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TOTALLY REAL SUBMANIFOLDS OF COMPLEX SPACE FORMS II

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Introduction.

Let $\bar{M}$ be a Kaehler manifold of complex dimension $n+p$, $p\geqq 0$, and M be a Riemannian manifold of real dimension n. Let J be the almost complex structure of $\bar{M}$. We call M a totally real submanifold of $\bar{M}$ if M admits an isometric immersion into $\bar{M}$ such that $JT_x(M)\subset T_x(M)^{\perp}$ where $T_x(M)$ denotes the tangent space of M at x and $T_x(M)^{\perp}$ the normal space of M at x. When $p=0$, we see that $JT_x(M)=T_x(M)^{\perp}$, for which case many interesting properties of totally real submanifolds have been studied by different authors (see [1], [2], [4], [5], [6], [7], [9] and [12]). For the case $p>0$, one of the present authors proved in [10] some theorems for totally real, totally umbilical submanifolds of a Kaehler manifold. On the other hand, Ludden-Okumura-Yano [6] proved a pinching theorem for a compact minimal totally real submanifold of a complex space form also for the case $p>0$.

The purpose of the present paper is to generalize some of theorems proved in [5], [6], [7], [10] and [12].

In § 1 we derive some fundamental formulas for a totally real submanifold M of a Kaehler manifold $\bar{M}$. In § 2 we study the f-structure in the normal bundle of a totally real submanifold (see [6], [8], [10]). In § 3 we consider an n-dimensional compact totally real submanifold of a complex space form $\bar{M}(c)$ of complex dimension $n+p$ and of constant holomorphic sectional curvature c and give some integral formulas computing the Laplacian of the square of the second fundamental form. As an application of these integral formulas we prove a pinching theorem for compact totally real submanifolds which is a generalization of theorems in [2] and [5]. In § 4 and § 5 we study generalizations of results proved in [12]. The purpose of the last section is to give a characterization of an n-dimensional compact flat totally real submanifold $S^1(r_1)\times S^1(r_2)\times \cdots \times S^1(r_n)$ in some C^n in C^{n+p}.

§ 1. Preliminaries.

Let $\bar{M}$ be a Kaehler manifold of complex dimension $n+p$. We denote by J the almost complex structure of $\bar{M}$. An n-dimensional Riemannian manifold

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M isometrically immersed in $\bar{M}$ is called a *totally real* submanifold of $\bar{M}$ if $JT_x(M)\perp T_x(M)$ for each $x\in M$ where $T_x(M)$ denotes the tangent space to M at $x\in M$. Here we have identified $T_x(M)$ with its image under the differential of the immersion because our computation is local. If $X\in T_x(M)$, then JX is a normal vector to M. Thus we see that $p\geqq 0$. Let $\bar{g}$ be the metric tensor field of $\bar{M}$ and g be the induced metric tensor field on M. We denote by $\bar{\nabla}$ (resp. ∇) the operator of covariant differentiation with respect to $\bar{g}$ (resp. g). Then the Gauss-Weingarten formulas are respectively given by

$$\bar{\nabla}_X Y=\nabla_X Y+B(X, Y), \qquad \bar{\nabla}_X N=-A_N X+D_X N$$

for any tangent vector fields X, Y and any normal vector field N on M, where D is the operator of covariant differentiation with respect to the linear connection induced in the normal bundle. Both A and B are called the second fundamental form of M and satisfy

$$\bar{g}(B(X, Y), N)=g(A_N X, Y).$$

A normal vector field N in the normal bundle is said to be *parallel* if $D_X N=0$ for any tangent vector field X on M. The mean curvature vector H is defined as $H=(1/n)\operatorname{Tr} B$, $\operatorname{Tr} B$ being defined by $\operatorname{Tr} B=\sum_i B(e_i, e_i)$ for an orthonormal frame $\{e_i\}$. If $H=0$, then M is said to be *minimal* and if the second fundamental form is of the form $B(X, Y)=g(X, Y)H$, then M is said to be *totally umbilical*. If the second fundamental form of M vanishes identically, i.e., $B=0$, then M is said to be *totally geodesic*.

We choose a local field of orthonormal frames $e_1, \cdots, e_n$; $e_{n+1}, \cdots, e_{n+p}$; $e_{1^*}=Je_1, \cdots, e_{n^*}=Je_n$; $e_{(n+1)^*}=Je_{n+1}, \cdots, e_{(n+p)^*}=Je_{n+p}$ in $\bar{M}$ in such a way that, restricted to M, $e_1, \cdots, e_n$ are tangent to M. With respect to this frame field of $\bar{M}$, let $\omega^1, \cdots, \omega^n$; $\omega^{n+1}, \cdots, \omega^{n+p}$; $\omega^{1^*}, \cdots, \omega^{n^*}$; $\omega^{(n+1)^*}, \cdots, \omega^{(n+p)^*}$ be the field of dual frames. Unless otherwise stated, we use the conventions that the ranges of indices are respectively:

$$A, B, C, D=1, \cdots, n+p, 1^*, \cdots, (n+p)^*,$$

$$i, j, k, l, t, s=1, \cdots, n,$$

$$a, b, c, d=n+1, \cdots, n+p, 1^*, \cdots, (n+p)^*,$$

$$\alpha, \beta, \gamma=n+1, \cdots, n+p,$$

$$\lambda, \mu, \nu=n+1, \cdots, n+p, (n+1)^*, \cdots, (n+p)^*,$$

and that when an index appears twice in any term as a subscript and a superscript, it is understood that this index is summed over its range. Then the structure equations of $\bar{M}$ are given by

