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ON THE CHARACTERISTIC FUNCTIONS OF HARMONIC QUATERNION KÄHLERIAN SPACES

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1. Introduction.

An analytic Riemannian space M is harmonic if every point p_0 is the origin of a normal neighbourhood N such that, if p is a point of N and Ω is the distance function $\Omega(p_0, p)=(1/2)s^2$, then the Laplacian $\Delta\Omega$, calculated for fixed p_0 and variable p, is a function depending upon Ω and not otherwise upon p, i. e., $\Delta\Omega=f(\Omega)$ where $f(\Omega)$ is called the characteristic function of M.

Its typical examples are the following: (1) spheres, (2) real projective spaces, (3) complex projective spaces, (4) quaternion projective spaces and (5) the Cayley projective planes ([8], [11]), and the characteristic functions $f(\Omega)$ of spheres, real projective spaces and complex projective spaces have been found ([9], [13], [14]). Moreover, we have already the following theorems in harmonic spaces.

THEOREM A (Lichnérowicz [9]). *In any harmonic Riemannian space M of dimension n, its characteristic function $f(\Omega)$ satisfies the inequality*

$$f^2(0)+\frac{5}{2}(n-1)\ddot{f}(0)\leqq 0\,.$$

The equality sign is valid if and only if M is of constant curvature.

THEOREM B (Tachibana [13]). *In any harmonic Kählerian space M of dimension $2m$, its characteristic function $f(\Omega)$ satisfies the inequality*

$$f^2(0)+\frac{5(m+1)^2}{m+7}\ddot{f}(0)\leqq 0\,.$$

The equality sign is valid if and only if M is of constant holomorphic curvature.

THEOREM C (Watanabe [15]). *In any harmonic Kählerian space M of dimension $2m$, its characteristic function $f(\Omega)$ satisfies the inequality*

$$2\dot{f}^3(0)+(13m+28)\dot{f}(0)\ddot{f}(0)+7(m+1)(m+2)\dddot{f}(0)\leqq 0\,.$$

The equality sign is valid if and only if M is locally symmetric.

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A quaternion Kählerian space is defined as a Riemannian space whose holonomy gronp is a subgroup of $Sp(m)\cdot Sp(1)$. Its typical example is a quaternion projective space. Several authors (Alekseevskii [1], Gray [2], Ishihara [3], [4], [5], Ishihara and Konishi [6], Krainse [7] and Wolf [16]) have studied quaternion Kählerian spaces, and obtained many interesting results. As we stated in the first place, a quaternion projective space is also a harmonic space. So in the present paper, we study harmonic quaternion Kählerian space by using tensor calculus developed in [3], [4] and [5]. In § 2, we give some preliminaries. § 3 is devoted to establish some formulas in a quaternion Kählerian space. In § 4, we give an equation, which plays an important role in any harmonic quaternion Kählerian space. The last section is devoted to prove our main theorems 5.1 and 5.5.

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2. Preliminaries.

Let (M, g) be a Riemannian space with Levi-Civita connection ∇. By $R=(R^{i}{}_{jkl})$, we denote the Riemannian curvature tensor. Then $R_1=(R^{a}{}_{ija})=(R_{ij})$ and $S=g^{ij}R_{ij}$ are Ricci tensor and scalar curvature respectively. Let (;) denote the covariant differentiation, and put $\nabla R=(R^{i}{}_{jkl;h})$. For a tensor field $T=(T_{ijk})$, for example, we denote $|T|=T_{ijk}T^{ijk}$. We put

$$\alpha=|R|^2,\quad \beta=R^{abcd}R_{ab}{}^{uv}R_{cduv}\quad\text{and}\quad \gamma=R^{abcd}R_{a}{}^{u}{}_{c}{}^{v}R_{budv}\,.$$

Then they satisfy the following fundamental formulas (cf. [12]).

