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# ASYMPTOTIC BEHAVIOR AND DEGENERACY OF BIHARMONIC FUNCTIONS ON RIEMANNIAN MANIFOLDS

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One of the most fascinating results in harmonic classification theory is the is the identity $O_{HD}^N = O_{HC}^N$, where $H$ stands for the class of harmonic functions $h$, $\Delta h=0$, with $\Delta = d\delta + \delta d$ the Laplace-Beltrami operator, and $HD$, $HC$ are the subclasses of functions which are Dirichlet finite, or bounded Dirichlet finite, respectively. For any class $F$ of functions, $O_F$, $\tilde{O}_F$ denote the classes of Riemannian manifolds on which $F \subset \boldsymbol{R}$ or $F \not\subset \boldsymbol{R}$ respectively, and $O_F^N$, $\tilde{O}_F^N$ are the corresponding subclasses of manifolds of dimension $N \geqq 2$.

A striking phenomenon in biharmonic classification theory is that, in contrast with the harmonic case, the inclusion $O_{H^2D} \subset O_{H^2C}$ is strict, with $H^2$ the class of nonharmonic biharmonic functions. This has been, however, known only in the 2-dimensional case, in which it was established by undoubtedly the most intricate counterexample in all classification theory (Nakai-Sario [6]). The technique of complex analysis used therein is not available for an arbitrarily high dimension.

Combining certain recent results in the biharmonic classification of the Poincaré $N$-ball for the subclasses $H^2D$, $H^2B$ of $H^2$ functions which are Dirichlet finite or bounded, respectively (Hada-Sario-Wang [2], [3]), one can draw the conclusion that $O_{H^2D}^N \subset O_{H^2C}^N$ is strict for $N \geqq 5$. However, for $N=3, 4$, the reasoning fails and the question remains unsettled.

The first purpose of the present paper is to give a complete and unified solution to this problem by proving the strict inclusion

$$O_{H^2D}^N < O_{H^2C}^N$$

for any dimension $N \geqq 2$. We shall, in fact, show more generally that $O_{H^2B}^N \not\subset O_{H^2D}^N$. On the other hand, from recent results on the Poincaré $N$-ball (Hada-Sario-Wang [2], [3]), we infer that $O_{H^2D}^N \not\subset O_{H^2B}^N$. In summary, we have the following string of strict inclusion relations:

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$$\begin{array}{ccccc} & & O^N_{H^2B} & & \\ & \angle & & \nwarrow & \\ O^N_{H^2B\cup H^2D} & & & & O^N_{H^2C}\,. \\ & \nwarrow & & \angle & \\ & & O^N_{H^2D} & & \end{array}$$

Proceeding from the special to the general, we state our most general result which will be the content of our paper:

$$\begin{array}{ccccc} & & O^N_X & & \\ & \angle & & \nwarrow & \\ O^N_{X\cup Y} & & & & O^N_{XY} \\ & \nwarrow & & \angle & \\ & & O^N_Y & & \end{array}$$

for and $N\geqq 2$; $X=H^2B$, $\Gamma$, $G$, $HP$, $HB$, $HD$, $HC$; $Y=H^2D$, $H^2L^p$. Here $HF=H\frown F$, $H^2F=H^2\frown F$; $1\leqq p<\infty$; $\Gamma$ is the class of biharmonic Green's functions (Sario [9]); $G$ is the class of harmonic Green's functions; and $P$ is the class of positive functions. Of these relations, the following cases, in addition to the aforementioned partial relations on $H^2B$ and $H^2D$, have been previously known: $(X, Y)=(HP, H^2D)$, (Sario-Wang [11]); $(X, Y)=(HD, H^2D)$, $(HB, H^2D)$, (Sario-Wang [13]); $(X, Y)=(G; H^2D)$, (Nakai-Sario [8], Sario-Wang [12]); $(X, Y)=(\Gamma, H^2D)$, (Wang [14]). The rest are new: in addition to the aforementioned unsettled relation between $H^2B$ and $H^2D$, the cases $(X, H^2L^p)$, where $X=G$, $HP$, $HB$, $HD$, $HC$, $\Gamma$, and $H^2B$.