(1.1) $$d\omega^A=-\omega^A_B\omega^B, \qquad \omega^A_B+\omega^B_A=0,$$

$$\omega^i_j+\omega^j_i=0, \qquad \omega^i_j=\omega^{i^*}_{j^*}, \qquad \omega^{i^*}_j=\omega^{j^*}_i,$$

(1.2) $$\omega_\beta^\alpha+\omega_\alpha^\beta=0\,,\qquad \omega_\beta^\alpha=\omega_{\beta^*}^{\alpha^*}\,,\qquad \omega_\beta^{\alpha^*}=\omega_\alpha^{\beta^*}\,,$$
$$\omega_\alpha^i+\omega_i^\alpha=0\,,\qquad \omega_\alpha^i=\omega_{\alpha^*}^{i^*}\,,\qquad \omega_\alpha^{i^*}=\omega_i^{\alpha^*}\,,$$

(1.3) $$d\omega_B^A=-\omega_C^A\omega_B^C+\Phi_B^A\,,\qquad \Phi_B^A=\frac{1}{2}K_{BCD}^A\omega^C\wedge\omega^D\,.$$

Restricting these forms to M, we have

(1.4) $$\omega^a=0\,,$$

(1.5) $$d\omega^i=-\omega_k^i\wedge\omega^k\,,$$

(1.6) $$d\omega_j^i=-\omega_k^i\wedge\omega_j^k+\Omega_j^i\,,\qquad \Omega_j^i=\frac{1}{2}R_{jkl}^i\omega^k\wedge\omega^l\,.$$

Since $0=d\omega^a=-\omega_i^a\wedge\omega^i$, by Cartan's lemma we have

(1.7) $$\omega_i^a=h_{ij}^a\omega^j\,,\qquad h_{ij}^a=h_{ji}^a\,.$$

We see that $g(A_ae_i, e_j)=h_{ij}^a$. The Gauss-equation is given by

(1.8) $$R_{jkl}^i=K_{jkl}^i+\sum_a(h_{ik}^ah_{jl}^a-h_{il}^ah_{jk}^a)\,.$$

Moreover we have

(1.9) $$d\omega_b^a=-\omega_c^a\wedge\omega_b^c+\Omega_b^a\,,\qquad \Omega_b^a=\frac{1}{2}R_{bkl}^a\omega^k\wedge\omega^l\,,$$

and the Ricci-equation is given by

(1.10) $$R_{bkl}^a=K_{bkl}^a+\sum_i(h_{ik}^ah_{il}^b-h_{il}^ah_{ik}^b)\,.$$

From (1.2) and (1.7) we have

(1.11) $$h_{jk}^{i^*}=h_{ik}^{j^*}=h_{ij}^{k^*}\,.$$

We define the covariant derivative h_{ijk}^a of h_{ij}^a by setting

(1.12) $$h_{ijk}^a\omega^k=dh_{ij}^a-h_{il}^a\,\omega_j^l-h_{lj}^a\omega_i^l+h_{ij}^b\omega_b^a\,.$$

The Laplacian Δh_{ij}^a of h_{ij}^a is defined as

(1.13) $$\Delta h_{ij}^a=\sum_k h_{ijkk}^a\,,$$

where we have put

(1.14) $$h_{ijkl}^a\omega^l=dh_{ijk}^a-h_{ljk}^a\omega_i^l-h_{ilk}^a\omega_j^l-h_{ijl}^a\omega_k^l+h_{ijk}^b\omega_b^a\,.$$

The forms (ω_j^i) define the Riemannian connection of M and the forms (ω_b^a) define the connection induced in the normal bundle. If $h_{ijk}^a=0$ for all a, i, j and k, then the second fundamental form of M is said to be *parallel*.

If a Kaehler manifold $\bar{M}$ is of constant holomorphic sectional curvature c, then we have

(1.15) $$K_{BCD}^A=\frac{1}{4}c(\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC}+J_{AC}J_{BD}-J_{AD}J_{BC}+2J_{AB}J_{CD})\,.$$

We call such a manifold a complex space form and denote it by $\bar{M}(c)$. If a Riemannian manifold M is of constant curvature k, then we have

$$R^i_{jkl}=k(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}). \tag{1.16}$$

We call such a manifold a real space form and denote it by $M(k)$.

§ 2. f-structure in the normal bundle.