$$R_{ijkh}+R_{ikhj}+R_{ihjk}=0\,, \tag{2.1}$$

$$\left\{\begin{aligned}
&\text{(a)}\quad R^{abcd}R_{a}{}^{u}{}_{b}{}^{v}R_{cudv}=\frac{1}{4}\beta\,,\\
&\text{(b)}\quad R^{abcd}R_{aub}{}^{v}R_{cvdu}=-\frac{1}{4}\beta\,,\\
&\text{(c)}\quad R^{abcd}R_{a}{}^{u}{}_{c}{}^{v}R_{bduv}=R^{abcd}R_{ac}{}^{uv}R_{budv}=\frac{1}{4}\beta\,,\\
&\text{(d)}\quad R^{abcd}R_{a}{}^{u}{}_{c}{}^{v}R_{bvdu}=R^{abcd}R_{a}{}^{v}{}_{c}{}^{u}R_{budv}=\gamma-\frac{1}{4}\beta\,.
\end{aligned}\right. \tag{2.2}$$

Lichnérowicz [10], [17] proved the following formula.

$$\frac{1}{2}\Delta\alpha=|\nabla R|^2-4R^{ijkh}R_{ik,h,j}+2R_{ij}R^{ihkl}R^{j}{}_{hkl}+\beta+4\gamma\,. \tag{2.3}$$

Let M be a harmonic Riemannian space. Then it is well known ([9], [11]) that M satisfies the following curvature conditions:

$$R_{ij}=-\frac{3}{2}\dot{f}(0)g_{ij}\,,\qquad S=-\frac{3n}{2}\dot{f}(0)\,, \tag{2.4}$$

$$P(R^p{}_{ijq}R^q{}_{klp})=-\frac{45}{8}\ddot{f}(0)P(g_{ij}g_{kl})\,, \tag{2.5}$$

equivalently,

$$\begin{aligned}R^p{}_{ijq}(R^q{}_{klp}+R^q{}_{lkp})+R^p{}_{ikq}(R^q{}_{ljp}+R^q{}_{jlp})+R^p{}_{ijq}(R^q{}_{jkp}+R^q{}_{kjp})\\ =-\frac{45}{4}\ddot{f}(0)(g_{ij}g_{kl}+g_{ik}g_{lj}+g_{il}g_{jk})\end{aligned} \tag{2.5$'$}$$

and

$$P(9R^p{}_{ijq,k}R^q{}_{lmp;n}-32R^p{}_{ijq}R^q{}_{klr}R^r{}_{mnp})=315\dddot{f}(0)P(g_{ij}g_{kl}g_{mn})\,, \tag{2.6}$$

where P denotes the sum of terms obtained by permuting the given free indices, and $(\dot{})$ means the operator taking the derivative with respect to Ω.

(2.4) and (2.5)′ give

$$\begin{aligned}\alpha&=-\frac{3n}{2}\Big\{f^2(0)+\frac{5(n+2)}{2}\ddot{f}(0)\Big\}\,,\\ \ddot{f}(0)&=-\frac{4}{15n(n+2)}\Big(\alpha+\frac{2}{3n}S^2\Big)\,.\end{aligned} \tag{2.7}$$

Taking account of (2.4) and (2.7), we have from (2.3)

$$|\nabla R|^2+\frac{2}{n}S\alpha+\beta+4\gamma=0\,. \tag{2.8}$$

From (2.6), it follows ([15]) that

$$\begin{aligned}27|\nabla R|^2-32\Big(\frac{S^3}{n^2}+\frac{9S}{2n}\alpha-\frac{7}{2}\beta+\gamma\Big)\\ =315n(n+2)(n+4)\dddot{f}(0)\,.\end{aligned} \tag{2.9}$$

3. Quaternion Kählerian spaces.

Let M be a differentiable manifold of dimension n and assume that there is a 3-dimensional vector bundle V consisting of tensors of type (1, 1) over M satisfying the following conditions:

In any coordinate neighbourhood U of M, there is a local base $\{F, G, H\}$ of V such that

$$\begin{aligned}&F^2=G^2=H^2=-I\,;\\ &GH=-HG=F,\qquad HF=-FH=G,\qquad FG=-GF=H\,,\end{aligned} \tag{3.1}$$

I denoting the identity tensor field of type (1, 1) in M.

Such a local base $\{F, G, H\}$ is called a canonical local base of the bundle

V in U. Then the bundle V is called an almost quaternion structure in M and (M, V) an almost quaternion space. Thus, an almost quaternion space is necessarily of dimension $n=4m(m\geqq 1)$.

In any almost quaternion space (M, V), there is a Riemannian metric g such that $g(\phi X, Y)+g(X, \phi Y)=0$ for any cross-section ϕ of V, local or global, X and Y being arbitrary vector fields in M. Such a set (g, V) is called an almost quaternion metric structure and the set (M, g, V) an almost quaternion metric space. Thus a manifold M admits an almost quaternion (metric) structure if and only if the structure group of the tangent bundle over M is reducible to $Sp(m)\cdot Sp(1)$.

