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SOME CONDITIONS FOR CONSTANCY OF THE HOLOMORPHIC SECTIONAL CURVATURE

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1. Introduction. Let M be a Kaehler manifold with complex structure J and Riemann metric $\langle\ ,\ \rangle$. By a plane section σ we mean a 2-dimensional linear subspace of a tangent space of M. If σ is invariant under the action of J, it is called *holomorphic section*. If σ is perpendicular to $J\sigma$, it is called *anti-holomorphic section*. A Kaehler manifold of constant holomorphic sectional curvature is called a *complex space form*. Then the following result is known.

THEOREM A (B. Y. Chen and K. Ogiue). *Let M be a Kaehler manifold. If the anti-holomorphic sectional curvatures of M are constant and if* $\dim M \geqq 3$, *then M is a complex space form.*

In Section 2, we shall define the angle between two linear subspaces of a vector space. Then we shall call a plane section σ a *θ-holomorphic section* if the angle between σ and $J\sigma$ is θ and the sectional curvature for a θ-holomorphic section the *θ-holomorphic sectional curvature.* It is clear that a complex space form has constant θ-holomorphic sectional curvatures. Conversely, in Section 3, we shall prove

THEOREM 3. *Let M be a Kaehler manifold. If the θ-holomorphic sectional curvatures of M are constant and if* $\cos\theta \neq 0$, *then M is a complex space form.*

Next, the holomorphic bisectional curvature $H(\sigma, \sigma')$ for holomorphic planes σ and σ' is defined by S. I. Goldberg and S. Kobayashi ([2]) as $H(\sigma, \sigma') = \langle R(X, JX)JY, Y\rangle$, where R is the curvature tensor of M and X, Y are unit vector in σ and σ' respectively. We shall call $H(\sigma, \sigma')$ a *holomorphic τ-bisectional curvature* if the angle between σ and σ' is τ. Then it is clear that a complex space form has constant holomorphic τ-bisectional curvatures. We shall prove

THEOREM 4. *Let M be a Kaehler manifold. If the holomorphic τ-bisectional curvatures of M are canstant and if $\tau \neq \frac{\pi}{2}$, then M is a complex space form.*

THEOREM 5. *Let M be a Kaehler manifold. If the holomorphic $\frac{\pi}{2}$-bisectional curvatures of M are constant and if* $\dim M \geqq 3$, *then M is a complex space form.*

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2. The angle between two subspaces. Throughout this section, V denotes an n-dimensional real vector space with inner product $\langle\ ,\ \rangle$. Consider two arbitrary k-dimensional subspaces S and T of V. We are now going to define inductively sets

$$(*)\qquad \Theta_m=\{\theta_1,\cdots,\theta_m\,;\ X_1,\cdots,X_m\,;\ Y_1,\cdots,Y_m\}\,,\qquad (1\leqq m\leqq k)\,.$$

where $0\leqq\theta_1\leqq\cdots\leqq\theta_m\leqq\frac{\pi}{2}$, and where $X_1,\cdots,X_m$ and $Y_1,\cdots,Y_m$ are orthonormal respectively in S and T. First we put

$$\theta_1=\inf\{\sphericalangle(X,Y)\,;\ X\in S,\ Y\in T,\ X\neq 0,\ Y\neq 0\}\,,$$

where $\sphericalangle(X,Y)$ denotes the angle between X and Y. Then we can take unit vectors X_1 and Y_1 such that $X_1\in S$ and $Y_1\in T$ with $\sphericalangle(X_1,Y_1)=\theta_1$. Next, assuming that Θ_m $(1\leqq m<k)$ is already defined, we put

$$S_m=\{X\in S\,;\ \langle X,X_i\rangle=0,\ (i=1,\cdots,m)\}\,,$$

$$T_m=\{Y\in T\,;\ \langle Y,Y_i\rangle=0,\ (i=1,\cdots,m)\}$$

and define θ_{m+1} by

$$\theta_{m+1}=\inf\{\sphericalangle(X,Y)\,;\ X\in S_m,\ Y\in T_m\}\,.$$

Then we can take unit vectors $X_{m+1}\in S$, $Y_{m+1}\in T$ such that $\sphericalangle(X_{m+1},Y_{m+1})=\theta_{m+1}$. So, we now have a set $\Theta_{m+1}=\{\theta_1,\cdots,\theta_{m+1}\,;\ X_1,\cdots,X_{m+1}\,;\ Y_1,\cdots,Y_{m+1}\}$. Therefore we can define inductively sets Θ_m $(1\leqq m\leqq k)$. We can now prove that the set $\{X\in S\,;$ there exists $Y\in T$ such that $\sphericalangle(X,Y)=\theta\}\cup\{0\}$ is a vector subspace of S and hence the set $\{\theta_1,\cdots,\theta_k\}$ is independent of the choice of $X_1,\cdots,X_k\in S$ and $Y_1,\cdots,Y_k\in T$.