Let M be a totally real submanifold of real dimension n of a Kaehler manifold $\bar{M}$ of complex dimension $n+p$. We denote by $T_x(M)$ the tangent space of M at $x\in M$ and by $T_x(M)^\perp$ the normal space of M at $x\in M$. Then we see that $JT_x(M)\subset T_x(M)^\perp$. Let $N_x(M)$ be an orthogonal complement of $JT_x(M)$ in $T_x(M)^\perp$. Then we have the decomposition:

$$T_x(M)^\perp=JT_x(M)\oplus N_x(M).$$

If $N\in N_x(M)$, we obtain $JN\in N_x(M)$. If N is a vector field in the normal bundle $T(M)^\perp$, we put

$$JN=PN+fN, \tag{2.1}$$

where PN is the tangential part of JN and fN the normal part of JN. Then P is a tangent bundle valued 1-form on the normal bundle and f is an endomorphism of the normal bundle. Then, putting $N=JX$ in (2.1) and applying J to (2.1), we find [6], [10]:

$$\begin{aligned} &PfN=0, \qquad f^2N=-N-JPN,\\ &PJX=-X, \qquad fJX=0, \end{aligned} \tag{2.2}$$

where X is a tangent vector field to M and N is a vector field in the normal bundle. Equations (2.2) imply that

$$f^3+f=0.$$

Therefore, f being of constant rank, if f does not vanish, then it defines an f-structure in the normal bundle [8]. From (2.1), using the Gauss-Weingarten formulas, we have

$$-JA_NX+fD_XN=B(X, PN)+D_X(fN), \tag{2.3}$$

from which

$$(D_Xf)N=-B(X, PN)-JA_NX. \tag{2.4}$$

If $D_Xf=0$ for any tangent vector field X, then the f-structure in the normal bundle is said to be *parallel.*

LEMMA 2.1. *Let M be a totally real submanifold of real dimension n of a Kaehler manifold $\bar{M}$ of complex dimension $n+p$. If the f-structure in the normal* *bundle is parallel, then we have*

$$A_N=0 \quad \text{for} \quad N\in N_x(M). \tag{2.5}$$

Proof. If $N\in N_x(M)$, then we have $PN=0$. Thus by the assumption and (2.4) we have (2.5).

Remark. We can take a frame $e_{1^*},\cdots,e_{n^*}$ for $JT_x(M)$ and a frame $e_{n+1},\cdots,e_{n+p}, e_{(n+1)^*},\cdots,e_{(n+p)^*}$ for $N_x(M)$. Therefore if the f-structure in the normal bundle is parallel, then we have

$$A_\lambda=0, \quad \text{i.e.,} \quad h_{ij}^\lambda=0. \tag{2.6}$$

§ 3. Integral formulas.

Let $\bar{M}(c)$ be a complex space form of complex dimension $n+p$ and of constant holomorphic sectional curvature c and let M be a totally real submanifold of real dimension n of $\bar{M}(c)$.

LEMMA 3.1. *Let M be a totally real submanifold of a complex space form $\bar{M}(c)$. Then we have*

$$\begin{aligned}\sum_{a,i,j} h_{ij}^a \Delta h_{ij}^a = \sum_{a,i,j,k} h_{ij}^a h_{kkij}^a &+ \sum_a\Big[\frac{1}{4}nc\operatorname{Tr}A_a^2-\frac{1}{4}c(\operatorname{Tr}A_a)^2\Big]\\ &+\sum_t\Big[\frac{1}{4}c\operatorname{Tr}A_t^2-\frac{1}{4}c(\operatorname{Tr}A_t)^2\Big]\\ &+\sum_{a,b}\{\operatorname{Tr}(A_aA_b-A_bA_a)^2-[\operatorname{Tr}(A_aA_b)]^2-\operatorname{Tr}A_b\operatorname{Tr}(A_aA_bA_a)\}.\end{aligned} \tag{3.1}$$

where we have put $A_t=A_{t^*}$.

Proof. First of all, by a straightforward computation, we have (see [3; p. 63]):

$$\begin{aligned}\sum_{a,i,j} h_{ij}^a \Delta h_{ij}^a = \sum_{a,i,j,k}(h_{ij}^a h_{kkij}^a &- K_{ijb}^a h_{ij}^a h_{kk}^b + 4K_{bki}^a h_{jk}^b h_{ij}^a\\ &- K_{kbk}^a h_{ij}^a h_{ij}^b + 2K_{kik}^l h_{lj}^a h_{ij}^a + 2K_{ijk}^l h_{lk}^a h_{ij}^a)\\ - \sum_{a,b,i,j,k,l}[(h_{ik}^a h_{jk}^b - h_{jk}^a h_{ik}^b)&(h_{il}^a h_{jl}^b - h_{jl}^a h_{il}^b) + h_{ij}^a h_{kl}^a h_{ij}^b h_{kl}^b - h_{ji}^a h_{ki}^a h_{kj}^b h_{ll}^b].\end{aligned}$$

From this and (1.15) we have (3.1).

Using Lemma 2.1 and (3.1), we obtain the following

LEMMA 3.2. *Let M be a totally real submanifold of a complex space form $\bar{M}(c)$. If the f-structure in the normal bundle is parallel, then we have*

$$(3.2)\qquad \sum_{a,i,j} h^a_{ij}\Delta h^a_{ij}=\sum_{a,i,j,k} h^a_{ij}h^a_{kkij}+\sum_t\Big[\frac{1}{4}(n+1)c\operatorname{Tr}A_t^2-\frac{1}{4}c(\operatorname{Tr}A_t)^2\Big]$$
$$+\sum_{t,s}\{\operatorname{Tr}(A_tA_s-A_sA_t)^2-[\operatorname{Tr}(A_tA_s)]^2+\operatorname{Tr}A_s\operatorname{Tr}(A_tA_sA_t)\}\,.$$

In the sequel, we need the following lemma proved in [3].