An essential aspect of our paper is that all the above inclusion relations, old and new, are obtained in a simple and unified manner. The $N$-cylinder, endowed with various simple metrics, is the only manifold we will need as a counterexample. This unification of approach is made possible by a systematic use of the asymptotic behavior of solutions of differential equations.

The proof of the above statement on the classes $O_X$ and $O_Y$ will be presented in Lemmas 1-25 and § 5.

**1. Consider the $N$-cylinder.**

$$M=\boldsymbol{R}\times S^{N-1}=\{|x|<\infty,\ |y_i|\leqq\pi,\ i=1, \cdots, N-1\}$$

with the faces $y_i=\pi$ and $y_i=-\pi$ identified, for each $i$, by a parellel translation perpendicular to the $x$-axis. Endow $M$ with the metric

$$ds^2=\varphi^2(x)dx^2+\psi^2(x)dy_1^2+\sum_{i=2}^{N-1}dy_i^2\,,$$

where $\varphi$, $\psi\in C^\infty(-\infty, \infty)$. The proof of our theorem will consist, in essence, of two parts. First we show that for a suitable choice of $\varphi$, $\psi$,

$$M\in O^N_G\frown O^N_{HF}\frown O^N_\Gamma\frown O^N_{H^2B}\frown\tilde{O}^N_{H^2D}\frown\tilde{O}^N_{H^2L^p}\,,$$

and then that for another choice of $\varphi$, $\psi$,

$$M_1 \in \tilde{O}_G^N \cap \tilde{O}_{HF}^N \cap \tilde{O}_\Gamma^N \cap \tilde{O}_{H^2B}^N \cap O_{H^2D}^N \cap O_{H^2LP}^N ,$$

where $M_1$ is the manifold with the new metric, and $F=P$, $B$, $D$, $C$. This will establish our claims $O_X^N \cap \tilde{O}_Y^N \neq \phi$ and $\tilde{O}_X^N \cap O_Y^N \neq \phi$. The remaining relations $\tilde{O}_X^N \cap \tilde{O}_Y^N \neq \phi$ and $O_X^N \cap O_Y^N \neq \phi$ will then follow from other quite trivial choices of $\varphi$ and $\psi$.

**2. To establish the first string of relations in § 1, we choose $\varphi=\psi$ on $(-\infty, \infty)$, $\varphi(x)=|x|^{-3}$ for $|x|>1$.**

LEMMA 1. *A harmonic function $h(x, y)$, $y=(y_1, \cdots, y_{N-1})$, has a representation $h(x, y)=f_0(x)+\sum_{n=1}^{\infty} f_n(x)G_n(y)$, where $G_n(y)=\prod_{i=1}^{N-1} G_n^i(y_i)$ with $G_n^i(y_i)=\pm \sin n_i y_i$ or $\pm \cos n_i y_i$ for some integer $n_i$. The series converges absolutely and uniformly on compact sets.*

In fact, by a standard application of the Peter-Weyl theorem, we obtain for any $x_0$, $h(x_0, y)=f_0(x_0)+\sum_{n=1}^{\infty} f_n(x_0)G_n(y)$. Here the $G_n$ are invariant under varying $x_0$ by virtue of continuity. The convergence follows by a standard argument using differentiation with respect to $y$.

LEMMA 2. *$f(x)$ is harmonic if and only if $f(x)=ax+b$.*

For the proof, solve the equation $\Delta f=-g^{-1/2}f''=0$, where $\sqrt{g}\,dxdy$ is the volume element.

LEMMA 3. *$M \in O_X$ with $X=\Gamma$, $G$, $HP$, $HB$, $HD$, $HC$.*

From the harmonic classification theory, we have the the inclusions $O_G < O_{HP} < O_{HB} < O_{HD} = O_{HC}$. Moreover, $O_G < O_\Gamma$ (Wang [15]). Thus it suffices to show that $M \in O_G$. The harmonic measure $\omega$ of $\{x=c>0\}$ on $\{0<x<c\}$ is $x/c$ in view of Lemma 2. As $c \to \infty$, $\omega \to 0$. Similarly, the harmonic measure of the boundary component at $x=-\infty$ vanishes. Therefore, $M \in O_G$.

