

Taking $s_0\in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\sin s_0=\beta(p)$, we have

$$c_1=\langle Z_p, Z_p\rangle^{1/2}=\sqrt{1-c_2{}^2}=\cos s_0$$

and therefore from (1.10), (1.9) and (1.8), we obtain

$$l(s)=\cos(s+s_0),$$

$$\beta(\gamma(s))=\sin(s+s_0), \tag{1.11}$$

$$Z_{\gamma(s)}=\cos(s+s_0)\dot{\gamma}(s) \qquad \text{for} \quad s\in(a, b). \tag{1.12}$$

Thus the non-vanishing of Z on (a, b) implies the interval (a, b) to be finite. Remark here that the continuous function $h(\gamma(s))$ on $\boldsymbol{R}$ must approach to zero as s tending to a or b.

Now substituting (1.11) and (1.12) into (1.6), we have

$$\cos(s+s_0)\frac{dh\cdot\gamma}{ds}(s)=\sin(s+s_0)h\cdot\gamma(s), \qquad \text{for} \quad s\in(a, b),$$

which shows immediately that

$$h(\gamma(s))=\frac{c_p}{\cos(s+s_0)} \qquad \text{for} \quad s\in(a, b),$$

where c_p is some constant. Therefore by the above remark, c_p and hence $h(\gamma(s))$ must be identically zero on (a, b), which is a contradiction. Finally we have $M'=\phi$, and the proof is completed. Q. E. D.

§2. An example in the non-complete case.

We give an example of non-complete hypersurface M in S^{n+1} whose normal vectors are always orthogonal to a fixed direction and hence so is the mean curvature vector. But in this case, we show that M is not necessarily a minimal hypersurface.

Let $\varphi: N\to S^n$ be an immersion of an $(n-1)$-dimensional manifold N into a great hypersphere S^n in S^{n+1}. Let a be the unit vector orthogonal to the hyperplane containing S^n in E^{n+2} and ω the angle on the unit circle S^1. Then we define a geometric suspension ψ: $N\times S^1\to S^{n+1}$ of φ by

$$\psi(p, \omega)=\cos\omega\cdot\varphi(p)+\sin\omega\cdot a.$$

Choosing local coordinates $(x_1, x_2, \cdots, x_{n-1})$ on N, we see that

$$\begin{aligned} \psi_*\left(\frac{\partial}{\partial x_i}\right)&=\cos\omega\frac{\partial\varphi}{\partial x_i}, \qquad 1\leqq i\leqq n-1, \\ \psi_*\left(\frac{\partial}{\partial\omega}\right)&=-\sin\omega\cdot\varphi+\cos\omega\cdot a. \end{aligned} \tag{2.1}$$

Thus ψ immerses $N':=\{(p, \omega)\in N\times S^1 \mid \omega\neq\text{odd multiple of } \pi/2\}$ into S^{n+1}. We denote by M one of the connected components of N'.

In the neighborhood of coordinates $(x_1, x_2, \cdots, x_{n-1})$ on N and $(x_1, x_2, \cdots, x_{n-1}, \omega)$ on M chosen as above, let η and ξ be local unit vector fields normal to N in S^n and M in S^{n+1}, respectively. Then as ξ is orthogonal to $\psi(p, \omega)$, $\psi_*(\partial/\partial x_i)$ and $\psi_*(\partial/\partial\omega)$, and therefore to $\varphi(p)$, a and $\partial\varphi/\partial x_i$ $(1\leqq i\leqq n-1)$, choosing the direction of ξ suitably, we have

$$\xi_{\psi(p,\omega)}=\eta_{\varphi(p)} \tag{2.2}$$

in this neighborhood. In particular, we note that

$$\langle \xi, a \rangle = 0 . \tag{2.3}$$

Now by B and A we denote the matrices of the second fundamental forms of N and M respectively in the coordinates above. Then from

$$\frac{\partial^2 \psi}{\partial x_i \partial x_j} = \cos \omega \frac{\partial^2 \varphi}{\partial x_i \partial x_j}$$

$$\frac{\partial^2 \psi}{\partial x_i \partial \omega} = -\sin \omega \frac{\partial \varphi}{\partial x_i}$$

$$\frac{\partial^2 \psi}{\partial \omega^2} = -\cos \omega \cdot \varphi - \sin \omega \cdot a$$

and from (2.2), it follows that

$$A_{\psi(p,\omega)} = \cos \omega \begin{pmatrix} B_{\varphi(p)} & 0 \\ 0 & 0 \end{pmatrix} .$$