LEMMA 3.3 ([3]). *Let A and B be symmetric (n, n)-matrices. Then*

$$-\operatorname{Tr}(AB-BA)^2\leqq 2\operatorname{Tr}A^2\operatorname{Tr}B^2\,,$$

and the equality holds for non-zero matrices A and B if and only if A and B can be transformed by an orthogonal matrix simultaneously into scalar multiples of $\bar{A}$ and $\bar{B}$ respectively, where

$$\bar{A}=\left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right),\qquad \bar{B}=\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right).$$

Moreover, if A_1, A_2, A_3 are three symmetric (n, n)-matrices such that

$$-\operatorname{Tr}(A_aA_b-A_bA_a)^2=2\operatorname{Tr}A_a^2\operatorname{Tr}A_b^2\,,\qquad 1\leqq a,b\leqq 3\,,\quad a\neq b\,,$$

then at least one of the matrices A_a must be zero.

We next put

$$S_{ab}=\sum_{i,j}h^a_{ij}h^b_{ij}=\operatorname{Tr}A_aA_b\,,\qquad S_a=S_{aa}\,,\qquad S=\sum_a S_a\,,$$

so that S_{ab} is a symmetric (n, n)-matrix and can be assumed to be diagonal for a suitable frame. S is the square of the length of the second fundamental form. When the f-structure in the normal bundle is parallel, using these notations, we can rewrite (3.2) in the following form:

$$(3.3)\qquad \sum_{a,i,j} h^a_{ij}\Delta h^a_{ij}=\sum_{a,i,j,k} h^a_{ij}h^a_{kkij}+\frac{1}{4}(n+1)cS-\sum_t S_t^2$$
$$+\sum_{t,s}\operatorname{Tr}(A_tA_s-A_sA_t)^2-\frac{1}{2}c\sum_t(\operatorname{Tr}A_t)^2+\sum_{t,s}\operatorname{Tr}A_s\operatorname{Tr}(A_tA_sA_t)\,.$$

On the other hand, using Lemma 3.3, we have

$$(3.4)\qquad -\sum_{t,s}\operatorname{Tr}(A_tA_s-A_sA_t)^2+\sum_t S_t^2-\frac{1}{4}(n+1)cS$$
$$\leqq 2\sum_{t\neq s}S_tS_s+\sum_t S_t^2-\frac{1}{4}(n+1)cS$$
$$=\Big[\Big(2-\frac{1}{n}\Big)S-\frac{1}{4}(n+1)c\Big]S-\frac{1}{n}\sum_{t>s}(S_t-S_s)^2\,.$$

From (3.3) and (3.4) we find

$$-\sum_{a,i,j} h_{ij}^a h_{ij}^a \leqq W - \sum_{a,i,j,k} h_{ij}^a h_{kkij}^a , \tag{3.5}$$

where we have put

$$W=\Big[\Big(2-\frac{1}{n}\Big)S-\frac{1}{4}(n+1)c\Big]S \tag{3.6}$$
$$+\frac{1}{2}c\sum_t(\operatorname{Tr}A_t)^2-\sum_{t,s}\operatorname{Tr}A_s\operatorname{Tr}(A_tA_sA_t)\,.$$

Now assume that M is compact and orientable, then we have the integral formulas (cf. [5]):

$$\int_M \sum_{a,i,j,k}(h_{ijk}^a)^2*1=-\int_M \sum_{a,i,j} h_{ij}^a \Delta h_{ij}^a*1\,,$$

$$\int_M \sum_{a,i,j,k} h_{ij}^a h_{kkij}^a*1=\int_M \sum_a(\operatorname{Tr}A_a)\Delta(\operatorname{Tr}A_a)*1\,.$$

Inequality (3.5) and these integral formular imply the following

THEOREM 3.1. *Let M be a compact orientable totally real submanifold of a complex space form $\bar{M}(c)$. If the f-structure in the normal bundle is parallel, then*

$$\int_M[W-\sum_a(\operatorname{Tr}A_a)\Delta(\operatorname{Tr}A_a)]*1\geqq\int_M \sum_{a,i,j,k}(h_{ijk}^a)^2*1\geqq 0\,. \tag{3.7}$$

THEOREM 3.2. *Let M be a compact orientable totally real minimal submanifold of a complex space form $\bar{M}(c)$. If the f-structure in the normal bundle is parallel, then*

$$\int_M\Big[\Big(2-\frac{1}{n}\Big)S-\frac{1}{4}(n+1)c\Big]S*1\geqq\int_M \sum_{a,i,j,k}(h_{ijk}^a)^2*1\geqq 0\,. \tag{3.8}$$

COROLLARY 3.1. *Let M be a compact orientable totally real minimal submanifold of real dimension n of a complex space form $\bar{M}(c)$ of complex dimension $n+p$. If the f-structure in the normal bundle is parallel and if $S<n(n+1)c/4(2n-1)$, then M is totally geodesic.*