DEFINITION. The set $(\theta_1,\cdots,\theta_k)$ is called the angle between S and T. It is denoted by $\sphericalangle(S,T)=(\theta_1,\cdots,\theta_k)$.

LEMMA. *Let S, S', T, T' be k-dimensional subspaces of V such that $\sphericalangle(S,T)=\sphericalangle(S',T')=(\theta_1,\cdots,\theta_k)$. Then there exists an orthogonal transformation u of V such that $u(S)=S'$ and $u(T)=T'$.*

Now, we assume that V has a complex structure J and an hermitian metric $\langle\ ,\ \rangle$ in the sequel and in Theorems 1 and 2.

THEOREM 1. *Let S and T be two $2m$-dimensional subspaces of V. Then*

$$\sphericalangle(S,T)=(\theta_1,\theta_1,\theta_2,\theta_2,\cdots,\theta_m,\theta_m)\,.$$

THEOREM 2. *If S is a k-dimensional subspace of V, then we have*

(i) $\sphericalangle(S,JS)=\left(\theta_1,\theta_1,\cdots,\theta_m,\theta_m,\frac{\pi}{2}\right)$ *for* $k=2m+1$,

(ii) $\sphericalangle(S,JS)=(\theta_1,\theta_1,\cdots,\theta_m,\theta_m)$ *for* $k=2m$.

Since Theorems 1 and 2 are established, we can put

$$\sphericalangle(S, T)=(\theta_1, \cdots, \theta_m) \quad \text{and} \quad \sphericalangle(S, JS)=(\theta_1, \cdots, \theta_m).$$

If σ, σ' are J-invariant plane sections, we have $\sphericalangle(\sigma, \sigma')=(\theta_1)=\theta_1$ where $\theta_1=\inf\{\sphericalangle(X, X'); X\in\sigma, X'\in\sigma'\}$, and $\sphericalangle(\sigma, J\sigma)=(\tau_1)=\tau_1$ where $\tau_1=\inf\{\sphericalangle(X, X'); X\in\sigma, X'\in J\sigma\}$.

3. Proof of Theorem 3. Throughout this section, we denote a plane section spanned by orthonormal vectors X and Y by $\{X, Y\}$. Let X and Y be orthonormal vectors which span a θ-holomorphic section such that $\cos\theta=\langle JX, Y\rangle$. We put $\alpha=\cos\theta$ $(=\langle JX, Y\rangle)$ and $\lambda=K(X, Y)$. Then, since $\{X, -Y+2\alpha JX\}$ is a θ-holomorphic section, we have

$$\begin{aligned}\lambda&=\langle R(X, -Y+2\alpha JX)(-Y+2\alpha JX), X\rangle\\&=K(X, Y)-4\alpha\langle R(X, Y)JX, X\rangle+4\alpha^2H(X), \quad \text{i.e.,}\end{aligned}$$

$$\langle R(X, Y)JX, X\rangle=\alpha H(X). \tag{1}$$

Similarly we have

$$\langle R(X, Y)JY, Y\rangle=\alpha H(Y). \tag{2}$$

Also, since $\left\{\frac{\alpha}{\sqrt{1-\alpha^2}}X+\frac{1}{\sqrt{1-\alpha^2}}JY, \frac{1}{\sqrt{1-\alpha^2}}JX-\frac{\alpha}{\sqrt{1-\alpha^2}}Y\right\}$ is a θ-holomorphic section, we have

$$\begin{aligned}\lambda&=\frac{1}{(1-\alpha^2)^2}\langle R(\alpha X+JY, JX-\alpha Y)(JX-\alpha Y), \alpha X+JY\rangle\\&=\frac{2}{(1-\alpha^2)^2}\{\alpha^2(H(X)+H(Y))+(\alpha^4+2\alpha^2+1)\lambda\\&\quad-2\alpha(\alpha^2+1)(\langle R(X, Y)JX, X\rangle+\langle R(X, Y)JY, Y\rangle)-2\alpha^2\langle R(X, JX)Y, JY\rangle\}.\end{aligned}$$