Let $\{F, G, H\}$ be a canonical local base of the bundle V in a coordinate neighborhood U of an almost quaternion metric space (M, g, V). Since each of F, G and H is an almost Hermitian with respect to g, Φ, Ψ and θ are local 2-forms in U, where they are defined respectively by

$$\Phi(X, Y)=g(FX, Y), \qquad \Psi(X, Y)=g(GX, Y), \qquad \theta(X, Y)=g(HX, Y)\,, \tag{3.2}$$

X and Y being arbitray vector fields. However,

$$\omega=\Phi\wedge\Phi+\Psi\wedge\Psi+\theta\wedge\theta \tag{3.3}$$

is also a 4-form defined globally in M.

If an almost quaternion metric space (M, g, V) satisfies the condition,

$$\nabla\omega=0\,, \tag{3.4}$$

then (M, g, V) is called a quaternion Kählerian space and (g, V) a quaternion Kählerian structure.

The following formulas (3.5)~(3.9) were proved in [4] and [5] if m is greater than 2:

$$R_{ji}=\frac{S}{4m}g_{ji}\,, \tag{3.5}$$

$$\begin{aligned} R_{kjih}F^{ih}&=-\frac{S}{2(m+2)}F_{kj}\,,\\ R_{kjih}G^{ih}&=-\frac{S}{2(m+2)}G_{kj}\,,\\ R_{kjih}H^{ih}&=-\frac{S}{2(m+2)}H_{kj}\,, \end{aligned} \tag{3.6}$$

$$\begin{aligned} R_{ktsh}F^{ts}&=\frac{S}{4(m+2)}F_{kh}\,,\\ R_{ktsh}G^{ts}&=\frac{S}{4(m+2)}G_{kj}\,,\\ R_{ktsh}H^{ts}&=\frac{S}{4(m+2)}H_{kj}\,. \end{aligned} \tag{3.7}$$

$$R_{kjts}F_i{}^tF_h{}^s=R_{kjih}+\frac{S}{4m(m+2)}(G_{kj}G_{ih}+H_{kj}H_{ih}),$$

(3.8)
$$R_{kjts}G_i{}^tG_h{}^s=R_{kjih}+\frac{S}{4m(m+2)}(H_{kj}H_{ih}+F_{kj}F_{ih}),$$

$$R_{kjts}H_i{}^tH_h{}^s=R_{kjih}+\frac{S}{4m(m+2)}(F_{kj}F_{ih}+G_{kj}G_{ih}).$$

$$R_{kjt}{}^hF_i{}^t=R_{kji}{}^sF_s{}^h-\frac{S}{4m(m+2)}(-H_{kj}G_i{}^h+G_{kj}H_i{}^h),$$

(3.9)
$$R_{kjt}{}^hG_i{}^t=R_{kji}{}^sG_s{}^h-\frac{S}{4m(m+2)}(-F_{kj}H_i{}^h+H_{kj}F_i{}^h),$$

$$R_{kjt}{}^hH_i{}^t=R_{kji}{}^sH_s{}^h-\frac{S}{4m(m+2)}(-G_{kj}F_i{}^h+F_{kj}G_i{}^h).$$

We take a point p in a quaternion Kählerian space (M, g, V) of dimension $4m$ and a vector X tangent to M at p. Putting

(3.10)
$$Q(X)=\{Y \mid Y=aX+bFX+cGX+dHX\},$$

a, b, c and d being arbitrary real numbers, we call $Q(X)$ the Q-section determined by X, where $Q(X)$ is a 4-dimensional subspace of the tangent space of M at p. When for any $Y, Z \in Q(X)$, the sectional curvature $\sigma(Y, Z)$ is a constant $\rho(X)$, then $\rho(X)$ is called the Q-sectional curvature of (M, g, V) with respect to X. A quaternion Kählerian space is said to be of constant Q-sectional curvature κ when any Q-section $Q(X)$ has its Q-sectional curvature $\rho(X)$ and $\rho(X)$ is a constant κ independent of X at each point p. The following proposition is well known:

PROPOSITION 3.1 (Ishihara [4]). *A quaternion Kählerian space is of constant Q-sectional curvature κ, if and only if its curvature tensor has components of the form*

(3.11)
$$\begin{aligned}R_{kjih}=\frac{\kappa}{4}[&(g_{kh}g_{ji}-g_{jh}g_{ki})+(F_{kh}F_{ji}-F_{jh}F_{ki}-2F_{kj}F_{ih})\\&+(G_{kh}G_{ji}-G_{jh}G_{ki}-2G_{kj}G_{ih})+(H_{kh}H_{ji}-H_{jh}H_{ki}-2H_{kj}H_{ih})]\end{aligned}$$

A typical example of quaternion Kählerian space of constant Q-sectional curvature is a quaternion projective space $HP(m)$ of dimension m, whose Q-sectional curvature is equal to 4. When $m=1$, $HP(1)$ is a natural sphere with constant curvature.

Let $A=T_{ijkl}T^{ijkl}$ be the square of the tensor T_{ijkl} defined by

$$\begin{aligned}T_{kjih}=R_{kjih}-\frac{\kappa}{4}[&g_{kh}g_{ji}-g_{jh}g_{ki}+F_{kh}F_{ji}-F_{jh}F_{ki}-2F_{kj}F_{ih}\\&+G_{kh}G_{ji}-G_{jh}G_{ki}-2G_{kj}G_{ih}+H_{kh}H_{ji}-H_{jh}H_{ki}-2H_{kj}H_{ih}]\end{aligned}$$

where $\kappa=\frac{S}{4m(m+2)}$. Then after some calculations, we get

$$A=\alpha-\frac{5m+1}{4m(m+2)^2}S^2 .$$

If A vanishes identically, then M is of constant Q-sectional curvature. Thus we have

PROPOSITION 3.2. *In any quaternion Kählerian space M of dimension $4m$, we have*

$$\alpha-\frac{5m+1}{4m(m+2)^2}S^2\geqq 0 . \tag{3.12}$$

The equality sign is valid if and only if M is of constant Q-sectional curvature. Especially when $m=1$, M is of constant curvature.

Now, we prove two lemmas for later use.

LEMMA 3.3. *In any quaternion Kählerian space of dimension $4m(m>1)$,*

$$R_{abcd}R^{aucv}G_u{}^dG_v{}^b=-\frac{1}{2}\alpha+\frac{1}{(m+2)^2}S^2 . \tag{3.13}$$

Proof. From (3.1) and (3.9), we get

$$\begin{aligned}
&R_{abcd}R^{aucv}G_u{}^dG_v{}^b\\
&\quad=-R_{abdc}G_u{}^dR^{aucv}G_v{}^b\\
&\quad=-\left\{-R_{abur}G_c{}^r+\frac{S}{4m(m+2)}(F_{ab}H_{uc}-H_{ab}F_{uc})R^{aucv}G_v{}^b\right\}\\
&\quad=R_{abur}G_c{}^rG_v{}^bR^{aucv}+\frac{S}{2m(m+2)}H_{av}H_{uc}R^{aucv}\\
&\quad=R_{abur}\left\{R^{aurb}+\frac{S}{2m(m+2)}H^{au}H^{rb}\right\}+\frac{S}{2m(m+2)}H_{av}H_{uc}R^{aucv}\\
&\quad=-\frac{1}{2}\alpha+\frac{1}{(m+2)^2}S^2 .
\end{aligned}$$

LEMMA 3.4. *In any quaternion Kählerian space of dimension $4m(m>1)$,*

$$H_{bh}G_{ik}F_{qr}R^{qkhp}R_p{}^{ibr}=\frac{1}{2}\alpha-\frac{2m+1}{2m(m+2)^2}S^2 . \tag{3.14}$$