Let CP^{n+p} be a complex projective space of constant holomorphic sectional curvature 4 and of complex dimension $n+p$. We would like to study a compact orientable totally real submanifold M of real dimension n of CP^{n+p} such that the f-structure in the normal bundle is parallel and satisfies

$$\int_M[W-\sum_a(\operatorname{Tr}A_a)\Delta(\operatorname{Tr}A_a)]*1=0\,. \tag{3.9}$$

In the following we assume that M is not totally geodesic. From (3.7) and (3.9) the second fundamental form of M is parallel, i.e., $h_{ijk}^a=0$. Then (3.3), (3.4) and (3.5) imply

(3.10) $$\sum_{t>s}(S_t-S_s)^2=0\,,$$

(3.11) $$-\mathrm{Tr}(A_tA_s-A_sA_t)^2=2\,\mathrm{Tr}\,A_t^2\,\mathrm{Tr}\,A_s^2$$

for any $t\neq s$. By Lemma 3.3 we may assume that $A_t=0$ for $t=3,\cdots,n$, which means that $S_t=0$ for $t=3,\cdots,n$. On the other hand, we have $S_t=S_s$ for any t, s by (3.10). Therefore, using Lemma 3.3, we can assume that

(3.12) $$A_1=\lambda\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\qquad A_2=\mu\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Consequently M is minimally immersed in CP^{2+p}. Since $h_{ij}^{\lambda}=0$, (1.7) implies that

(3.13) $$\omega_i^{\lambda}=0\,.$$

From (1.12) we also have the following

(3.14) $$dh_{ij}^a=h_{il}^a\omega_j^l+h_{lj}^a\omega_i^l-h_{ij}^b\omega_b^a\,.$$

From (3.14) we have $h_{ij}^{k^*}\omega_{k^*}^{\lambda}=0$, which implies that

(3.15) $$\omega_{i^*}^{\lambda}=0\,.$$

Setting $a=1^*$, $i=1$ and $j=2$ in (3.14), we see that $d\lambda=dh_{12}^{1^*}=0$, which means that λ is constant. Similarly, setting $a=2^*$ and $i=j=1$, we see that μ is also constant and by (3.10) we get $\lambda^2=\mu^2$. Since M is not totally geodesic, $\lambda\neq0$. This shows that

(3.16) $$\omega_i^{t^*}\neq0\,,\qquad t=1,2\,.$$

From (3.13), (3.15) and (3.16) we can consider a distribution L defined by

$$\omega^{\lambda}=0\,,\qquad \omega_i^{\lambda}=0\,,\qquad \omega_{i^*}^{\lambda}=0\,.$$

Then it easily follows from the structure equations that

$$d\omega^{\lambda}=0\,,\qquad d\omega_i^{\lambda}=0\,,\qquad d\omega_{i^*}^{\lambda}=0\,.$$

Therefore the distribution L is a 4-dimensional completely integrable distribution. We consider the maximal integral submanifold $\bar{M}(x)$ of L through $x\in M$. Then $\bar{M}(x)$ is of dimension 4 and by construction it is totally geodesic and is a complex submanifold in CP^{2+p}. Moreover M is immersed in $\bar{M}(x)$. Thus we can consider that M is minimally immersed in CP^2. From these considerations, combined with the theorems of [5], [7], we have the following

THEOREM 3.3. *Let M be an n-dimensional compact orientable totally real submanifold of a complex projective space, CP^{n+p} $(n>1)$ and suppose that M is not totally geodesic but satisfies the condition* (3.9). *If the f-structure in the normal bundle is parallel, then M is $S^1\times S^1$ in some CP^2 in CP^{2+p}.*

THEOREM 3.4. *Let M be an n-dimensional compact orientable totally real* *minimal submanifold of a complex projective space* CP^{n+p} $(n>1)$ *with* $S=n(n+1)/(2n-1)$. *If the f-structure in the normal bundle is parallel, then M is* $S^1\times S^1$ *in some* CP^2 *in* CP^{2+p}.

§4. Totally real submanifolds with commutative second fundamental form.

Let M be a real n-dimensional totally real submanifold of a complex $(n+p)$-dimensional Kaehler manifold $\bar{M}$. If the second fundamental form of M satisfies $A_aA_b=A_bA_a$ for all a and b, then the second fundamental form of M is said to be commutative, which is equivalent to that $\sum_j h_{ij}^a h_{jk}^b=\sum_j h_{jk}^a h_{ij}^b$ for all a, b, i and k.

LEMMA 4.1 ([10]). *Let M be a real n-dimensional totally real submanifold of a complex (n+p)-dimensional Kaehler manifold* $\bar{M}$. *If the f-structure in the normal bundle is parallel, then M is flat if and only if the normal connection of M is flat, i.e.,* $R_{bkl}^a=0$.

Proof. From Lemma 2.1, we get $h_{ij}^\lambda=0$, which shows that $\omega_i^\lambda=0$. Then (1.2), (1.6) and (1.9) imply

$$\Omega_{j^*}^{i^*}=d\omega_{j^*}^{i^*}+\omega_a^{i^*}\wedge\omega_{j^*}^a=d\omega_j^i+\omega_k^i\wedge\omega_j^k=\Omega_j^i\,,$$

$$\Omega_\lambda^{i^*}=0\,,\qquad \Omega_\mu^\lambda=0\,,$$

which show that $R_{j^*kl}^{i^*}=R_{jkl}^i$ and $R_{\mu kl}^\lambda=R_{j^*kl}^\lambda=R_{\mu kl}^{i^*}=0$. Thus we have our assertion.