**3. Having discussed the spaces $O_G$, $O_{HF}$, $O_\Gamma$ of the first string of relations in § 2, we turn to the spaces related to biharmonic functions.** First we present some preparatory results.

LEMMA 4. *If $f(x)G(y)$ is harmonic, then $f$ is strictly monotone.*

Suppose the claim false. Then for $c_1<c_2$, say, $f|\{c_1<x<c_2\}$ is not strictly monotone, and $f$ takes on its maximum or minimum on $\{c_1<x<c_2\}$ at some point of this open interval. So does, a fortiori, $fG$, in violation of the maximum principle for harmonic functions.

LEMMA 5. *If* $f(x)G(y_1)$ *is harmonic,* $G(y_1)=\pm\sin n_1y_1$ *or* $\pm\cos n_1y_1$ *with* $n_1\neq 0$, *then* $f(x)=ae^{-n_1x}+be^{n_1x}$.

We obtain successively

$$\Delta(fG)=(\Delta f)G+f\Delta G=0\,,$$

$$\Delta f=-g^{-1/2}f''\,,$$

$$\Delta G=g^{-1/2}(\varphi^{-2}g^{1/2}n_1^2G)=n_1^2g^{-1/2}G\,,$$

$$\Delta(fG)=(-g^{-1/2}f''+n_1^2g^{-1/2}f)G=0\,,$$

with the fundamental solutions $f_1(x)=e^{n_1x}$ and $f_2(x)=e^{-n_1x}$.

LEMMA 6. *If* $f(x)G(y_i)$ *is harmonic with* $G(y_i)$ *not constant,* $i\neq 1$, *then*

$$f(x)=ax(1+o(1))+b(1+o(1))\,,\qquad a\neq 0$$

*either as* $x\to\infty$ *or else as* $x\to-\infty$.

This time we have

$$\Delta(fG)=\Big(-\frac{1}{\sqrt{g}}f''+n_i^2f\Big)G=0\,,$$

hence

$$f''=n_i^2\sqrt{g}f\,.$$

We now make use of the following theorem of Haupt [4] and Hille [5]:

A necessary and sufficient condition for the equation

$$f''(x)=p(x)f(x)$$

on $(0,\infty)$ to have the fundamental solutions

$$f_1(x)=x(1+o(1))\,,$$

$$f_2(x)=1+o(1)$$

as $x\to\infty$ is that

$$xp(x)\in L^1(0,\infty)\,.$$

Since $n_i^2\sqrt{g}=n_i^2|x|^{-3}$ for $|x|>1$, the condition of the theorem of Haupt and Hille is satisfied, and we conclude that

$$f(x)=a_1x(1+o(1))+b_1(1+o(1))\qquad\text{as}\quad x\to\infty\,,$$

or

$$f(x)=a_2x(1+o(1))+b_2(1+o(1))\qquad\text{as}\quad x\to-\infty\,.$$

By Lemma 3, $fG$ is not bounded and the same is true of $f$. Consequently $a_1\neq 0$ or $a_2\neq 0$.

LEMMA 7. *If* $f(x)G(y_2, y_3, \cdots, y_{N-1})$ *is harmonic, with* $G$ *not constant, then*

$$f(x)=ax(1+o(1))+b(1+o(1))$$

*either as* $x\to\infty$ *or else as* $x\to-\infty$.

The proof is the same as for Lemma 6, the equation

$$f''=(\sum_{i=2}^{N-1} n_i^2)\sqrt{g}f$$

again satisfying the Haupt-Hille condition.

LEMMA 8. *If* $f(x)G(y)$ *is harmonic with* $G(y)=\prod_{i=1}^{N-1} G^i(y_i)$, $G^1(y_1)$ *not constant, then*

$$f(x)\sim ae^{n_1|x|} \qquad \text{with} \quad a\neq 0$$

*either as* $x\to\infty$ *or else as* $x\to-\infty$.