On the other hand, from (2.1), the matrices of the first fundamental forms G of N and $\tilde{G}$ of M are related by

$$\tilde{G}_{\psi(p,\omega)} = \begin{pmatrix} \cos^2 \omega \, G_{\varphi(p)} & 0 \\ 0 & 1 \end{pmatrix} ,$$

therefore we obtain

$$\operatorname{Tr} A_{\psi(p,\omega)} = \frac{1}{\cos \omega} \operatorname{Tr} B_{\varphi(p)} . \tag{2.4}$$

Thus it turns out that M is minimal if and only if N is minimal.

We now observe that as M is generated by a family of semi-great circles of S^{n+1} through the fixed pair of antipodal points $\pm a$, the following definition is somewhat reasonable.

DEFINITION. By a locally cylindrical hypersurface in S^{n+1}, we mean some open piece of such a hypersurface as M constructed above.

§ 3. Characterizations of locally cylindrical hypersurfaces in S^{n+1}.

Here we come to prove the following:

LEMMA. *Let M be a Riemannian manifold of dimension $n \geqq 2$ isometrically immersed in S^{n+1}. Then M is locally cylindrical if and only if its normal directions are always orthogonal to a fixed direction.*

Proof. As the necessity was shown above by (2.3), we prove the sufficiency. The property to prove is local, so we may assume M to be orientable. Let ξ be a unit vector field normal to M in S^{n+1}, which satisfies $\langle \xi, a \rangle = 0$ for some fixed unit vector a in E^{n+2}. Here we note that the argument on M' in § 1 is all available in this case on M because the condition $\langle \xi, a\rangle=0$ is essential in the process up to (1.6) in § 1, and further because the remove of the two vanishing points $\pm a$ of Z from M, if necessary, gives no effects on the conclusion of this lemma. Thus we may consider $Z/|Z|$ a geodesic vector field on M just as in § 1 on M'. Moreover as we have

$$\langle \tilde{\nabla}_Z Z, \xi\rangle=\langle Z, AZ\rangle=0$$

by (1.4), the geodesic γ through $p\in M$ tangent to Z_p is in fact an arc of a great circle in S^{n+1}. By the definition of Z, this great circle passes through $\pm a$ for any $p\in M$. Thus our proof is almost accomplished. In fact if we cut the family of such semi-great circles with its two ends $\pm a$ that intersect M, by the hyperplane E_a^{n+1} orthogonal to a through the origin, then we have a hypersurface N in $S_a^n: S^{n+1}\cap E_a^{n+1}$, from which we can reconstruct M by the same procedure as is described in the previous section. Thus M is proved to be locally cylindrical.

Q. E. D.

§ 4. A results in the case $n=2$.

In the case $n=2$, eliminating the completeness of M, we can prove the following theorem by using some special properties of surfaces.

THEOREM II. *Let M be a surface of class C^2 in S^3 whose mean curvature vector is always orthogonal to a fixed direction. Then M is either a minimal surface, or else a locally cylindrical surface in S^3.*

Proof. As usual let $S^3=\{x\in E^4 \mid |x|=1\}$. Handling local properties, we may assume M to be orientable, and further in this case a conformally immersed Riemann surface since there always exist isothermal coordinates on surfaces of class C^2. Now just as in § 1, let ξ be a unit vector field normal to M in S^3 and $H=h\xi$ be the mean curvature vector field satisfying $\langle H, a\rangle=h\langle \xi, a\rangle=0$ for some fixed unit vector a in E^4. Let M' be the open subset of M defined by $M':=\{p\in M \mid h(p)\neq 0\}$ as before. If $M'=\phi$ then M is minimal and if $M'=M$ then M is locally cylindrical by Lemma in § 3, so let $S:=M-\bar{M}'$ and assume both $M'\neq\phi$ and $S\neq\phi$. We claim in this case that $\langle \xi, a\rangle=0$ holds not only on M', but throughout M.