Proof. From (3.1) and (3.9), we get

$$\begin{aligned}
&H_{bh}G_{ik}F_{qr}R^{qkhp}R_p{}^{ibr}\\
&\quad=-F_r{}^qG_i{}^kR_{hpqk}H_b{}^hR^{pibr}
\end{aligned}$$

$$=-G_i{}^k H_b{}^h R^{pibr}\left\{-R_{hprs}F_k{}^s+\frac{S}{4m(m+2)}(H_{hp}G_{rk}-G_{hp}H_{rk})\right\}$$

$$=-H_i{}^s H_b{}^h R^{pibr}R_{hprs}+\frac{S}{4m(m+2)}(g_{bp}g_{ir}+F_{ir}F_{bp})R^{pibr}$$

$$=\frac{1}{2}\alpha-\frac{1}{(m+2)^2}S^2-\frac{1}{2m(m+2)^2}S^2$$

$$=\frac{1}{2}\alpha-\frac{2m+1}{2m(m+2)^2}S^2 .$$

4. Harmonic quaternion Kählerian space.

In the present section, we shall give an important equality in any harmonic quaternion Kählerian space. Transvecting (2.5)′ with $F_a{}^i F_b{}^j R^{kabl}$, we have

$$(4.1)\quad \begin{aligned} &F_a{}^i F_b{}^j R^{kabl}\{R^p{}_{ijq}(R^q{}_{klp}+R^p{}_{lkp}) \\ &\quad +R^p{}_{ikq}(R^q{}_{ljp}+R^q{}_{jlp})+R^p{}_{ilq}(R^q{}_{jkp}+R^q{}_{kjp})\} \\ &=-\frac{45(2m+1)}{2(m+2)}S\ddot{f}(0) . \end{aligned}$$

We now are going to show by using (3.1), (3.5), (3.6), (3.7), (3.8) and (3.9) that the left hand side of (4.1) reduces to that of (4.2). To do so, we have by using (2.2) and (3.13)

$$F_a{}^i F_b{}^j R_{pijq}R^q{}_{kl}R^{kabl}$$

$$=F_a{}^i R^q{}_{kl}{}^p R^{kabl}\left\{-R_{pibr}F_q{}^r+\frac{S}{4m(m+2)}(H_{pi}G_{bq}-G_{pi}H_{bq})\right\}$$

$$=R^{qklp}R_p{}^{ib}{}_r F_q{}^r\left\{R_{blis}F_k{}^s-\frac{S}{4m(m+2)}(H_{bl}G_{ik}-G_{bl}H_{ik})\right\}$$

$$+\frac{S}{2m(m+2)}G_{pa}G_{bq}R^q{}_{kl}{}^p R^{kabl}$$

$$=R_{blis}R_p{}^{ib}{}_r\left\{R^{rslp}+\frac{S}{4m(m+2)}(G^{rs}G^{lp}+H^{rs}H^{lp})\right\}$$

$$-\frac{S}{4m(m+2)}\{H_{bl}G_{ik}F_q{}^r-G_{bl}H_{ik}F_q{}^r\}R^{qklp}R_p{}^{ib}{}_r$$

$$+\frac{S}{2m(m+2)}G_{pa}G_{bq}R^q{}_{kl}{}^p R^{kabl}$$

$$=\gamma-\frac{1}{4}\beta+\frac{S}{m(m+2)}R_{blis}R^{pibr}G_r{}^s G^l{}_p$$

$$-\frac{S}{2m(m+2)}H_{bl}G_{ik}F_q{}^r R^{qklp}R_p{}^{ib}{}_r$$

$$= \gamma-\frac{1}{4}\beta-\frac{3S}{4m(m+2)}\alpha+\frac{6m+1}{4m^2(m+2)^3}S^3 .$$

By (2.1) and (2.2), we get

$$R_{pijq}R^q{}_{lk}{}^p F_a{}^i F_b{}^j R^{kabl}$$

$$=R_{pijq}R^q{}_{lk}{}^p F_a{}^i F_b{}^j R^{baka}-R_{pijq}R^q{}_{lk}{}^p F_a{}^i F_b{}^j R^{klab}$$

$$=R_{pijq}R^q{}_{kl}{}^p F_a{}^i F_b{}^j R^{lbak}$$

$$-R_{pijq}R^q{}_{lk}{}^p\left\{R^{klij}+\frac{S}{2m(m+2)}G^{kl}G^{ij}\right\}$$

$$= \gamma-\frac{1}{4}\beta-\frac{3S}{4m(m+2)}\alpha+\frac{6m+1}{4m^2(m+2)^3}S^3$$

$$-\frac{1}{4}\beta-\frac{1}{8(m+2)^3}S^3$$

$$= \gamma-\frac{1}{2}\beta-\frac{3S}{4m(m+2)}\alpha-\frac{m^2-12m-2}{8m^2(m+2)^3}S^3 .$$