The equation $\Delta(fG)=0$ gives

$$f''=\sqrt{g}\,(n_1^2\varphi^{-2}+\sum_{i=2}^{N-1} n_i^2)f\,,$$

which for $|x|>1$ is reduced by the transformation $f(x)=f(n_1x)$ to the form

$$f''(x)=(1+c|x|^{-3})f(x)\,.$$

We now make use of the following theorem of Bellman [1]:

If $p(x)\to 0$ as $x\to\infty$ and if $\int_0^\infty p^2dx<\infty$, then the equation $f''=(1+p)f$ in $(0,\infty)$ has the fundamental solutions

$$f_1(x)\sim\exp\Big[-\Big(x+\frac{1}{2}\int_{x_0}^x p(x)dx+o(1)\Big)\Big]\,,$$

$$f_2(x)\sim\exp\Big[x+\frac{1}{2}\int_{x_0}^x p(x)dx+o(1)\Big]\,.$$

In the present case, $p(x)=c|x|^{-3}$ satisfies the condition of Bellman's theorem, and we obtain $f=a_1f_1+b_1f_2$ for $x>1$ and $f=a_2f_1+b_2f_2$ for $x<-1$ with

$$f_1(x)\sim\exp\Big[-\Big(|x|+\frac{1}{2}\int_{x_0}^x pdx+o(1)\Big)\Big]\,,$$

$$f_2(x)\sim\exp\Big[|x|+\frac{1}{2}\int_{x_0}^x pdx+o(1)\Big]\,.$$

If $b_1=b_2=0$, then $f(x)\to 0$ as $|x|\to\infty$, in violation of the maximum principle. Therefore either $b_1\neq 0$ or $b_2\neq 0$, and in view of the above transformation, we have the lemma.

LEMMA 9. *A solution of* $\Delta q=1$ *is* $q_0(x)=\int_0^x\int_0^t\sqrt{g(s)}\,dsdt$. *The general solution of* $\Delta q=c$ *is* $cq_0(x)+h(x,y)$ *where* $h\in H$. *Every* $q$ *is unbounded.*

The only part of the lemma that needs proving is the unboundedness of $q$. Suppose that there exists a bounded $q$. Then the transform $(Tq)(x)=\int_y q(x, y)dy =aq_0(x)+bx+c$ is bounded. Since $q_0\to-\infty$ as $|x|\to\infty$, whereas $bx$ changes its sign with $x$, we have a contradiction.

LEMMA 10. *A solution of* $\Delta^2u(x)=0$ *is* $u_0(x)=\int_0^x\int_{-\infty}^t s\sqrt{g(s)}\,dsdt$. *It satisfies* $u_0(x)\sim\pm a\log|x|$ *for some constant* $a$ *as* $x\to\pm\infty$, *respectively. The general solution* $c_0u_0(x)+c_1q_0(x)+c_2x+c_3$ *is unbounded.*

The proof is analogous to that of Lemma 9.

LEMMA 11. $M\in\tilde{O}_{H^2D}$.

The function $u_0(x)$ of Lemma 10 is Dirichlet finite:

$$D(u_0)=c\int_{-\infty}^{\infty}(u_0')^2\varphi^{-2}\varphi^2dx$$

$$=c_1+c\Big(\int_{-\infty}^{-1}+\int_1^{\infty}\Big)|x|^{-2}dx<\infty\,.$$

LEMMA 12. $M\in\tilde{O}_{H^2L^p}$.

If fact,

$$\|u_0\|_p^p=c\int_{-\infty}^{\infty}|u_0|^p\sqrt{g}\,dx<\infty\,,$$

since $|u_0(x)|\sim|a\log|x||$ but $\sqrt{g}\sim|x|^{-3}$ as $x\to\pm\infty$.