LEMMA 4.2. *Let M be a real n-dimensional totally real submanifold of a complex (n+p)-dimensional Kaehler manifold* $\bar{M}$. *If the second fundamental form of M is commutative and if the f-structure in the normal bundle is parallel, then we can choose an orthonormal frame for which* A_a *is of the form*

$$A_t=\begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \lambda_t & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix} t\,,\qquad t=1,\cdots,n\,,\qquad A_\lambda=0\,,$$

i.e., $h_{ij}^\lambda=0$ *and* $h_{ij}^{t^*}=0$ *unless* $t=i=j$.

Proof. By the assumption we have $h_{ij}^\lambda=0$. If $A_aA_b=A_bA_a$, we can choose an orthonormal frame $e_1,\cdots,e_n$ for $T_x(M)$ in such a way that all A_a's are simultaneously diagonal, i.e., $h_{ij}^a=0$ when $i\neq j$, that is, $h_{ij}^{t^*}=0$ when $i\neq j$. From (1.11) we see that $h_{ij}^{t^*}=0$ unless $t=i=j$.

COROLLARY 4.1. *Let M be a real n-dimensional totally real minimal submanifold of a complex $(n+p)$-dimensional Kaehler manifold $\bar{M}$ with commutative second fundamental form. If the f-structure in the normal bundle is parallel, then M is totally geodesic.*

Proof. From Lemma 4.2 we have $\lambda_i=0$ for all i, by the fact that $\mathrm{Tr}\, A_i=0$. On the other hand, we have already $A_\lambda=0$. Thus M is totally geodesic.

COROLLARY 4.2. *Let M be a real n-dimensional $(n>1)$ totally real, totally umbilical submanifold of a complex $(n+p)$-dimensional Kaehler manifold $\bar{M}$. If the f-structure in the normal bundle is parallel, then M is totally geodesic.*

Proof. Since M is umbilical, we have $h_{ij}^{k^*}=\delta_{ij}(\mathrm{Tr}\, A_k)/n$ and $A_\lambda=0$ by Lemma 2.1. Therefore the second fundamental form of M is commutative. Thus Lemma 4.2 implies that $h_{ij}^{k^*}=0$ unless $i=j=k$. On the other hand, we have $h_{ij}^{k^*}=\lambda_k\delta_{ij}/n$. Setting $i=j\neq k$, we have $\lambda_k=0$ and hence M is totally geodesic.

LEMMA 4.3. *Let M be a real n-dimensional totally real submanifold of a complex space form $\bar{M}^{n+p}(c)$ with parallel f-structure in the normal bundle. Then M is a real space form of constant curvature $(1/4)c$ if and only if the second fundamental form of M is commutative.*

Proof. First of all, we have $h_{ij}^\lambda=0$ by Lemma 2.1. Then (1.8), (1.11) and (1.15) imply

$$R^i_{jkl}=K^i_{jkl}+\sum_a (h^a_{ik}h^a_{jl}-h^a_{il}h^a_{jk})$$
$$=\frac{1}{4}c(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+\sum_t (h^{t^*}_{ik}h^{t^*}_{jl}-h^{t^*}_{il}h^{t^*}_{jk})\ ,$$

which proves our assertion.

LEMMA 4.4. *Let M be a real n-dimensional totally real submanifold of a complex $(n+p)$-dimensional Kaehler manifold $\bar{M}$. Then we have*

$$\sum_{t,s}\mathrm{Tr}\, A_t^2A_s^2=\sum_{t,s}(\mathrm{Tr}\, A_tA_s)^2\,. \tag{4.1}$$

Proof. Since $h^{t^*}_{ij}=h^{i^*}_{tj}$, we have

$$\sum_{t,s}\mathrm{Tr}\, A_t^2A_s^2=\sum_{t,s,i,j,k,l} h^{t^*}_{kl}h^{t^*}_{li}h^{s^*}_{ij}h^{s^*}_{jk}$$
$$=\sum_{t,s,i,j,k,l} h^{k^*}_{tl}h^{i^*}_{lt}h^{i^*}_{sj}h^{k^*}_{js}=\sum_{k,i}(\mathrm{Tr}\, A_kA_i)^2\,.$$

LEMMA 4.5. *Let M be a real n-dimensional totally real submanifold with constant curvature k of a complex space form $\bar{M}^{n+p}(c)$. If the f-structure in the normal bundle is parallel, then we have*

$$\left(\frac{1}{4}c-k\right)\sum_t[\mathrm{Tr}\, A_t^2-(\mathrm{Tr}\, A_t)^2]=\sum_{t,s}[\mathrm{Tr}\, A_t^2A_s^2-\mathrm{Tr}\,(A_tA_s)^2]\,. \tag{4.2}$$

Proof. From (1.8), (1.15) and (1.16) we have

$$(4.3)\qquad \left(\frac{1}{4}c-k\right)(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})=\sum_t (h_{il}^{t^*}h_{jk}^{t^*}-h_{ik}^{t^*}h_{jl}^{t^*}),$$

where we have used the fact that $h_{ij}^{\lambda}=0$ as is seen from Lemma 2.1. Multiplying the both sides of (4.3) by $\sum_s h_{il}^{s^*}h_{jk}^{s^*}$ and summing up with respect to i, j, k and l we have (4.2) by using (4.1).