Similarly, we get

$$R_{pikq}R^q{}_{lj}{}^p F_a{}^i F_b{}^j R^{kabl}=-\frac{1}{4}\beta-\frac{S}{2m(m+2)}\alpha+\frac{3m+1}{4m^2(m+2)^3}S^3 ,$$

$$R_{pikq}R^q{}_{jl}{}^p F_a{}^i F_b{}^j R^{kabl}=-\frac{1}{4}\beta-\frac{S}{2m(m+2)}\alpha-\frac{6m^2-6m-2}{8m^2(m+2)^3}S^3 ,$$

$$R_{pilq}R^q{}_{jk}{}^p F_a{}^i F_b{}^j R^{kabl}= \gamma-\frac{1}{2}\beta-\frac{3S}{4m(m+2)}\alpha-\frac{m^2-10m-2}{8m^2(m+2)^3}S^3 ,$$

and

$$R_{pilq}R^q{}_{kj}{}^p F_a{}^i F_b{}^j R^{kabl}= \gamma-\frac{1}{4}\beta-\frac{S}{2m(m+2)}\alpha+\frac{4m+1}{4m^2(m+2)^3}S^3 .$$

Substituting these into (4.1), we get

$$\text{(4.2)}\qquad 4\gamma-2\beta-\frac{15S}{4m(m+2)}\alpha-\frac{3(m^2-18m-4)}{8m^2(m+2)^3}S^3=-\frac{45(2m+1)S}{2(m+2)}\ddot{f}(0) .$$

Thus, we obtain from (2.7)

PROPOSITION 4.1. *In any harmonic quaternion Kählerian space of dimension* $4m(m>1)$,

$$\text{(4.3)}\qquad 4\gamma-2\beta=\frac{9S}{2m(m+2)}\alpha+\frac{2m^2-25m-4}{4m^2(m+2)^3}S^3 .$$

5. Some theorems.

In this section, we give some results by combining (2.9) with Proposition 4.1. Substituting (2.4) and (2.7) into (3.12), we get

$$f^2(0)+\frac{10(m+2)^2}{m+11}\ddot{f}(0)\leqq 0\,,$$

from which

THEOREM 5.1. *In any harmonic quaternion Kählerian space M of dimension 4m(m>1), the inequality*

$$f^2(0)+\frac{10(m+2)^2}{m+11}\ddot{f}(0)\leqq 0 \tag{5.1}$$

holds. The equality sign is valid if and only if M is of constant Q-sectional curvature.

By (2.8) and (4.3), we get

$$|\nabla R|^2+3\beta+\frac{(m+11)S}{2m(m+2)}\alpha+\frac{2m^2-25m-4}{8m^2(m+2)^3}S^3=0\,. \tag{5.2}$$

Similarly, we get

$$|\nabla R|^2+6\gamma+\frac{(2m-5)S}{4m(m+2)}\alpha-\frac{2m^2-25m-4}{8m^2(m+2)^3}S^3=0\,. \tag{5.3}$$

From (5.2) and (5.3), we have

PROPOSITION 5.2. *Any harmonic quaternion Kählerian space M of dimension 4m(m>1) satisfies the following inequalities:*

$$\beta\leqq-\frac{S}{6m(m+2)}\left\{(m+11)\alpha+\frac{2m^2-25m-4}{2m(m+2)^2}S^2\right\}, \tag{5.4}$$

and

$$\gamma\leqq-\frac{S}{24m(m+2)}\left\{(2m-5)\alpha-\frac{2m^2-25m-4}{2m(m+2)^2}S^2\right\}. \tag{5.5}$$

Each equality sign is valid if and only if M is locally symmetric.

When $S\geqq 0$ in (4.3) and (5.5), (3.12) gives

$$4\gamma-2\beta=\frac{9S}{2m(m+2)}\alpha+\frac{2m^2-25m-4}{4m^2(m+2)^3}S^3$$

$$\geqq\frac{(4m-1)(m-1)}{8m^2(m+2)^3}S^3\,.$$

and

$$\gamma\leqq-\frac{2m^2+9m+1}{32m^2(m+2)^3}S^3 \qquad \text{(resp.)}$$

Thus, we have $0 \geqq 2\gamma \geqq \beta$ if $S \geqq 0$. Similarly, we have $2\gamma > \beta$ if $S<0$. Summing up, we have

PROPOSITION 3.3. *Let M be a harmonic quaternion Kählerian space of dimension $4m(m>1)$. Then*