Now we denote by $\psi: M\rightarrow S^3$ a conformal immersion of M and let $z=x_1+ix_2$ be an associated local isothermal coordinate on M where $i=\sqrt{-1}$. Setting $\partial=(1/2)(\partial/\partial x_1-i(\partial/\partial x_2))$, we have for the metric induced by ψ from S^3,

$$ds^2=2F|dz|^2$$

where

$$F=\langle \partial\psi, \bar{\partial}\psi\rangle=\frac{1}{2}\left|\frac{\partial\psi}{\partial x_1}\right|^2=\frac{1}{2}\left|\frac{\partial\psi}{\partial x_2}\right|^2 \tag{4.1}$$

by using the complex linearly extended inner product. Since $\langle \psi, \psi\rangle=1$ and ψ

is of class C^2, we have

$$\langle\phi, \partial^k\phi\rangle=\langle\phi, \bar{\partial}^k\phi\rangle=0=\langle\partial\phi, \partial^k\phi\rangle=\langle\bar{\partial}\phi, \bar{\partial}^k\phi\rangle \quad \text{for} \quad k=1, 2. \tag{4.2}$$

From now on we denote $\partial\phi/\partial x_i$ and $\partial^2\phi/\partial x_i\partial x_j$ by ϕ_i and ϕ_{ij}, respectively. Let D be the connection of E^4 as in § 1. Then the vector-valued second fundamental form B of M is given by

$$B(X, Y)=D_XY-\langle D_XY, \phi\rangle\phi-\frac{1}{2F}\sum_{k=1}^{2}\langle D_XY, \phi_k\rangle\phi_k$$

where X and Y are any tangent vector fields of M. Then identifying $\partial/\partial x_i$ with $\phi_*(\partial/\partial x_i)$, we define

$$B_{ij}:=B\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)=\phi_{ij}-\langle\phi_{ij}, \phi\rangle\phi-\frac{1}{2F}\sum_{k=1}^{2}\langle\phi_{ij}, \phi_k\rangle\phi_k$$

and

$$\beta_{ij}:=\langle B_{ij}, \xi\rangle=\langle\phi_{ij}, \xi\rangle\cdot$$

Choosing $\xi=(1/2F)\phi\wedge\phi_1\wedge\phi_2=(1/iF)\phi\wedge\partial\phi\wedge\bar{\partial}\phi$ as the unit normal vector of M, we have

$$\beta_{ij}=\frac{1}{iF}\phi\wedge\partial\phi\wedge\bar{\partial}\phi\wedge\phi_{ij}.$$

Now we define a quadratic differential $\omega=\alpha dz^2$ on M by

$$\alpha:=\frac{1}{iF}\phi\wedge\partial\phi\wedge\bar{\partial}\phi\wedge\partial^2\phi=\frac{1}{4}(\beta_{11}-\beta_{22}-2i\beta_{12}),$$

which is well-defined since for another associated isothermal coordinate $\tilde{z}=\tilde{x}_1+i\tilde{x}_2$, setting $\tilde{\partial}=(1/2)(\partial/\partial\tilde{x}_1-i(\partial/\partial\tilde{x}_2))$, we can easily show that

$$\partial^2=\left(\frac{d\tilde{z}}{dz}\right)^2\tilde{\partial}^2+\partial\left(\frac{d\tilde{z}}{dz}\right)\tilde{\partial}.$$