LEMMA 13. *Let* $v(x)$ *satisfy the equation* $\Delta(v(x)G(y_1))=f(x)G(y_1)\in H$ *with* $fG\neq0$ *and* $G(y_i)$ *not constant. Then* $v$ *is unbounded.*

We have

$$\Big(-\frac{1}{\sqrt{g}}v''+\frac{n_1^2}{\sqrt{g}}v\Big)G=fG\,,$$

hence

$$v''=n_1^2v-\sqrt{g}f\,.$$

By Lemma 5, $f(x)=ae^{n_1x}+be^{-n_1x}$ with $|a|+|b|\neq0$. We may assume $a<0$; the proof for the other case is analogous.

Suppose $v$ is bounded. For sufficiently large $x>0$, $n_1^2v-\sqrt{g}f$ grows at the rate of $x^{-3}e^{n_1x}$. We thus have

$$v'(x)=v'(x_0)+\int_{x_0}^x(n_1^2v-ax^{-3}e^{n_1x})dx\,,$$

where we may choose $x_0>1$. It follows that

$$v(x)=v(x_0)+v'(x_0)(x-x_0)+\int_{x_0}^x\int_{x_0}^t(n_1^2v(s)-as^{-3}e^{n_1s})dsdt$$

which is clearly not bounded.

LEMMA 14. *Let $v(x)$ satisfy the exuation $\Delta(v(x)G(y_i))=f(x)G(y_i)\in H$ with $fG\neq 0$, $i>1$, and $G(y_i)$ not constant. Then $v$ is unbounded.*

The proof is analogous to that of Lemma 13, with

$$v''=n_i^2\sqrt{g}\,v-\sqrt{g}\,f\,.$$

In applying Lemma 6, we may assume that $f(x)=ax(1+o(1))+b(1+o(1))$ as $x\to-\infty$ with $a<0$. If $v$ is bounded, we have for $x<-1$,

$$v'(x)=\int_{-\infty}^{x} n_1^2|t|^{-3}v(t)dt-\int_{-\infty}^{x}|t|^{-3}f dt\,,$$

$$v(x_1)-v(x_2)=\int_{x_2}^{x_1}\int_{-\infty}^{x} n_1^2|t|^{-3}v(t)dtdx-\int_{x_2}^{x_1}\int_{-\infty}^{x}|t|^{-3}fdtdx$$

for $x_2<x<x_1<-1$. As $x_2\to-\infty$, the first integral converges but the second does not. Thus $v(x_2)$ cannot be bounded as $x_2\to-\infty$, in violation of the assumption.

LEMMA 15. *Let $v(x)$ satisfy $\Delta(v(x)G(y))=f(x)G(y)\in H$, with $f(x)G(y)$ not constant. Then $v(x)$ is not bounded.*

We may assume $n_1\neq 0$ and at least one $n_i\neq 0$, $i>1$. We now have

$$v''=(n_1^2+\sum_{i=2}^{N-1}n_i^2\sqrt{g})v-\sqrt{g}\,f\,.$$

Since $fG$ is harmonic, $f\sim ae^{n_1|x|}$ for either $x\to\infty$ or else $x\to-\infty$. We may assume the former. Clearly $|\sqrt{g}f|\to\infty$ as $x\to\infty$. If $v$ is bounded, then $\sqrt{g}f$ will dominate the right-hand side of the equation. On integrating as in the proof of Lemmas 13 and 14, we arrive at the contradiction that $v$ is both bounded and unbounded.

LEMMA 16. *$M\in O_{H^2B}$.*

Suppose there exists a $u(x,y)\in H^2B$. Write $u(x,y)=v_0(x)+\sum_{n=1}^{\infty}v_n(x)G_n(y)$ with $G_n\neq G_m$ for $n\neq m$. Either $v_0(x)$ or some $v_nG_n$ is not harmonic. Suppose this is true of $v_{n_0}G_{n_0}$. Then the thransform

$$(Tu)(x)=\int_y uG_{n_0}dy=cv_{n_0}(x)$$

is bounded, in violation of Lemma 15.

With Lemma 16, the proof of the first string of inclusion relations in § 1 is complete.