(1) *If $S \geqq 0$, then $0 \geqq 2\gamma \geqq \beta$.*

(2) *If $S<0$, then $2\gamma < \beta$.*

Next, (2.9) and (4.3) give by using (5.2)

$$-32\cdot 315m(2m+1)(m+1)\dddot{f}(0)$$
$$=5|\nabla R|^2+\frac{4(13m+71)S}{m(m+2)}\alpha+\frac{2(m^3+16m^2-113m-12)}{m^2(m+2)^3}S^3\,. \tag{5.6}$$

Thus, when $S>0$, we obtain from (3.12) and (5.6)

PROPOSITION 5.4. *A harmonic quaternion Kählerian space of dimension $4m(m>1)$ with positive scalar curvature satisfise*

$$\dddot{f}(0)<0\,.$$

Lastly, making use of (2.4) and (2.7), we can prove that (5.6) takes

$$|\nabla R|^2+\frac{144m(2m+1)}{(m+2)^3}\{(m^2+7m+64)\dot{f}^3(0)$$
$$+(13m+71)(m+2)^2 f(0)\ddot{f}(0)+14(m+1)(m+2)^3\dddot{f}(0)\}=0\,,$$

from which

THEOREM 5.5. *In any harmonic quaternion Kählerian space M of dimension $4m(m>1)$, its characteristic function $f(\Omega)$ satisfies the inequality*

$$(m^2+7m+64)f^3(0)+(13m+71)(m+2)^2 f(0)\ddot{f}(0)$$
$$+14(m+1)(m+2)^3\dddot{f}(0)\leqq 0\,.$$

The equality sign is valid if and only if M is locally symmetric.

BIBLIOGRAPHY

[1] D.V. ALEKSEEVSKII, Riemannian spaces with exceptional holonomy groups, Functionl'nyi Analiz i Ego Prilozhenyia 2 (1968), 1-10.

[2] A. GRAY, A note on manifolds whose holonomy group is a subgroup of Sp (m)•Sp(1), Michigan Math. J. (1969), 125-128.

[3] S. ISHIHARA, Quaternion Kählerian manifolds and fibred Riemannian space with Sasakian 3-structure, to appear in Kōdai Math. Sem. Rep.

[4] S. ISHIHARA, Notes on quaternion Kählerian manifolds, J. Differential Geo-

metry **9** (1974), 483-500.

[5] S. ISHIHARA, Integral formulas and their applications ın quaternion Kählerian manifolds, to appear.

[6] S. ISHIHARA AND M. KONISHI, Fibred Riemannian spaces with Sasakian 3-structure, Differential Geometry, in honor of K. Yano, Kinokuniya, Tokyo, 1972. 179-194.

[7] V.Y. KRAINSE, Topology of quaternionic manifolds, Trans, Amer. Math. Soc., **122** (1966), 357-367.

[8] A.J. LEDGER, Harmonic homogeous spaces of Lie groups, Journal London Math. Soc., **29** (1954), 345-347.

[9] A. LICHNÉROWICZ, Sur les espaces Riemanniens complĕtment harmoniques, Bull. Soc. Math., Fr. **72** (1944) 146-168.

[10] A. LICHNÉROWICZ, Geometrie des groupes de transformations, Dund, Paris, 1958.

[11] H.S. RUSE, A.G. WALKER AND T.J. WILLMORE, Harmonic Spaces, Edizioni Cremonese, Roma, 1961.

[12] T. SAKAI, On eigenvalues of Laplacian and curvature of Riemannian manifolds, Tôhoku Math. Jour., **23** (1971), 589-603.

[13] S. TACHIBANA, On the characteristic function of spaces of constant holomorphic curvature, Colloq. **26** (1972), 145-155.

[14] T.J. WILLMORE, Some properties of harmonic Riemannian manifolds, Convegno di Geometria Differenzial, Venice, (1953) 141-147.

[15] Y. WATANABE, On the characteristic function of harmonic Kählerian spaces, Tôhoku Math. Jour., **27** (1975), 13-24.

[16] J.A. WOLF, Complex homogeneous contact manifolds and quaternionic symmetric spaces, J. Math. Mech., **14** (1963), 1033-1047.

[17] K. YANO, Differential geometry on complex and almost complex spaces, Pergamon Press, 1965.

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