By virtue of (4.1) and (4.2) we can compute α^2, which we need later, as follows:

$$\alpha^2=-\frac{1}{F^2}\begin{vmatrix} \langle\phi, \phi\rangle & \langle\phi, \partial\phi\rangle & \langle\phi, \bar{\partial}\phi\rangle & \langle\phi, \partial^2\phi\rangle \\ \langle\partial\phi, \phi\rangle & \langle\partial\phi, \partial\phi\rangle & \langle\partial\phi, \bar{\partial}\phi\rangle & \langle\partial\phi, \partial^2\phi\rangle \\ \langle\bar{\partial}\phi, \phi\rangle & \langle\bar{\partial}\phi, \partial\phi\rangle & \langle\bar{\partial}\phi, \bar{\partial}\phi\rangle & \langle\bar{\partial}\phi, \partial^2\phi\rangle \\ \langle\partial^2\phi, \phi\rangle & \langle\partial^2\phi, \partial\phi\rangle & \langle\partial^2\phi, \bar{\partial}\phi\rangle & \langle\partial^2\phi, \partial^2\phi\rangle \end{vmatrix} \tag{4.3}$$

$$=-\frac{1}{F^2}\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & F & 0 \\ 0 & F & 0 & \partial F \\ 0 & 0 & \partial F & \langle\partial^2\phi, \partial^2\phi\rangle \end{vmatrix}$$

$$=\langle\partial^2\phi, \partial^2\phi\rangle.$$

On the other hand, since S is minimal and

$$h=\frac{1}{2F}(\beta_{11}+\beta_{22})=\frac{1}{2F}\frac{4}{iF}\psi\wedge\partial\psi\wedge\bar{\partial}\psi\wedge\partial\bar{\partial}\psi\,,$$

we have

$$\beta_{11}+\beta_{22}=0 \qquad \text{on} \quad S \tag{4.4}$$

or equivalently

$$\partial\bar{\partial}\psi=-F\psi \qquad \text{on} \quad S\,. \tag{4.5}$$

(The equivalence of (4.4) and (4.5) easily follows from (4.1) and (4.2); while (4.5) is just $\Delta\psi=-2\psi$ where Δ is the Laplace-Beltrami operator of M. cf. § 5.) Accordingly, by (4.4) we obtain

$$\alpha=\frac{1}{2}(\beta_{11}-i\beta_{12}) \qquad \text{on} \quad S\,. \tag{4.6}$$

Moreover, noting that ψ is real analytic on S [4, Lemma 1.1], we can show that ω is holomorphic on S. In fact as

$$\begin{aligned}\bar{\partial}\alpha^2&=\bar{\partial}\langle\partial^2\psi,\,\partial^2\psi\rangle=2\langle\partial(\partial\bar{\partial}\psi),\,\partial^2\psi\rangle=-2\langle\partial(F\psi),\,\partial^2\psi\rangle\\&=-2\partial F\langle\psi,\,\partial^2\psi\rangle-2F\langle\partial\psi,\,\partial^2\psi\rangle=0 \qquad \text{on} \quad S\end{aligned}$$

by (4.3), (4.5) and (4.2), we see that ω^2 and so ω are holomorphic on S.

Now we go back to the proof of the theorem. Take the universal covering $\tilde{M}$ of M. Then $\tilde{M}$ is conformally equivalent to one of the unit 2-sphere, the unit disk and the entire plane. As we can apply Theorem I in the compact case, it is sufficient to consider the latter two cases, both of which are nice since we can choose a fixed parameter $\zeta=u_1+iu_2$ all over $\tilde{M}$. We denote by $\tilde{S}$ the open subset of $\tilde{M}$ which projects onto S. Then the coefficient function $\tilde{\alpha}$ of the lifted differential $\tilde{\omega}=\tilde{\alpha}d\zeta^2$ of ω is holomorphic when restricted to $\tilde{S}$. We extend this holomorphic function $\tilde{\alpha}|_{\tilde{S}}$ on $\tilde{S}$ to a function $\tilde{F}$ on $\tilde{M}$ as follows:

$$\tilde{F}(\zeta)=\begin{cases}\tilde{\alpha}(\zeta) & \text{on} \quad \tilde{S}\\ 0 & \text{on} \quad \tilde{M}-\tilde{S}\,.\end{cases}$$