Hence, substituting (1) and (2) into the above equation, we have

$$\langle R(X, JX)Y, JY\rangle=2\lambda-\frac{1}{2}(2\alpha^2+1)(H(X)+H(Y)). \tag{3}$$

Moreover, since $\{\alpha X-\alpha JX+JY, X\}$ and $\{X, \alpha X+\alpha JX+JY\}$ are θ-holomorphic sections, we have

$$\begin{aligned}\lambda&=\langle R(\alpha X-\alpha JX+JY, X)X, \alpha X-\alpha JX+JY\rangle\\&=\alpha^2H(X)-2\alpha\langle R(X, JX)JX, Y\rangle+\langle R(X, JY)JY, X\rangle,\\\lambda&=\langle R(X, \alpha X+\alpha JX+JY)(\alpha X+\alpha JX+JY), X\rangle\\&=\alpha^2H(X)+2\alpha\langle R(X, JX)JX, Y\rangle+\langle R(X, JY)JY, X\rangle,\end{aligned}$$

from which

$$\langle R(X, JY)JY, X\rangle=\lambda-\alpha^2H(X)\,. \tag{4}$$

Similarly we have $\langle R(Y, JX)JX, Y\rangle=\lambda-\alpha^2H(Y)$. This, together with (4), implies $H(X)=H(Y)$. Consequently (3) reduces to

$$\langle R(X, JX)Y, JY\rangle=2\lambda-(2\alpha^2+1)H(X)\,. \tag{5}$$

Therefore, substituting (4) and (5) into the first Bianchi's identity, we have

$$2\lambda-(2\alpha^2+1)H(X)+\lambda-\alpha^2H(X)+\lambda=0\,, \quad \text{i.e.,}$$

$$H(X)=\frac{4\lambda}{3\alpha^2+1}\,.$$

Thus M is a space of constant holomorphic sectional curvature. q. e. d.

4. Proof of Theorem 4 and Theorem 5. Throughout this section, we denote a holomorphic section determined by a unit vector X by $\{X\}$ and the angle between σ and σ' by $\measuredangle(\sigma, \sigma')$. Let X and Y be orthonormal vectors such that $\measuredangle(\{X\}, \{Y\})=\tau$. Then we may set $\cos\tau=\langle JX, Y\rangle$.

Proof of Theorem 4. We put $\beta=\cos\tau$ $(=\langle JX, Y\rangle)$ and $\mu=H(\sigma, \sigma')$, where $\measuredangle(\sigma, \sigma')=\tau$. Then, since $\measuredangle(\{X\}, \{-Y+2\beta JX\})=\tau$, we have

$$\begin{aligned}\mu&=\langle R(X, JX)(-JY-2\beta X), -Y+2\beta JX\rangle\\ &=\langle R(X, JX)JY, Y\rangle-4\beta\langle R(X, Y)JX, X\rangle+4\beta^2H(X)\,, \quad \text{i.e.,}\end{aligned}$$

$$\langle R(X, Y)JX, X\rangle=\beta H(X)\,. \tag{6}$$

Similarly we have

$$\langle R(X, Y)JY, Y\rangle=\beta H(Y)\,. \tag{7}$$

Also, since $\measuredangle\left(\left\{\frac{\beta}{\sqrt{1-\beta^2}}X+\frac{1}{\sqrt{1-\beta^2}}JY\right\}, \left\{\frac{1}{\sqrt{1-\beta^2}}JX-\frac{\beta}{\sqrt{1-\beta^2}}Y\right\}\right)=\tau$, we have by using (6) and (7)

$$\begin{aligned}\mu&=\frac{1}{(1-\beta^2)^2}\langle R(\beta X+JY, \beta JX-Y)(-X-\beta JY), JX-\beta Y\rangle\\ &=\frac{1}{(1-\beta^2)^2}\{-\beta^2(1+2\beta^2)(H(X)+H(Y))+(\beta^4+1)\mu+4\beta^2K(X, Y)\}\,,\end{aligned}$$