**4. We turn to the second string of relations in § 1.** We now choose $\varphi\equiv 1$ and $\psi$ a positive symmetric $C^\infty$ function with $\psi(x)=\exp e^{|x|}$ for $|x|>1$, and denote the resulting manifold by $M_1$.

The same proof as for Lemma 1 shows that every harmonic function $h$ on $M_1$ has a representation

$$h(x, y)=f_0(x)+\sum_{n=1}^{\infty} f_n(x)G_n(y) .$$

LEMMA 17. *$f(x)$ is harmonic if and only if $f(x)=a\int_0^x \psi^{-1}dx+b$.*

This is seen by solving the harmonic equation $\Delta f(x)=-\psi^{-1}(\psi f')'=0$.

LEMMA 18. *$M_1 \in \tilde{O}_G \cap \tilde{O}_{HX}$, where $X=P, B, D, C$.*

The function $f(x)=\int_0^x \psi^{-1}dx$ is bounded and its Dirichlet integral is

$$D(f)=\int_{M_1}(f')^2\psi dxdy=c\int_{-\infty}^{\infty}\psi^{-1}dx<\infty .$$

LEMMA 19. *The function*

$$q_0(x)=-\int_0^x \psi^{-1}(t)\int_0^t \psi(s)dsdt$$

*is quasiharmonic, that is, $\Delta q_0=1$. Every quasiharmonic function has the form $q=q_0+h$ with $h\in H$.*

This is verified by direct computation of $\Delta q_0$.

LEMMA 20. *$M_1\in\tilde{O}_{H^2B}$.*

In fact, $q_0\in H^2B$, since

$$\left|\psi^{-1}(t)\int_0^t \psi(s)ds\right|\sim e^{-|t|} \qquad \text{as} \quad |t|\to\infty .$$

For verification, first apply l'Hospital's rule to the left-hand side to see that it goes to 0 as $|t|\to\infty$, and then show, again by l'Hospital's rule, that $\left|e^{|t|}\psi^{-1}(t)\int_0^t \psi(s)ds\right|\to 1$ as $|t|\to\infty$.

LEMMA 21. *For $h(x)\in H$, the function*

$$u_0(x)=\int_0^x \psi^{-1}(t)\int_0^t \psi(s)h(s)dsdt$$

*is biharmonic. Every biharmonic function of the form $u(x)$ can be written $u(x)=u_0(x)+c$.*

The proof is again by direct computation.

LEMMA 22. *Every nonharmonic biharmonic function of the form $u(x)$ has infinite Dirichlet and $L^p$ norms.*

An estimate similar to that in the proof of Lemma 20 shows that $|u'(t)|\sim e^{-|t|}$ either as $t\to\infty$ or else as $t\to-\infty$. The Dirichlet integral is

$$D(u)=c\int_{-\infty}^{\infty}(u')^2\phi dx=\infty\,.$$

Since $u'(t)$ does not decrease faster than $e^{-|t|}$ at least in one direction, the same is true of $u(t)$. Therefore,

$$\|u\|_p^p=c\int_{-\infty}^{\infty}|u|^p\phi dx=\infty\,.$$

LEMMA 23. *If $v(x)G(y)$ is a nonharmonic biharmonic function, with $G(y)$ not constant, then $v\notin L^p$.*

Suppose $v\in L^p$ for some $1\leqq p<\infty$. Then $|v(x)|^p\phi(x)$ is integrable and decreases to 0. Let $\Delta(vG)=fG$. Since $vG$ is nonharmonic biharmonic, $f$ does not vanish in the neighborhood of at least one component of the ideal boundary, say $x=\infty$. As in Lemma 15, $\Delta(vG)=fG$ gives

$$(\phi v')'=(n_1^2\phi^{-1}+\sum_{i=2}^{N-1}n_i^2\phi)v-\phi f\,.$$

For large $x>0$, we may assume $f(x)<\varepsilon<0$, by changing the sign of $G$ if necessary. Since $|v|\to0$ rapidly, the dominating term on the right-hand side is $-\phi f$, and we obtain

$$(\phi v')'\geqq c\phi>0$$

for all sufficiently large $x>0$. On integrating from a sufficiently large $x_0$ to a larger $x$, we obtain

$$\phi v'\geqq c\int_{x_0}^{x}\phi dx\,.$$

An estimation exactly as that in the proof of Lemma 20 yields

$$v'\geqq ce^{-x}\,.$$

Thus $v$ can not be decreasing faster than $ce^{-x}$. This contradicts $|v|^p\phi(x)\to0$ and completes the proof of the lemma.