We next show that $\tilde{F}(\zeta)$ is continuous on $\tilde{M}$. To do this we return to M and consider a continuous function G on M given by

$$G(p)=\frac{\beta_{11}\beta_{22}-\beta_{12}^2}{(2F)^2}(p) \qquad \text{for} \quad p\in M\,,$$

which is well-defined since the right hand side is independent of the choice of coordinates. In particusar, $G(p)=0$ on M' because $\langle B(X, Y), \xi\rangle=\langle AX, Y\rangle$ for X, $Y\in T_pM$, and we have $AZ=0$ with $Z\neq 0$ on M' for the tangent vector field Z on M defined in § 1. On the other hand, as we have

$$G(p)=-\frac{\beta_{11}^2+\beta_{12}^2}{(2F)^2}(p) \quad \text{on } S$$

by (4.4), the continuity of G implies that both β_{11} and β_{12} approach to zero as $p \in M$ goes to the boundary ∂S. Therefore noting (4.6), we see that ω vanishes on ∂S and does also $\tilde{\omega}$ on $\partial \tilde{S}$. The continuity of $\tilde{F}(\zeta)$ is thus obtained.

Now we recall the well-known theorem of Radó-Behnke-Stein-Cartan [2]: if a continuous complex valued function f on a complex analytic manifold N is holomorphic wherever $f(z)\neq 0$, $z \in N$, then f is holomorphic all over N.

Applying this to $\tilde{F}(\zeta)$, we have $\tilde{F}(\zeta)\equiv 0$ on $\tilde{M}$ since $\tilde{F}(\zeta)$ is holomorphic on $\tilde{M}$ and vanishes on the non-empty interior of $\tilde{M}-\tilde{S}$.

Finally we have $\beta_{11}=\beta_{12}\equiv 0$ or $B\equiv 0$ on S which shows that each connected component of S lies in some great hypersphere of S^3 and hence the normal vector ξ is constant on each component. In particular as $\langle \xi, a\rangle=0$ on M', the connectedness of M shows $\langle \xi, a\rangle=0$ holds throughout M. Then the theorem follows immediately from Lemma. Q. E. D.

Note. In the proof above, it is not essential to take the universal covering. The argument on $\tilde{M}$ is merely for the local argument on a coordinate neighborhood of each point of M.

§ 5. Remarks.

1. For a submanifold M^n of S^{n+p},

$$\Delta\langle x, a\rangle=\langle H, a\rangle-n\langle x, a\rangle$$

holds where Δ is the Laplace-Beltrami operator of M and a is any constant unit vector in E^{n+p+1}, [1]. Thus if M is minimal, then

$$\Delta\langle x, a\rangle=-n\langle x, a\rangle \tag{5.1}$$

holds for all unit vector a in E^{n+p+1}. When $p=1$ and M is complete, (5.1) for one unit vector a in E^{n+p+1} is sufficient for M to be minimal by Theorem I.

2. It may not be so easy to derive something in the case when the codimension p is larger than 1 in Theorem I or II with an added condition such as H is contained in some great sphere of S^{n+p} or as the normal connection is flat. For the case of surfaces in E^{2+p}, see L. Jonker [3].

3. It was proved by K. Nomizu and B. Smyth [5] that a complete orientable locally cylindrical hypersurface in S^{n+1} is a great hypersphere.

BIBLIOGRAPHY

[1] S. S. CHERN, Minimal Submanifolds in a Riemannian Manifold, University of Kansas, Technical Report 19, 1968.

[2] E. HEINZ, Ein elementarer Beweis des Satzes von Radó-Behnke-Stein-Cartan über analytische Funktionen, Math. Ann., Vol. 131, 1956, pp. 258-259.

[3] L. JONKER, A theorem on minimal surfaces, J. Diff. Geom., **3**, 1969, pp. 351-360.
[4] H.B. LAWSON, Complete minimal surfaces in S^3, Ann. of Math., **92**, pp. 335-374.
[5] K. NOMIZU AND B. SMYTH, On the Gauss mapping for hypersurfaces of constant mean curvature in the sphere, Comm. Math. Helv., **44**, 1969, pp. 484-490.
[6] R. OSSERMAN, Remarks on minimal surfaces, Comm. pure and Appl. Math., Vol. **12**, 1959, pp. 233-239.

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