LEMMA 4.6. *Let M be a real n-dimensional totally real submanifold with constant curvature k of a complex space from $\bar{M}^{n+p}(c)$. If the f-structure in the normal bundle is parallel, then we have*

$$(4.4)\qquad (n-1)\left(\frac{1}{4}c-k\right)\sum_t \operatorname{Tr} A_t^2=\sum_{t,s}[\operatorname{Tr} A_t^2A_s^2-\operatorname{Tr} A_s \operatorname{Tr}(A_tA_sA_t)].$$

Proof. From Lemma 2.1 and (4.3) we have

$$(4.5)\qquad (n-1)\left(\frac{1}{4}c-k\right)\delta_{jl}=\sum_{t,i}(h_{il}^{t^*}h_{ij}^{t^*}-h_{ii}^{t^*}h_{jl}^{t^*}).$$

Multiplying the both sides of (4.5) by $\sum_s h_{jk}^{s^*}h_{kl}^{s^*}$ and summing up with respect to i, k and l we obtain (4.4).

§5. Totally real submanifolds of constant curvature.

PROPOSITION 5.1. *Let M be a real n-dimensional totally real submanifold of a complex space form $\bar{M}^{n+p}(c)$ with parallel mean curvature vector. If M is of constant curvature k and if the f-structure in the normal bundle is parallel, then*

$$(5.1)\qquad \sum_{a,i,j,k}(h_{ijk}^a)^2=-k\sum_t[(n+1)\operatorname{Tr} A_t^2-2(\operatorname{Tr} A_t)^2].$$

Proof. By the assumption we see that $\sum_a \operatorname{Tr} A_a^2$ is constant. Thus we have

$$\sum_{a,i,j} h_{ij}^a \Delta h_{ij}^a=\frac{1}{2}\Delta\sum_a \operatorname{Tr} A_a^2-\sum_{a,i,j,k}(h_{ijk}^a)^2=-\sum_{a,i,j,k}(h_{ijk}^a)^2.$$

Therefore (3.2) becomes

$$(5.2)\qquad \sum_{a,i,j,k}(h_{ijk}^a)^2=-\sum_t\left[\frac{1}{4}(n+1)c\operatorname{Tr} A_t^2-\frac{1}{2}c(\operatorname{Tr} A_t)^2\right]$$
$$-\sum_{t,s}\{\operatorname{Tr}(A_tA_s-A_sA_t)^2-[\operatorname{Tr}(A_tA_s)]^2+\operatorname{Tr} A_s\operatorname{Tr}(A_tA_sA_t)\}.$$

Substituting (4.2) and (4.4) into (5.2) and using (4.1) we have (5.1).

PROPOSITION 5.2. *Let M be a real n-dimensional totally real submanifold of a complex space form $\bar{M}^{n+p}(c)$ $(n>1)$ and M be with parallel mean curvature vector and of constant curvature k. If $\frac{1}{4}c\geqq k$ and if the f-structure in the* *normal bundle is parallel, then $k\leqq 0$ or M is totally geodesic $\left(\frac{1}{4}c=k\right)$.*

Proof. From (4.5) we have

$$\left(\frac{1}{4}c-k\right)n(n-1)=\sum_t[\mathrm{Tr}\,A_t^2-(\mathrm{Tr}\,A_t)^2].$$

Since $\frac{1}{4}c\geqq k$, we have

$$\sum_t \mathrm{Tr}\,A_t^2\geqq\sum_t(\mathrm{Tr}\,A_t)^2. \tag{5.3}$$

If $k>0$, (5.1) implies that

$$0=\sum_t\{(n-1)\mathrm{Tr}\,A_t^2+2[\mathrm{Tr}\,A_t^2-(\mathrm{Tr}\,A_t)^2]\},$$

which implies that $\sum \mathrm{Tr}\,A_t^2=0$ and hence that M is totally geodesic. Except for this possibility we have $k\leqq 0$.

PROPOSITION 5.3. *Let M be a real n-dimensional totally real submanifold of a complex space form $\bar{M}^{n+p}(c)$ $(n>1)$ and M be with parallel second fundamental form and of constant curvature k. If $\frac{1}{4}c\geqq k$ and if the f-structure in the normal bundle is parallel, then either M is totally geodesic $\left(\frac{1}{4}c=k\right)$ or flat $(k=0)$.*

COROLLARY 5.1. *Let M be a real n-dimensional totally real minimal submanifold with constant curvature k of a complex space form $\bar{M}^{n+p}(c)$. If the f-structure in the normal bundle is parallel, then either $k\leqq 0$ or M is totally geodesic.*

COROLLARY 5.2. *Let M be a real n-dimensional totally real minimal submanifold of a complex space form $\bar{M}^{n+p}(c)$ and M be with constant curvature k and parallel second fundamental form. If the f-structure in the normal bundle is parallel, then either M is totally geodesic or flat.*

PROPOSITION 5.4. *Let M be a real n-dimensional totally real submanifold with parallel mean curvature vector of a complex space form $\bar{M}^{n+p}(c)$. If the second fundamental form of M is commutative and if the f-structure in the normal bundle is parallel, then we have*

$$\sum_{a,i,j,k}(h_{ijk}^a)^2=-\frac{1}{4}c(n-1)\sum_t \mathrm{Tr}\,A_t^2. \tag{5.4}$$

Proof. Using Lemma 4.2 and Lemma 4.3, we can transform (5.1) into (5.4).