LEMMA 24. *The Dirichlet and $L^p$ norms of the function $vG$ of Lemma 23 are infinite.*

By Lemma 23, $v\notin L^p$. Therefore

$$\|vG\|_p^p=c\int_{-\infty}^{\infty}|v|^p\phi dx=\infty\,.$$

By the proof of Lemma 23, $|v'|\geqq ce^{-|x|}$ either as $x\to\infty$ or else as $x\to-\infty$. Therefore,

$$D(vG)=\int_{M_1}(v'G)^2\psi dxdy+\int_{M_1}\sum_{i=1}^{N-1}\left(v\frac{\partial G}{\partial y^i}\right)^2 g^{ii}dxdy$$
$$\geqq c\int_{-\infty}^{\infty}(v')^2\psi dx=\infty\ .$$

LEMMA 25. $M_1\in O_{H^2L^p}\cap O_{H^2D}$.

Let $u(x, y)$ be a nonharmonic biharmonic function. Write $u(x, y)=v_0(x)+\sum_{n=1}^{\infty}v_n(x)G_n(y)$. By Lemmas 22 and 24, neither $v_0$ nor any $v_nG_n$ belongs to $D\cup L^p$ if it is nonharmonic. By the Dirichlet orthogonality of $v_0$ and the $v_nG_n$, we conclude that $v_0+\sum_{n=1}^{\infty}v_nG_n$ is Dirichlet infinite.

Suppose $v_0+\sum_{n=1}^{\infty}v_nG_n\in L^p$. Choose a nonharmonic term $v_{n_0}G_{n_0}$. Since $v_{n_0}G_{n_0}\notin L^p$, there exists an $L^q$ function $fG_{n_0}$ such that $(v_{n_0}G_{n_0}, fG_{n_0})=\int_{M_1}v_{n_0}G_{n_0}fG_{n_0}dV=\infty$. On the other hand, $(v_{n_0}G_{n_0}, fG_{n_0})=(v_0+\sum_{n=1}^{\infty}v_nG_n, fG_{n_0})<\infty$, a contradiction.

With Lemma 25, the proof of the second string of relations in § 1 is complete.

**5. It remains to show that $O_X\cap O_Y\neq\phi$ and $\tilde{O}_X\cap\tilde{O}_Y\neq\phi$.** The metrics we shall choose will result in simple computations which also are completely analogous to those in §§ 2-4, and we can be brief.

To show that $\tilde{O}_X\cap\tilde{O}_Y\neq\phi$, we choose $\psi=1$ and $\varphi(x)=|x|^{-4}$ for $|x|>1$. Then the solutions $\Delta(f(x))=0$ and $\Delta(q(x))=1$ turn out to belong to the desired function classes $X$, $Y$.

To prove $O_X\cap O_Y\neq\phi$, let $\varphi=\psi=1$. It is easy to explicitly solve the equation $\Delta^2u=0$ in all cases and to show that the solutions do not belong to $X$ or $Y$.

**6. We have completed, by Lemmas 1-25, and § 5, the proof of the following result:**

THEOREM. *The classification scheme*

$$\begin{array}{ccccc} & & O_X^N & & \\ & \angle & & \nwarrow & \\ O_{X\cup Y}^N & & & & O_{XY}^N \\ & \nwarrow & & \angle & \\ & & O_Y^N & & \end{array}$$

*holds for* $X=G, HP, HB, HD, HC, \Gamma, H^2B$; $Y=H^2D, H^2L^p$.

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