from which

$$K(X, Y)=\frac{2\beta^2+1}{4}(H(X)+H(Y))-\frac{1}{2}\mu\,. \tag{8}$$

Similarly, since $\sphericalangle\left(\left\{\frac{1}{\sqrt{2}}Y-\frac{1}{\sqrt{2}}(\sqrt{1+\beta^2}+\beta)JX\right\}, \left\{\frac{1}{\sqrt{2}}X-\frac{1}{\sqrt{2}}(\sqrt{1+\beta^2}-\beta)JY\right\}\right)$ $=\tau$, we can derive

$$K(X, Y)=\frac{(\sqrt{1+\beta^2}+\beta)^2}{4}H(X)+\frac{(\sqrt{1+\beta^2}-\beta)^2}{4}H(Y)-\frac{1}{2}\mu. \tag{9}$$

This, together with (8), implies $H(X)=H(Y)$. Consequently (8) or (9) reduces to

$$K(X, Y)=\frac{2\beta^2+1}{2}H(X)-\frac{1}{2}\mu. \tag{10}$$

Again, noticing that $\sphericalangle(\{\beta X-\beta JX+JY\}, \{X\})=\tau$, we have by using (6)

$$\mu=\langle R(\beta X-\beta JX+JY, \beta JX+\beta X-Y)JX, X\rangle=2\beta\langle R(JX, Y)JX, X\rangle+\mu,$$

from which

$$\langle R(JX, Y)JX, X\rangle=0. \tag{11}$$

Similarly, we have

$$\langle R(JY, X)JY, Y\rangle=0. \tag{12}$$

Hence, since $\sphericalangle(\{2\beta X+JY\}, \{(1-2\beta^2)X-\beta Y+2\beta^2 JX-\beta JY\})=\tau$, we can derive, using (6), (7), (8), (9), (11) and (12),

$$\langle R(X, Y)JX, Y\rangle=0. \tag{13}$$

Finally, since $\sphericalangle\left(\left\{\frac{1}{2}Y-\beta JX-\frac{\sqrt{3}}{2}JY\right\}, \left\{\left(1+\frac{\beta^2}{2}\right)X+\frac{\sqrt{3}}{2}\beta^2 JX+\beta JY\right\}\right)=\tau$, using (6), (7), (8), (9), (11), (12), (13) and the first Bianchi identity, we can derive

$$H(X)=\frac{2\mu}{\beta^2+1}.$$

Thus M is a space of constant holomorphic sectional curvature. q. e. d.

Proof of Theorem 5. We put $\nu=H(\sigma, \sigma')$, where $\sphericalangle(\sigma, \sigma')=\frac{\pi}{2}$. Let X, Y and JY be orthonormal vectors. Then, since $\sphericalangle\left(\left\{\frac{1}{\sqrt{2}}(X+Y)\right\}, \left\{\frac{1}{\sqrt{2}}(JX-JY)\right\}\right)$ $=\frac{\pi}{2}$, we have

$$\begin{aligned}\nu&=\frac{1}{4}\langle R(X+Y, JX+JY)(-X+Y), JX-JY\rangle\\&=\frac{1}{4}\{-2\nu+4K(X, Y)+H(X)+H(Y)\},\end{aligned}$$

from which

$$H(X)+H(Y)=6\nu-4K(X, Y). \tag{14}$$

Also, since $\sphericalangle\left(\left\{\frac{1}{\sqrt{2}}(X-JY)\right\}, \left\{\frac{1}{\sqrt{2}}(Y-JX)\right\}\right)=\frac{\pi}{2}$, we have

$$\nu=\frac{1}{4}\langle R(X-JY, JX+Y)(JY+X), Y-JX\rangle$$

$$=\frac{1}{4}\{2\nu-4K(X, Y)+H(X)+H(Y)\},$$

from which

$$H(X)+H(Y)=2\nu+4K(X, Y). \tag{15}$$

This, together with (14), implies

$$K(X, Y)=\frac{1}{2}\nu, \qquad H(X)+H(Y)=4\nu.$$

Hence the anti-holomorphic sectional curvature of M is a constant $\frac{1}{2}\nu$. Therefore, by Theorem A, we see that M is a space of constant holomorphic sectional curvature 2ν. q. e. d.

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