PROPOSITION 5.5. *Let M be a real n-dimensional totally real submanifold of a complex space form $\bar{M}^{n+p}(c)$ $(n>1)$ and M be with parallel mean curvature vector and with commutative second fundamental form. If the f-structure in the normal bundle is parallel, then either M is totally geodesic or $c\leqq 0$.*

PROPOSITION 5.6. *Let M be a real n-dimensional totally real submanifold of a complex space form $\bar{M}^{n+p}(c)$ $(n>1)$ and M be with parallel and commutative* *second fundamental form. If the f-structure in the normal bundle is parallel, then M is either totally geodesic or flat.*

Proof. By the assumption and Lemma 4.3, M is of constant curvature $\frac{1}{4}c$. On the other hand, by (5.4), M is totally geodesic or $c=0$ in which case M is flat.

PROPOSITION 5.7. *Let M be a real n-dimensional flat totally real submanifold with parallel mean curvature vector of a complex $(n+p)$-dimensional flat Kaehler manifold $\bar{M}$. If the f-structure in the normal bundle is parallel, then the second fundamental form of M is parallel.*

Proof. From Lemma 4.3 and (5.4) we have our assertion.

§ 6. Flat totally real submanifolds.

A simply connected complete Kaehler manifold of constant holomorphic sectional curvature c and of complex dimension n can be identified with the complex projective space CP^n, the open unit ball D^n in C^n or C^n according as $c>0$, $c<0$ or $c=0$. In [12] we gave an example of a flat totally real submanifold of C^n, that is, we showed that $S^1(r_1)\times S^1(r_2)\times\cdots\times S^1(r_n)$ is a flat totally real submanifold in C^n, where we put $S^1(r_i)=\{z_i\in C: |z_i|^2=r_i^2\}$, $i=1,\cdots,n$. Moreover an n-dimensional plane R^n is a totally real, totally geodesic submanifold in C^n and a pythagorean product $S^1(r_1)\times\cdots\times S^1(r_p)\times R^{n-p}$ is also a flat totally real submanifold of C^n where R^{n-p} denotes an $(n-p)$-dimensional $(p\geqq 1)$ plane.

THEOREM 6.1. *Let M be a real n-dimensional complete totally real submanifold of C^{n+p} $(n>1)$ and M be with parallel mean curvature vector and commutative second fundamental form. If the f-structure in the normal bundle is parallel, then M is an n-dimensional plane R^n in some C^n in C^{n+p}, a pythagorean product of the form*

$$S^1(r_1)\times S^1(r_2)\times\cdots\times S^1(r_n) \text{ in some } C^n \text{ in } C^{n+p},$$

or a pythagorean product of the form

$$S^1(r_1)\times S^1(r_2)\times\cdots\times S^1(r_m)\times R^{n-m} \text{ in some } C^n \text{ in } C^{n+p},$$

where R^{n-m} is an $(n-m)$-dimensional plane and $n>m$, $m\geqq 1$.

Proof. By the assumption and Lemma 4.3, M is flat. Thus Proposition 5.7 shows that the second fundamental form of M is parallel. Moreover, by using Lemma 4.1, we see that the normal connection of M is flat. From Lemma 2.9 of Yano-Ishihara [11], M is immersed in some C^n in C^{n+p}. Then Theorem 3.1 in [11] proves our statement.

THEOREM 6.2. *Let M be a real n-dimensional complete totally real submanifold of a simply connected complete complex space form $\bar{M}^{n+p}(c)$ $(n>1)$ and*

M be with parallel and commutative second fundamental form. If M is not totally geodesic and if the f-structure in the normal bundle is parallel, then M is a pythagorean product of the form

$$S^1(r_1)\times S^1(r_2)\times\cdots\times S^1(r_n) \text{ in some } C^n \text{ in } C^{n+p},$$

or a pythagorean product of the form

$$S^1(r_1)\times S^1(r_2)\times\cdots\times S^1(r_m)\times R^{n-m} \text{ in some } C^n \text{ in } C^{n+p},$$

where $n>m$ and $m\geqq 1$.

Proof. By the assumption and Proposition 5.6, we have $c=0$. In this case we may consider that the ambient space $\bar{M}$ is C^{n+p}. Then Theorem 6.2 follows from Theorem 6.1.

COROLLARY 6.1. *Under the same assumption as in Theorem* 6.1, *if M is compact, then M is a pythagorean product of the form*

$$S^1(r_1)\times S^1(r_2)\times\cdots\times S^1(r_n) \text{ in some } C^n \text{ in } C^{n+p}.$$

COROLLARY 6.2. *Under the same situation as in Theorem* 6.2, *if M is compact, then M is a pythagorean product of the form*

$$S^1(r_1)\times S^1(r_2)\times\cdots\times S^1(r_n) \text{ in some } C^n \text{ in } C^{n+p}.$$

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